Assignment 9

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1. Let $\Omega \overset{o}{\subset} \mathbb{R}^n$ be bounded. Prove that
$$\big(-\triangle, \gamma_{\partial\Omega}\big) \in \mathcal{L}_{is}\big(\mathrm{H}^2(\Omega), \mathrm{L}_2(\Omega) \times \mathrm{H}^{3/2}(\partial\Omega)\big)\,,$$
that is, that the boundary value problem
$$\begin{cases} -\triangle u = f & \text{in } \Omega\,, \\ u = g & \text{on } \partial\Omega\,, \end{cases}$$
has a unique solution $u \in \mathrm{H}^2(\Omega)$ for any choice of data $(f, g) \in \mathrm{L}_2(\Omega) \times \mathrm{H}^{3/2}(\partial\Omega)$ satisfying
$$\|u\|_{\mathrm{H}^2(\Omega)} \leq c\big(\|f\|_{\mathrm{L}_2(\Omega)} + \|g\|_{\mathrm{H}^{3/2}(\partial\Omega)}\big)$$

2. Let $\Omega \overset{o}{\subset} \mathbb{R}^n$ be bounded and connected. Prove that any weak solution of
$$\begin{cases} -\triangle u = 0 & \text{in } \Omega\,, \\ \partial_\nu u = 0 & \text{on } \partial\Omega\,, \end{cases}$$
must be constant.

3. Let $\Omega \overset{o}{\subset} \mathbb{R}^n$ be bounded and $f : \mathbb{R} \to \mathbb{R}$ be smooth and bounded. A function $u \in \overset{0}{\mathrm{H}}{}^1(\Omega)$ satisfying
$$\int_\Omega \nabla u \nabla v\, dx = \int_\Omega f(u) v\, dx \;\forall\, v \in \mathcal{D}(\Omega)$$
is called weak solution of
$$\begin{cases} -\triangle u = f(u) & \text{in } \Omega\,, \\ u = 0 & \text{on } \partial\Omega\,. \end{cases}$$

Prove that any such weak solution u is in $\mathrm{H}^2(\Omega)$ and, then, in $\mathrm{C}^\infty(\overline{\Omega})$.

A linear operator $K \in \mathcal{L}(X, Y)$ between Banach spaces X, Y is said to be *compact* iff
$$(Kx_n)_{n\in\mathbb{N}} \text{ is relatively compact in } Y \text{ whenever } (x_n)_{n\in\mathbb{N}} \text{ is bounded in } X\,.$$

4. Let $K \in \mathcal{L}(H)$ be a compact operator on a Hilbert space H. Show that
(i) $u_n \rightharpoonup u \, (n \to \infty) \implies Kx_n \longrightarrow Ku \, (n \to \infty)$.
(ii) K^* is compact.
(iii) $\ker(1 - K)$ is finite dimensional.

5. Let $\mathcal{A}u = -\operatorname{div}\big(\Lambda\nabla u\big)+b\cdot\nabla u+cu$ define a uniformly elliptic operator with smooth coefficients. Show that the subspace for which

$$\begin{cases} \mathcal{A}u = 0 & \text{in } \Omega\,, \\ u = 0 & \text{on } \partial\Omega\,, \end{cases}$$

is finite dimensional.

Homework due by Friday, February 23 2007