You Ask a Question Week 2
Math 2A – Winter 2012

Limits are the most important concept of this class. For a real real-valued function f we say that it has *limit* $l \in \mathbb{R}$ as $x \neq x_0$ approaches x_0 iff $f(x)$ approaches the value l in the process. The function f needs not even be defined in x_0 for this to make sense. Mathematically the definition is the following:

$$\lim_{x \to x_0} f(x) = l \;:\Leftrightarrow$$
$$\forall \varepsilon > 0 \, \exists \delta > 0 \text{ s.t. } |f(x) - l| \leq \varepsilon \text{ provided } 0 < |x - x_0| \leq \delta \, .$$

Since $\epsilon > 0$ can be chosen arbitrarily, the above means that l is the limiting value of the function f as $x \neq x_0$ approaches x_0 if, no matter how small (and positive) ε is chosen, it is always possible to make sure that the values $f(x)$ are at most at distance ε from the limit value l by taking x close enough to x_0, i.e. by taking δ small enough and

$$x_0 - \delta \leq x < x_0 \text{ or } x_0 < x \leq x_0 + \delta \, .$$

Imagine yourself walking towards x_0 on the real axis, even maybe hopping from one side to other of it (but always avoiding it), with a reader in your hands which gives you the value $f(x)$ of the function f at the location x where you stand at any given instant. You would say that f as limit l, if your reading gets closer and closer to l in the process.

Computing limits. Let us first define *continuity* for a real real-valued function f at a point x_0 in its domain of definition. We say that f is continuous at x_0 if the following is true:

$$\lim_{x \to x_0} f(x) = f(x_0) \, ,$$

that is, if the limit on the left-hand side exists, if f can be evaluated at x_0 (i.e. is defined there), and if these two numbers coincide. If the above equality does not hold, the function is called discontinuous at x_0.

It is clear that, if a function is continuous at x_0, then computing the limit $\lim_{x \to x_0} f(x)$ becomes a triviality: you just need to plug in the argument x_0!

The function might, however, not be defined at x_0, in which case plugging in won't do it. This is the case when the function is either given through its graph and you would have to find the limit by inspection, or given through an expression that is not defined at the point of interest x_0 or defined only piecewise. In latter two situations, further investigation is needed. In the first of the last two cases take for example the function

$$f(x) = \frac{x-1}{\sqrt{x}-1}\,,\ x \in [0,\infty)\,,\ x \neq 1\,,$$

and compute the limit as x approaches 1. The function is clearly not defined there and so plugging in won't work. Even if the function were defined there, plugging in might not work as the function could possibly be discontinuous there. We could of course start taking x closer and closer to 1 and monitor the values $f(x)$ to see if they are converging anywhere. But we can do better than that since we have an expression for the function which we can try and *massage* (i.e. use some algebra on it). Just observe that in this case

$$f(x) = \frac{(\sqrt{x}-1)(\sqrt{x}+1)}{\sqrt{x}-1} = \sqrt{x}+1\,,\ 0 \leq x \neq 1\,.$$

This means that we can use this equivalent expression (away from $x = 1$) in order to compute the limit. This expression is, however, continuous, and we get the limit simply by plugging in

$$\lim_{x\to 1} \frac{x-1}{\sqrt{x}-1} = 1 + \sqrt{1} = 2\,.$$

In the second case, take for instance

$$f(x) = \begin{cases} x-1\,, & x < -2\,, \\ x+1\,, & x \geq -2 \end{cases}$$

This function is clearly defined by an expression if $x < -2$ and a different one if $x \geq -2$. In class we have defined *left* and *right limits* and have also seen that

$$\lim_{x\to x_0} f(x) = l \Leftrightarrow \lim_{x\to x_0-} f(x) = \lim_{x\to x_0+} f(x) = l\,,$$

or, in other words, that a limit is l if and only if both the corresponding left and right limits exist and coincide with l. This comes in handy for the computation of

$$\lim_{x\to -2} f(x)$$

for the above function. In view of the definition of f it is easy to get that

$$\lim_{x\to -2-} f(x) = -2-1 = -3 \text{ and } \lim_{x\to -2+} f(x) = -2+1 = -1\,,$$

and therefore conclude that the limit does not exist.
The original reason and goal for our interest in the computation of limits is to make sense of

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

whenever possible, because it would deliver the slope of the line tangent to the graph of f at the the point $\big(x_0, f(x_0)\big)$. This limit is tricky because plugging in leads to $\frac{0}{0}$. This is a situation, along with $\frac{\infty}{\infty}$ and $\infty - \infty$ where we need to be careful as these expressions could actually evaluate to anything at all. Take for instance the first situation and consider

$$\lim_{x \to 3} \frac{x-3}{(x-3)^2}\,,\ \lim_{x \to 3} \frac{(x-3)^2}{\pi (x-3)^2}\,,\ \lim_{x \to 3} \frac{(x-3)^2}{(x-3)^4}\,,\ \lim_{x \to 3} \frac{(x-3)^5}{(x-3)^4}\,,$$

which all evaluate to $\frac{0}{0}$. The various expressions can be massaged to yield

$$\frac{1}{x-3}\,,\ \frac{1}{\pi}\,,\ \frac{1}{(x-3)^2}\,,\ (x-3)\,,$$

repectively. In the first case the limit does not exist because the expression approaches $-\infty$ getting close to 3 from the left and approaches ∞ coming from the right. The second limit clearly evaluates to $\frac{1}{\pi}$, the third to ∞ because the square makes sure that the denominator is always positive no matter on which side of 3 the argument x is, the last to 0 by simple plugging in.