You Ask a Question Week 1
Math 2A – Winter 2012

A *function* f is a triple (X, Y, G) where X and Y are arbitrary sets and $G \subset X \times Y$, the graph of f, satisfies a condition (see below). This simply means that, in order to define a function, one needs three sets, one for its *arguments* and one for its *values*, and one, $G \subset X \times Y$, that establishes a relation between them. Notice that

$$X \times Y = \{(x, y) \,|\, x \in X \text{ and } y \in Y\}.$$

We require that the set G satisfy the following property: for any given $x \in X$ there exists at most a $y \in Y$ s.t. $(x, y) \in G$. If $(x, y) \in G$, we simply write $y = f(x)$. Observe that this is sensible since there is only one y for which this can be true. It might appear as though the knowledge of the values $f(x)$ of f is all we need to know but, in reality, the *domain of definition* $\mathrm{dom}(f)$ and the *image set* $\mathrm{im}(f)$ of a function f are crucial in determining it and its properties. These are given by

$$\mathrm{dom}(f) = \{x \in X \,|\, \exists\, y \in Y \text{ s.t. } (x, y) \in G\}$$

and

$$\mathrm{im}(f) = \{y \in Y \,|\, \exists\, x \in X \text{ s.t. } (x, y) \in G\}$$

respectively.

Examples.

a. The sets X and Y could be anything. Take for instance X to be the set of all students registered in this class and Y to be set of colors. Then a function $c : X \to Y$ can be defined by assigning to each students $x \in X$ the color $y = c(x) \in Y$ of his or her hair. Clearly there is at most one such color for any given student. The domain $\mathrm{dom}(f)$ would consist of all students who are not bold and the image $\mathrm{im}(f)$ of all color actually occurring.

b. In this course we shall typically consider only functions for which $X \subset \mathbb{R}$ and $Y \subset \mathbb{R}$, that is *real real-valued* functions. Such a function is for instance $f : \mathbb{R} \to \mathbb{R}\,,\ x \mapsto x^2$, where then

$$\mathrm{dom}(f) = \mathbb{R}\,,\ \mathrm{im}(f) = [0, \infty), \text{ and } G = \{(x, x^2) \,|\, x \in \mathbb{R}\}.$$

The graph consists clearly of the points lying in the plot of the function.

c. Special functions are so-called *affine functions*. Their graphs are all non-vertical lines in the coordinate plane. Their general form is

$$y = mx + b\,,\ x \in \mathbb{R} \text{ (for some } m, b \in \mathbb{R})$$

so that $\mathrm{dom}(f) = \mathbb{R}$ and $Y = \mathbb{R}$, if $m \neq 0$ or $Y = \{b\}$, if $m = 0$ (check!). The number m is the *slope* of the line whereas b is the *y-intercept*, i.e. the value at the origin ($x = 0$), which clearly has to lie on the y-axis – hence the name).

d. An affine function is clearly also determined whenever two points in its graph are given (that is two points through which the corresponding line passes). Let two such points be labeled by $P_0 = (x_0, y_0)$ and $P_1 = (x_1, y_1)$, respectively. What is then the equation of the affine function?
Well we simply need to find m and b such that the two points given satisfy

$$\begin{cases} mx_0 + b = y_0 \\ mx_1 + b = y_1 \end{cases}$$

which leads to

$$m = \frac{y_1 - y_0}{x_1 - x_0}\,,$$

and to

$$y = y_0 + \frac{y_1 - y_0}{x_1 - x_0}(x - x_0)\,.$$

e. Functions can clearly behave as they please and won't necessarily have a name. Some, however, do. Polynomials are for instance such functions. They have the form

$$y = p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$$

for *coefficients* $a_0, \ldots, a_n$ and a natural number $n \in \mathbb{N}$ called its *degree*. As you can see they are generalizations of affine functions. If you were asked to determine a polynomial p such that

$$p(-1) = p(0) = p(3) = 0 \text{ and } p(2) = 1\,,$$

you would have to choose a polynomial of at least degree 3 because you need to satisfy 4 conditions and therefore need at least 4 "free" parameters. In this case the latter would be the unknown coefficients of

$$p(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3\,.$$

Plugging in the information we have, would lead to a 4 by 4 system of equations which could be solved. But we don't need to go through those calculations because we can make sure that the first three conditions be satisfied by simply taking

$$p(x) = c\,x(x + 1)(x - 3)$$

where c is a free constant which we fix by using the last condition

$$1 = p(2) = c2 \cdot 3 \cdot (-1) = -6c \Leftrightarrow c = -\frac{1}{6}\,.$$

f. Many asked questions about *odd* and *even* functions. Please refer to the book. They are simply functions which are either centrally symmetric with respect to the origin (the point $(0,0)$)

$$f(-x) = -f(x)\,,\ x \geq 0\,,$$

or are symmetric with respect to a reflection through the y-axis, i.e.

$$f(-x) = f(x)\,,\ x \geq 0\,.$$

g. A function $f : X \to Y$ is said to be *onto* if every value $y \in Y$ is actually taken on, that is, if, for any $y \in Y$, there is an $x \in X$ such that $y = f(x)$. It is said to be *one-to-one* if any value $y \in Y$ is taken on at most once, or, equivalently, if different arguments $x_1 \neq x_2$ in $\mathrm{dom}(f)$ always lead to different values $f(x_1) \neq f(x_2)$ in Y.
The function $f : \mathbb{R} \to \mathbb{R}\,,\ x \mapsto x^2$ is not onto nor one-to-one. In fact there is no real number such that $x^2 = -1$, for instance, and one has that

$$f(-5) = f(5) = 25\,,$$

that is, two different arguments share the same value. The function

$$g : \mathbb{R} \to [0, \infty)\,,\ x \mapsto x^2\,,$$

however, is onto but not one-to-one. Why?
Finally the function

$$h : [0, \infty) \to [0, \infty)\,,\ x \mapsto x^2\,,$$

is both onto and one-to-one (such functions are often called *bijective*). These last examples should also show you the importance of the choice of X and Y in the definition of a function! If you simply talked of the function $y = x^2$ you would not be able to distinguish between f, g, and h.