1. (a) The homogeneous equation has constant coefficients and its general solution \( v \) can be computed by means of the characteristic equation
\[
r^2 + 2r + 1 = (r + 1)^2 = 0.
\]
It has the double root \( r = -1 \) and therefore
\[
v(t) = c_1 e^{-t} + c_2 t e^{-t}.
\]
A particular solution \( z \) to the inhomogeneous equation can now be computed either by the variation-of-parameters formula or by the Ansatz
\[
z(t) = A t^p t e^{-t}
\]
for \( p = 1, 2 \). The case \( p = 0 \) can be excluded right away since it leads to a solution of the homogeneous equation. It turns out that \( p = 2 \) is the right choice. In fact
\[
z'(t) = 3A t^2 e^{-t} - A t^3 e^{-t}, \quad z''(t) = 6A t e^{-t} - 6A t^2 e^{-t} + A t^3 e^{-t}
\]
and consequently
\[
z''(t) + 2z'(t) + z(t) = A e^{-t} [6t - 6t^2 + t^3 + 6t^2 - 2t^3 + t^3] = 6A t e^{-t}.
\]
Now, in order for the equation to be satisfied, we have to choose \( A = 1/6 \). Summarizing we obtain the general solution
\[
y(t) = c_1 e^{-t} + c_2 t e^{-t} + \frac{1}{6} t^3 e^{-t}.
\]

1. (b) This equation has non constant coefficients but doesn’t involve the function \( y \) at all. If we rename \( y' \) to \( u \) the equation reads
\[
u' - tu = t.
\]
Separating the variables we obtain
\[
\int \frac{(1 + u(t))'}{1 + u(t)} dt = c + \int t \, dt
\]
which leads to \( u(t) = e^c e^{0.5t^2} = c_1 e^{0.5t^2} - 1 \). Finally
\[
y(t) = \int u \, dt = c_2 + c_1 \int e^{0.5t^2} \, dt - t.
\]

2. Let us discuss the equation first. The initial value \( y_0 \) is assumed to be positive. By inspecting the right-hand-side we can see that
\[
\begin{align*}
y' &< 0, \quad \text{if } y > \beta/\alpha \\
y' &= 0, \quad \text{if } y = \beta/\alpha \\
y' &> 0, \quad \text{if } 0 < y < \beta/\alpha
\end{align*}
\]
This already tells us that, if the solution starts out above \( \beta/\alpha \), it is going to decrease, that it is going to remain constant if \( y_0 = \beta/\alpha \) and that it is going to grow if \( y_0 < \beta/\alpha \). This only leaves
\( \beta/\alpha \) as a candidate for \( \lim_{t \to \infty} y(t) \). To analyze the solution’s convexity we need to compute \( y'' \).

We can do so using the equation to obtain

\[
y'' = ((\alpha - \beta y) y')' = y(\alpha - 2\beta y)(\alpha - \beta y).
\]

We therefore see that

\[
\begin{align*}
y'' &< 0, \quad \text{if } \beta/2\alpha < y < \beta/\alpha \\
y'' &= 0, \quad \text{if } y = \beta/\alpha, \beta/2\alpha \\
y'' &> 0, \quad \text{if } 0 < y < \beta/2\alpha \text{ or } y > \alpha/\beta
\end{align*}
\]

To compute the solution we observe that the equation is separable and that

\[
\frac{1}{y(\alpha - \beta y)} = \frac{1}{\alpha y} + \frac{\beta}{\alpha \alpha - \beta y}.
\]

This leads to

\[
\alpha t = \alpha \int_0^t \frac{y'}{y(\alpha - \beta y)} d\tau = \int_0^t \frac{y'}{\alpha - \beta y} d\tau + \int_0^t \frac{\beta y'}{\alpha - \beta y} d\tau
\]

\[
= \ln \frac{|y(t)|}{|y_0|} - \ln \frac{|\alpha - \beta y(t)|}{|\alpha - \beta y_0|} = \ln \frac{|\alpha - \beta y_0|}{|\alpha - \beta y(t)|} \frac{|y(t)|}{|y_0|}.
\]

Noticing that \( 0 < y(t) < \alpha/\beta \) whenever \( 0 < y_0 < \alpha/\beta \) and that \( y(t) > \alpha/\beta \) whenever \( y_0 > \alpha/\beta \) we can eliminate the absolute values and solve for \( y(t) \) obtaining

\[
y(t) = \frac{\alpha y_0 e^{\alpha t}}{\alpha + \beta y_0 (e^{\alpha t} - 1)}.
\]

3. Assuming continuous compounding, the evolution of an initial amount of money \( K_0 \) invested in an instrument with yearly return rate \( r \) and annual costs of \( C \) is described by

\[
K'(t) = rK(t) - C, \ t > 0, \ K(0) = K_0.
\]

This equation was derived in class and its solution is given by

\[
K(t) = \frac{C}{r} + (K_0 - \frac{C}{r})e^{rt}.
\]

(a) Answering the first question amounts to finding \( t > 0 \) such that

\[
2K_0 = \frac{C}{r} + (K_0 - \frac{C}{r})e^{rt} = 750 + (K_0 - 750)e^{0.08t}.
\]

We therefore obtain

\[
t = 12.5 \ln\left(\frac{2K_0 - 750}{K_0 - 750}\right).
\]

(b) Here we need to compare

\[
K_1(10) = 750 + (K_0 - 750)e^{0.8}
\]

to

\[
K_2(10) = 1200 + (K_0 - 1200)e^1
\]

for \( K_0 = 2000 \). This gives

\[
K_1(10) \approx 3500 \text{ and } K_2(10) \approx 3360\]
We would therefore pick the first fund for our investment.

4. By mere inspection we see that

\[
\begin{cases}
    y' < 0, & \text{if } y > 1 \text{ or } y < -1 \\
    y' = 0, & \text{if } y = \pm 1 \\
    y' > 0, & \text{if } -1 < y < 1
\end{cases}
\]

We therefore conclude that

\[
\lim_{t \to \infty} y(t, y_0) = 1, \text{ provided } y_0 > -1
\]

as well as \(\lim_{t \to \infty} y(t, -1) = -1\). After computing the solution we shall see that it blows up in finite time in the remaining case \((y_0 < -1)\). The equation is separable, in fact

\[
\frac{y'}{1 - y^2} = 1 (y \neq \pm 1).
\]

In order to integrate the left-hand-side of the above expression first notice that

\[
\frac{1}{1 - y^2} = \frac{1}{(1 + y)(1 - y)} = \frac{1}{2} \left( \frac{1}{1 + y} + \frac{1}{1 - y} \right).
\]

Then

\[
2t = 2 \int_0^t \frac{y' (\tau)}{1 - y (\tau)^2} d\tau = \int_0^t \left( \frac{y' (\tau)}{1 + y (\tau)} + \frac{y' (\tau)}{1 - y (\tau)} \right) d\tau = \ln \left( \frac{|1 + y(t)|}{|1 + y_0|} \right)
\]

Noticing that \(1 \pm y(t)\) always has the same sign as \(1 \pm y_0\) we can suppress the absolute values to obtain

\[
\frac{1 + y(t)}{1 - y(t) (1 + y_0)} = e^{2t}
\]

from which follows that

\[
y(t) = \frac{y_0 - 1}{1 - y_0} + \frac{(1 + y_0)e^{2t}}{1 - y_0 (1 + y_0)}.
\]

When \(y_0 < -1\) the denominator becomes zero if \(e^{2t} = \frac{y_0 - 1}{1 + y_0}\), that is, when \(t = \frac{1}{2} \ln \frac{y_0 - 1}{1 + y_0}\).

5. The find the evolution of the fish population \(y\) we need to solve the given inhomogeneous first order differential equation. Since a solution of the homogeneous equation is given by \(e^t\) and the initial population amounts to a 1000 fish, we obtain

\[
y(t) = 1000 e^t - F e^t \int_0^t e^{-\tau} (1 + \cos(\tau)) d\tau.
\]

using the variation-of-parameters formula. The integral can be computed by double integration by parts, for instance, to give

\[
y(t) = (1000 - \frac{5}{2} F) e^t + F \left[ 1 - \frac{1}{2} \sin(t) + \frac{1}{2} \cos(t) \right].
\]
Now, the exponential term dominates for large times. To prevent extinction we therefore certainly need to at least impose that $1000 - \frac{5}{2}F \geq 0$ or, equivalently, that $F \leq 400$. Otherwise, in fact, if $F > 400$ extinction is unavoidable. What if $F = 400$? Then

$$y(t) = 400\left[1 - \frac{1}{2} \sin(t) + \frac{1}{2} \cos(t)\right]$$

but $1 - \frac{1}{2} \sin(t) + \frac{1}{2} \cos(t) > 0$ for all times and therefore the maximal allowed fishing quota is precisely $F = 400$.

6. Since the equation is inhomogeneous and $g(t)$ is not specified we can only compute the solution by the variation-of-parameters formula. First we compute the general solution of the homogeneous equation via the characteristic equation

$$r^2 - \alpha = 0$$

which leads to the two linearly independent solutions

$$y_1(t) = e^{\sqrt{\alpha}t} \quad \text{and} \quad y_2(t) = e^{-\sqrt{\alpha}t}.$$ 

Then a particular solution is given by

$$y_p(t) = -y_1(t) \int_0^t y_2(\tau)g(\tau) \, d\tau + y_2(t) \int_0^t y_1(\tau)g(\tau) \, d\tau$$

Since $W(y_1, y_2)(\tau) = -2\sqrt{\alpha}$ this amounts to

$$y_p(t) = \frac{1}{2\sqrt{\alpha}} \left[ e^{\sqrt{\alpha}t} \int_0^t e^{-\sqrt{\alpha} \tau} g(\tau) \, d\tau - e^{-\sqrt{\alpha}t} \int_0^t e^{\sqrt{\alpha} \tau} g(\tau) \, d\tau \right]$$

$$= \frac{1}{\sqrt{\alpha}} \int_0^t \left[ \frac{e^{\sqrt{\alpha}(t-\tau)} - e^{-\sqrt{\alpha}(t-\tau)}}{2} \right] g(\tau) \, d\tau = \frac{1}{\sqrt{\alpha}} \int_0^t \sinh(\sqrt{\alpha}(t-\tau)) g(\tau) \, d\tau$$

The general solution is therefore given by

$$y(t) = c_1 y_1(t) + c_2 y_2(t) + y_p(t).$$

Finally we use the boundary conditions to obtain the following system

$$\begin{cases} 0 = y(0) = c_1 + c_2 \\ 0 = y(1) = c_1 e^{\sqrt{\alpha}} + c_2 e^{-\sqrt{\alpha}} + \frac{1}{\sqrt{\alpha}} \int_0^1 \sinh(\sqrt{\alpha}(1-\tau)) g(\tau) \, d\tau \end{cases}$$

for $c_1$ and $c_2$. Solving it gives

$$c_1 = -c_2 = - \frac{1}{2\sqrt{\alpha} \sinh(\sqrt{\alpha})} \int_0^1 \sinh(\sqrt{\alpha}(1-\tau)) g(\tau) \, d\tau$$

and thus

$$y(t) = - \frac{1}{\sqrt{\alpha} \sinh(\sqrt{\alpha})} \int_0^1 \sinh(\sqrt{\alpha}(1-\tau)) g(\tau) \, d\tau + \frac{1}{\sqrt{\alpha}} \int_0^t \sinh(\sqrt{\alpha}(t-\tau)) g(\tau) \, d\tau$$

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