You Ask a Question Week 9
Math 2A – Winter 2012

The majority of questions this week were about *slant asymptotes*. Let me step back and consider a slightly more general problem. Asymptotes are used to describe the behavior of a function. It can be argued that linear and affine functions, i.e. functions the graph of which are lines, a very simple and understandable. When looking for an asymptote of a given function we are looking for lines which well describe the behavior of the function in some limit. This limit always involves a quantity going to infinity, be it x or y. More in general if you feel like a certain function g is particularly simple (in terms of its behavior in one of the limits mentioned), then you could try and describe the behavior of another given function f by comparing it to g in the liit of interest. Assuming that you are interested in the behavior at $\pm\infty$, you can look at

$$\lim_{x\to\infty}\big(f(x)-g(x)\big) \text{ or } \lim_{x\to\infty}\big(f(x)-g(x)\big)$$

and check whether it vanishes. If it does, it would be natural to say that f and g have the same behavior at $x=\infty$ or at $x=-\infty$, respectively. Take for instance the two functions $g(x)=x^2$ and $f(x)=\frac{x^4+1}{x^2}$. While the two functions are clearly different, if we care about their behavior at $x=\infty$, we see that

$$\lim_{x\to\infty}\big(f(x)-g(x)\big)=\lim_{x\to\infty}\big(\frac{x^4+1}{x^2}-x^2\big)=\lim_{x\to\infty}\frac{x^4+1-x^4}{x^2}=\lim_{x\to\infty}\frac{1}{x^2}=0\,.$$

This means that the two functions become undistinguishable as x gets larger and larger. We also say that f is asymptotic to g as $x\to\infty$.

Horizontal and slant are simple instances of the above where the function g is simply a line

$$g_{m,b}(x)=mx+b\,,\ x\in\mathbb{R}\,,$$

for some $m,b\in\mathbb{R}$. Thus a function f would be asymptotic to $g_{m,b}$ as $x\to\infty$ in this case if it looks like a line out at infinity. If $m=0$, clearly the line is horizontal, hence the name *horizontal asymptote*. In many cases a slant aymptote occurs for a rational function when the top polynomial is of one degree higher than the polynomial in the denominator because, in that case, the function will clearly behave linearly for both $x\to\infty$ and $x\to-\infty$. Take for instance

$$f(x)=\frac{4x^{101}+34x^{100}-x^{36}+5}{2x^{100}+x^8-3x}\,.$$

By pulling x^{100} out, we see that

$$f(x) = \frac{x^{100}}{x^{100}} \frac{4x + 34 - x^{-64} + 5x^{-100}}{2 + x^{-92} - 3x^{-99}} = \frac{4x + 34 - x^{-64} + 5x^{-100}}{2 + x^{-92} - 3x^{-99}}$$

Next neglecting the smaller terms (the ones with negative powers of x) in the above expression, we see that this function should behave as

$$y = 2x + 17$$

for x large (positive or negative). This is not a formally correct proof but, with this guess at hand, we can compute

$$f(x) - (2x + 17) = \frac{4x + 34 - x^{-64} + 5x^{-100}}{2 + x^{-92} - 3x^{-99}} - (2x + 17)\frac{2 + x^{-92} - 3x^{-99}}{2 + x^{-92} - 3x^{-99}}$$

$$= \frac{4x + 34 - x^{-64} + 5x^{-100} - 4x - 2x^{-91} + 6x^{-98} - 34 - 17x^{-92} - 51x^{-99}}{2 + x^{-92} - 3x^{-99}}$$

$$\frac{-x^{-64} + 5x^{-100} - 2x^{-91} + 6x^{-98}}{2 + x^{-92} - 3x^{-99} - 17x^{-92} - 51x^{-99}} \longrightarrow 0 \text{ as } x \to \pm\infty\,.$$

Rational functions of this type are, however, not the only functions which are asymptotic to a linear function as the following functions

$$f(x) = \frac{3x + 2}{1 + x^{-2}\cos(4x)}$$

$$g(x) = x(3 + e^{-x}) + 2 + \frac{1}{3 + x^4}$$

$$h(x) = \sqrt{9x^2 + 1} + 2\,,$$

which are all asymptotic to $y = 3x + 2$ as $x \to \infty$, clearly demonstrate. Check this claim for all three examples!

A *vertical asymptote* occur when a function looks like a vertical line through some point $(x_0, 0)$. This happens when the functions becomes larger and larger (negative or positive) as the point x_0 is approached, or if

$$\lim_{x \to x_0} f(x) = \pm\infty\,.$$

This x_0 would clearly be a point of discontinuity of f. Examples of vertical are obtained by taking functions with vanishing denominators such as

$$f(x) = \frac{2}{1 - x}$$

$$g(x) = \frac{e^x}{e^{x-1} - e^{-x+1}}$$

$$h(x) = \frac{x^2 + 1}{\cos(x - 1) - 1}\,,$$

which all have a vertical asymptote at $x = 1$.