Mid Term Solutions

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1. (a) The homogeneous equation has constant coefficients and its general solution v can be computed by means of the characteristic equation

$$r^2 + 2r + 1 = (r+1)^2 = 0\,.$$

It has the double root $r = -1$ and therefore

$$v(t) = c_1\, e^{-t} + c_2\, te^{-t}\,.$$

A particular solution z to the inhomogeneous equation can now be computed either by the variation-of-parameters formula or by the Ansatz $z(t) = A\, t^p\, te^{-t}$ for $p = 1, 2$. The case $p = 0$ can be excluded right away since it leads to a solution of the homogeneous equation. It turns out that $p = 2$ is the right choice. In fact

$$z'(t) = 3A\, t^2 e^{-t} - A\, t^3 e^{-t}\,,\ z''(t) = 6A\, te^{-t} - 6A\, t^2 e^{-t} + A\, t^3 e^{-t}$$

and consequently

$$z''(t) + 2z'(t) + z(t) = A\, e^{-t}\big[6t - 6t^2 + t^3 + 6t^2 - 2t^3 + t^3\big] = 6A\, te^{-t}\,.$$

Now, in order for the equation to be satisfied, we have to choose $A = 1/6$. Summarizing we obtain the general solution

$$y(t) = c_1\, e^{-t} + c_2\, te^{-t} + \frac{1}{6} t^3 e^{-t}\,.$$

1. (b) This equation has non constant coefficients but doesn't involve the function y at all. If we rename y' to u the equation reads

$$u' - tu = t\,.$$

Separating the variables we obtain

$$\int \frac{(1+u(t))'}{1+u(t)}\, dt = c + \int t\, dt$$

which leads to $u(t) = e^c e^{0.5t^2} = c_1\, e^{0.5t^2} - 1$. Finally

$$y(t) = \int u\, dt = c_2 + c_1 \int e^{0.5t^2}\, dt - t\,.$$

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2. Let us discuss the equation first. The initial value y_0 is assumed to be positive. By inspecting the right-hand-side we can see that

$$\begin{cases} y' < 0\,, & \text{if } y > \beta/\alpha \\ y' = 0\,, & \text{if } y = \beta/\alpha \\ y' > 0\,, & \text{if } 0 < y < \beta/\alpha \end{cases}$$

This already tells us that, if the solution starts out above β/α, it is going to decrease, that it is going to remain constant if $y_0 = \beta/\alpha$ and that it is going to grow if $y_0 < \beta/\alpha$. This only leaves

β/α as a candidate for $\lim_{t\to\infty} y(t)$. To analyze the solution's convexity we need to compute y''. We can do so using the equation to obtain

$$y'' = \big((\alpha - \beta y)\, y\big)' = y(\alpha - 2\beta y)(\alpha - \beta y)\,.$$

We therefore see that

$$\begin{cases} y'' < 0\,, & \text{if } \beta/2\alpha < y < \beta/\alpha \\ y'' = 0\,, & \text{if } y = \beta/\alpha\,,\ \beta/2\alpha \\ y'' > 0\,, & \text{if } 0 < y < \beta/2\alpha \text{ or } y > \alpha/\beta \end{cases}$$

To compute the solution we observe that the equation is separable and that

$$\frac{1}{y(\alpha - \beta y)} = \frac{1}{\alpha}\frac{1}{y} + \frac{\beta}{\alpha}\frac{1}{\alpha - \beta y}\,.$$

This leads to

$$\begin{aligned} \alpha\, t = \alpha \int_0^t \frac{y'}{y(\alpha - \beta y)}\, d\tau &= \int_0^t \frac{y'}{y}\, d\tau + \int_0^t \frac{\beta y'}{\alpha - \beta y}\, d\tau \\ &= \ln\frac{|y(t)|}{|y_0|} - \ln\frac{|\alpha - \beta y(t)|}{|\alpha - \beta y_0|} = \ln\frac{|\alpha - \beta y_0|}{|y_0|}\frac{|y(t)|}{|\alpha - \beta y(t)|} \end{aligned}$$

Noticing that $0 < y(t) < \alpha/\beta$ whenever $0 < y_0 < \alpha/\beta$ and that $y(t) > \alpha/\beta$ whenever $y_0 > \alpha/\beta$ we can eliminate the absolute values and solve for $y(t)$ obtaining

$$y(t) = \frac{\alpha y_0 e^{\alpha t}}{\alpha + \beta y_0 (e^{\alpha t} - 1)}$$

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3. Assuming continuous compounding, the evolution of an initial amount of money K_0 invested in an instrument with yearly return rate r and annual costs of C is described by

$$K'(t) = rK(t) - C\,,\ t > 0\,,\ K(0) = K_0\,.$$

This equation was derived in class and its solution is given by

$$K(t) = \frac{C}{r} + (K_0 - \frac{C}{r})e^{rt}\,.$$

(a) Answering the first question amounts to finding $t > 0$ such that

$$2K_0 = \frac{C}{r} + (K_0 - \frac{C}{r})e^{rt} = 750 + (K_0 - 750)e^{0.08t}\,.$$

We therefore obtain

$$t = 12.5 \ln\big(\frac{2K_0 - 750}{K_0 - 750}\big)\,.$$

(b) Here we need to compare

$$K_1(10) = 750 + (K_0 - 750)e^{0.8}$$

to

$$K_2(10) = 1200 + (K_0 - 1200)e^{1}$$

for $K_0 = 2000$. This gives

$$K_1(10) \approx 3500 \text{ and } K_2(10) \approx 3360$$

We would therefore pick the first fund for our investment.

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4. By mere inspection we see that

$$\begin{cases} y' < 0\,, & \text{if } y > 1 \text{ or } y < -1 \\ y' = 0\,, & \text{if } y = \pm 1 \\ y' > 0\,, & \text{if } -1 < y < 1 \end{cases}$$

We therefore conclude that

$$\lim_{t\to\infty} y(t, y_0) = 1, \text{ provided } y_0 > -1$$

as well as $\lim_{t\to\infty} y(t, -1) = -1$. After computing the solution we shall see that it blows up in finite time in the remaining case ($y_0 < -1$). The equation is separable, in fact

$$\frac{y'}{1-y^2} = 1\,(y \neq \pm 1)\,.$$

In order to integrate the left-hand-side of the above expression first notice that

$$\frac{1}{1-y^2} = \frac{1}{(1+y)(1-y)} = \frac{1}{2}\frac{1}{1+y} + \frac{1}{2}\frac{1}{1-y}\,.$$

Then

$$2t \;=\; 2\int_0^t \frac{y'(\tau)}{1-y(\tau)^2}\,d\tau \;=\; \int_0^t \Big(\frac{y'(\tau)}{1+y(\tau)} \;+\; \frac{y'(\tau)}{1-y(\tau)}\Big)\,d\tau \;=\; \ln\Big(\frac{|1+y(t)|}{|1+y_0|}\frac{|1-y_0|}{|1-y(t)|}\Big)$$

Noticing that $1 \pm y(t)$ always has the same sign as $1 \pm y_0$ we can suppress the absolute values to obtain

$$\frac{1+y(t)}{1-y(t)}\frac{1-y_0}{1+y_0} = e^{2t}$$

from which follows that

$$y(t) = \frac{y_0 - 1 + (1+y_0)e^{2t}}{1 - y_0 + (1+y_0)e^{2t}}\,.$$

When $y_0 < -1$ the denominator becomes zero if $e^{2t} = \frac{y_0-1}{1+y_0}$, that is, when $t = \frac{1}{2}\ln\frac{y_0-1}{1+y_0}$.

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5. The find the evolution of the fish population y we need to solve the given inhomogeneous first order differential equation. Since a solution of the homogeneous equation is given by e^t and the initial population amounts to a 1000 fish, we obtain

$$y(t) = 1000\,e^t - Fe^t\int_0^t e^{-\tau}\big(1+\cos(\tau)\big)\,d\tau\,.$$

using the variation-of-parameters formula. The integral can be computed by double integration by parts, for instance, to give

$$y(t) = (1000 - \frac{5}{2}F)e^t + F\big[1 - \frac{1}{2}\sin(t) + \frac{1}{2}\cos(t)\big]\,.$$

Now, the exponential term dominates for large times. To prevent extinction we therefore certainly need to at least impose that $1000 - \frac{5}{2}F \geq 0$ or, equivalently, that $F \leq 400$. Otherwise, in fact, if $F > 400$ extinction is unavoidable. What if $F = 400$? Then

$$y(t) = 400\big[1 - \frac{1}{2}\sin(t) + \frac{1}{2}\cos(t)\big]$$

but $1 - \frac{1}{2}\sin(t) + \frac{1}{2}\cos(t) > 0$ for all times and therefore the maximal allowed fishing quota is precisely $F = 400$.

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6. Since the equation is inhomogeneous and $g(t)$ is not specified we can only compute the solution by the variation-of-parameters formula. First we compute the general solution of the homogeneous equation via the characteristic equation

$$r^2 - \alpha = 0$$

which leads to the two linearly independent solutions

$$y_1(t) = e^{\sqrt{\alpha}t} \text{ and } y_2(t) = e^{-\sqrt{\alpha}t}\,.$$

Then a particular solution is given by

$$y_p(t) = -y_1(t)\int_0^t \frac{y_2(\tau)g(\tau)}{W(y_1,y_2)(\tau)}\,d\tau + y_2(t)\int_0^t \frac{y_1(\tau)g(\tau)}{W(y_1,y_2)(\tau)}\,d\tau$$

Since $W(y_1,y_2)(\tau) = -2\sqrt{\alpha}$ this amounts to

$$\begin{aligned} y_p(t) &= \frac{1}{2\sqrt{\alpha}}\Big[e^{\sqrt{\alpha}t}\int_0^t e^{-\sqrt{\alpha}\tau}g(\tau)\,d\tau - e^{-\sqrt{\alpha}t}\int_0^t e^{\sqrt{\alpha}\tau}g(\tau)\,d\tau\Big] \\ &= \frac{1}{\sqrt{\alpha}}\int_0^t \big[\frac{e^{\sqrt{\alpha}(t-\tau)} - e^{-\sqrt{\alpha}(t-\tau)}}{2}\big]g(\tau)\,d\tau = \frac{1}{\sqrt{\alpha}}\int_0^t \sinh\big(\sqrt{\alpha}(t-\tau)\big)g(\tau)\,d\tau \end{aligned}$$

The general solution is therefore given by

$$y(t) = c_1y_1(t) + c_2y_2(t) + y_p(t)\,.$$

Finally we use the boundary conditions to obtain the following system

$$\begin{cases} 0 = y(0) = c_1 + c_2 \\ 0 = y(1) = c_1e^{\sqrt{\alpha}} + c_2e^{-\sqrt{\alpha}} + \frac{1}{\sqrt{\alpha}}\int_0^1 \sinh\big(\sqrt{\alpha}(1-\tau)\big)g(\tau)\,d\tau \end{cases}$$

for c_1 and c_2. Solving it gives

$$c_1 = -c_2 = -\frac{1}{2\sqrt{\alpha}\sinh(\sqrt{\alpha})}\int_0^1 \sinh\big(\sqrt{\alpha}(1-\tau)\big)g(\tau)\,d\tau$$

and thus

$$y(t) = -\frac{1}{\sqrt{\alpha}}\frac{\sinh(\sqrt{\alpha}\,t)}{\sinh(\sqrt{\alpha})}\int_0^1 \sinh\big(\sqrt{\alpha}(1-\tau)\big)g(\tau)\,d\tau + \frac{1}{\sqrt{\alpha}}\int_0^t \sinh\big(\sqrt{\alpha}(t-\tau)\big)g(\tau)\,d\tau$$