You Ask a Question Week 5
Math 2A – Winter 2012

This week we focused our attention on the *chain rule* and on *implicit differentiation*. The chain rule helps us find derivatives of more involved functions which can be viewed as compositions of simpler functions, such as for instance $f(x) = \cos^5(x)$. Since we already know how to differentiate the functions $g(x) = \cos(x)$ and $h(x) = x^5$, it is conceivable that it be possible to find a relation between f', on the one hand, and g' and h', on the other. In fact it holds that

$$f'(x) = h'(g(x))g'(x) = 5\big(g(x)\big)^4 g'(x) = 5\cos^4(x)\sin(x)\,.$$

It goes without saying that one needs the differentiability of all functions involved to ensure validity of the rule. Notice that, while the rule is stated for the composition of two functions, it is actually valid for the composition of any finite number of functions. Take for instance the function $f(x) = \sin\big(\sqrt{\frac{x-1}{x+1}}\big)$ which can be viewed as

$$f(x) = h\Big(g\big(\frac{k(x)}{j(x)}\big)\Big)\,.$$

It is the composition of the three functions

$$h(x) = \sin(x)\,,\ g(x) = \sqrt{x}\,,\text{ and } l(x) = \frac{k(x)}{j(x)}\,.$$

In a first step just think of f as the composition of the two functions h and $g \circ l$ to obtain that

$$f'(x) = h'\big(g \circ l(x)\big)(g \circ l)'(x)\,,$$

and then apply the chain rule again to the term $(g \circ l)'$ to get

$$(g \circ l)'(x) = g'\big(l(x)\big)l'(x)\,.$$

Finally apply the quotient rule to see that

$$l'(x) = \frac{k'(x)j(x) - k(x)j'(x)}{j^2(x)}\,.$$

In summary we have that

$$f'(x) = h'\Big(g\big(l(x)\big)\Big)g'\big(\frac{k(x)}{j(x)}\big)\frac{k'(x)j(x) - k(x)j'(x)}{j^2(x)}\,,$$

or that

$$f'(x) = \cos\Big(\sqrt{\frac{x-1}{x+1}}\Big)\frac{1}{2\sqrt{\frac{x-1}{x+1}}}\frac{2}{(x+1)^2}$$
$$= \frac{1}{(x-1)^{1/2}(x+1)^{3/2}}\cos\Big(\sqrt{\frac{x-1}{x+1}}\Big)\,.$$

Let us now turn to implicit differentiation. So far we have been considering mainly functions and their graphs. Graphs of functions are special curves which have at most one intersection point with any vertical line $x = x_0$. This is because functions are required by definition to have at most one value per argument. Even simple curves as the circle, which is the totality of points (x, y) satisfying

$$x^2 + y^2 = 1\,, \tag{1}$$

violates this requirement and is therefore NOT the graph of a function. We used derivatives as a tool to obtain the slope of lines tangent to the graph of functions. Now we have a curve, the circle, which clearly should have a tangent line at any of its points but does NOT have a representation as the graph of a function. So how do we go about computing the slope of tangent lines in this case? Assume that x is given, then relation (1) should determine none, one or more values y for which it holds true. For instance, take $x = 5$, then clearly there is no way we can find a y such that (1) is satisfied. If $x = 1/\sqrt{2}$, however, then both $y = 1/\sqrt{2}$ and $y = -1/\sqrt{2}$ will do. For $x = \pm 1$ the only choice would be $y = 0$. Now, in spite of the fact that y cannot be written as a function of x, it still clearly depends on x. The relation can also be more complicated such as

$$\sin(x + y) = x^2 \cos(y)\,,$$

and make it virtually impossible to solve for y - unlike (1). Still the relation determines some set of points and we might be interested in finding an equation for some specific tangent line. As it turns out, this is often possible by taking a derivative of the relation at hand in order to generate a relation containing derivatives. The latter can then be used to try and determine them. Let us work out the examples above. We think of x as the independent variable and y as the dependent variable. Then we proceed by taking a derivative with respect to the independent variable x to obtain

$$2x + 2y\frac{dy}{dx} = \frac{d}{dx}x^2 + \frac{d}{dx}y^2 = \frac{d}{dx}1 = 0$$

in the first case and

$$\cos(x+y)\big(1+\frac{dy}{dx}\big) = \frac{d}{dx}\sin(x+y) = \frac{d}{dx}\big(x^2\cos(y)\big) = 2x\cos(y) - x^2\sin(y)\frac{dy}{dx}\,,$$

in the second. In both cases the relations can be solved for $\frac{dy}{dx}$ to yield

$$y' = \frac{dy}{dx} = -\frac{y}{x}\,,$$

and

$$y' = \frac{dy}{dx} = \frac{2x\cos(y) - \cos(x+y)}{\cos(x+y) - x^2\sin(y)}\,,$$

respectively. These relations can clearly deliver the slope of the tangent to the curve determined by the original relation at any point (x, y) of that curve where the expressions on the right-hand-side are defined. In the example of the circle, take the point $(x, y) = (-1/\sqrt{2}, -1/\sqrt{2})$, for instance, and you would get that

$$y' = -\frac{y}{x} = -\frac{-1/\sqrt{2}}{-1/\sqrt{2}} = -1\,.$$

Thus, by using implicit differentiation, we were able to establish that the slope of the tangent to the unit circle at the above point is -1 without needing to have an explicit representation of the circle as the graph of a function.