1. Let $\Omega \subset \mathbb{R}^n$ be bounded. Prove that 
\[
(-\Delta, \gamma_{\partial \Omega}) \in \mathcal{L}_{is}(H^2(\Omega), L_2(\Omega) \times H^{3/2}(\partial \Omega)),
\]
that is, that the boundary value problem
\[
\begin{cases}
-\Delta u = f & \text{in } \Omega, \\
u = g & \text{on } \partial \Omega,
\end{cases}
\]
has a unique solution $u \in H^2(\Omega)$ for any choice of data $(f, g) \in L_2(\Omega) \times H^{3/2}(\partial \Omega)$ satisfying
\[
\|u\|_{H^2(\Omega)} \leq c(\|f\|_{L_2(\Omega)} + \|g\|_{H^{3/2}(\partial \Omega)}).
\]

2. Let $\Omega \subset \mathbb{R}^n$ be bounded and connected. Prove that any weak solution of
\[
\begin{cases}
-\Delta u = 0 & \text{in } \Omega, \\
\partial_\nu u = 0 & \text{on } \partial \Omega.
\end{cases}
\]
must be constant.

3. Let $\Omega \subset \mathbb{R}^n$ be bounded and $f : \mathbb{R} \to \mathbb{R}$ be smooth and bounded. A function $u \in H^1(\Omega)$ satisfying
\[
\int_{\Omega} \nabla u \nabla v \, dx = \int_{\Omega} f(u)v \, dx \quad \forall \, v \in \mathcal{D}(\Omega)
\]
is called weak solution of
\[
\begin{cases}
-\Delta u = f(u) & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega.
\end{cases}
\]
Prove that any such weak solution $u$ is in $H^2(\Omega)$ and, then, in $C^\infty(\Omega)$.

A linear operator $K \in \mathcal{L}(X, Y)$ between Banach spaces $X, Y$ is said to be compact iff 
\[
(Kx_n)_{n \in \mathbb{N}} \text{ is relatively compact in } Y \text{ whenever } (x_n)_{n \in \mathbb{N}} \text{ is bounded in } X.
\]

4. Let $K \in \mathcal{L}(H)$ be a compact operator on a Hilbert space $H$. Show that 
(i) $u_n \rightharpoonup u \, (n \to \infty) \implies Kx_n \rightharpoonup Ku \, (n \to \infty)$. 
(ii) $K^*$ is compact. 
(iii) ker$(1 - K)$ is finite dimensional.
5. Let \( Au = -\text{div}(A \nabla u) + b \cdot \nabla u + cu \) define a uniformly elliptic operator with smooth coefficients. Show that the subspace for which

\[
\begin{align*}
Au &= 0 \quad \text{in } \Omega, \\
u &= 0 \quad \text{on } \partial \Omega,
\end{align*}
\]

is finite dimensional.

Homework due by Friday, February 23 2007