You Ask a Question Week 6
Math 2A – Winter 2012

This week's topics were *related rates* and finding *extrema of functions*. Essentially all related rates problems have the same basic structure. There are two evolving (changing in time or so) quantities x and y. You know how one of them is changing (that is, you know its derivative) and you need to determine how the other one is changing when the system (x, y) is in a certain state (x_0, y_0). Clearly it will only be possible to solve such a problem provided a *relation* between the quantities x and y can be found. This means that an equation relating the two must be available; typically it is given by some geometric formula such as Pythagoras' theorem relating the sides of a right triangle and its hypotenuse, a volume formula relating radius or base area and height to the object's volume, or by some balance between inflow and outflow in tank problems, or ... Schematically you would need to:

1. Determine the quantities of interest. The given information and the final question will typically point to those.
2. Find a relation between them.
3. Gather what is known and use the relation to determine what is not known.

Let us take a look at a specific example.
Problem: *Two people, X and Y, are* 50 *feet apart. Denote by O the point at which X is standing initially and by B the one where Y is standing. X starts walking north at such a rate that the angle θ in B of the triangle OBC (C is X's current position) is changing at a constant rate of* 0.01 *rad/min. At what rate is distance between the two people changing when $\theta = 0.5$ rad?*

Solution: Notice that we are given information about an angle and are asked information about a distance. Thus the **quantities of interest** are **the angle OBC** (θ) and the **distance between X and Y** (d). Next we need to find a relation between them. The triangle OBC is a right triangle the hypotenuse of which is precisely d and where the angle opposite to OC is θ. The side OB of the triangle measures 50 feet and thus trigonometry tells us that

$$50/d = \cos(\theta) \text{ or } 50 = d\cos(\theta)\,.$$

Clearly the two quantities are changing by the minute and we indeed know that the angle is changing at the constant rate of 0.01 rad/min, that is,

$$\frac{d}{dt}\theta = \theta' = 0.01 \text{ (rad/min)}.$$

We are interested in figuring out the rate at which d is changing when $\theta = 0.5$ rad. Whenever it is that this happens the relation tells us that

$$d = 50/\cos(0.5) \approx 56.9747 \text{ (feet)},$$

and taking one derivative with respect to time of the relation yields

$$0 = d' \cos(\theta) - d \sin(\theta)\theta' \text{ or } d' = d \tan(\theta)\theta'.$$

Plugging in the above information, we arrive at

$$d' \approx 0.3113 \text{ (ft/min)}$$

for rate of change of distance.

Next let us turn to finding extrema of functions. It clearly would be useful to know that a given functions has a maximum and a minimum before even trying to find one! This is ensured by the following theorem.
Any **continuous** *function* $f : [a, b] \to \mathbb{R}$ *assumes its maximum and its minimum.* In class we have seen that continuity is essential as is the requirement that the end points be included! We also proved that $f'(x_0) = 0$ *for any function* $f : (a, b) \to \mathbb{R}$ *which has a* **(local) maximum or minimum at** x_0 *and is* **differentiable there** (Fermat's theorem). Combining the two results we obtain a recipe for finding maxima and minima of continuous functions on closed intervals $[a, b]$:

1. Compute $f(x_0)$ at all points $x_0 \in (a, b)$ for which $f'(x_0) = 0$.

2. Compute $f(a)$ and $f(b)$.

3. The largest value will correspond to a point of maximum and the smallest one to a point of minimum.

We should keep in mind that x_0 is a *point of maximum* for $f : [a, b] \to \mathbb{R}$ if

$$f(x) \leq f(x_0)\, \forall\, x \in [a, b],$$

whereas it is a point of *local maximum* if the same inequality is valid in a region around x_0, that is, if a $\delta > 0$ can be found such that

$$f(x) \leq f(x_0)\, \forall\, x \in (x_0 - \delta, x_0 + \delta).$$

Minima and local minima are defined similarly by reversing the inequality.