Final Examination

1. Find the general solution of the following equation:
   
   A. \( y''' - 3y' - 2y = 0 \)
   B. \( y''' + 3y'' - 4y = 0 \)

   **Solution**
   
   Looking for a solution in the form \( y(t) = e^{\lambda t} \) we are led to the characteristic equation:
   
   A. \( \lambda^3 - 3\lambda - 2 = 0 \) and B. \( \lambda^3 + 3\lambda^2 - 4 = 0 \).

   By inspection we see that the first equation admits the solution \( \lambda = -1 \) and, similarly, that the second admits the solution \( \lambda = 1 \). Thus both polynomials can be factored to obtain

   A. \( (\lambda + 1)(\lambda^2 - \lambda - 2) = (\lambda + 1)(\lambda + 1)(\lambda - 2) \)
   
   and B. \( (\lambda - 1)(\lambda^2 + 4\lambda + 4) = (\lambda - 1)(\lambda + 2)^2 \)

   This leads to the solutions

   A. \( y_1(t) = e^{-t}, \ y_2(t) = te^{-t}, \ y_3(t) = e^{2t} \).
   B. \( y_1(t) = e^t, \ y_2(t) = e^{-2t}, \ y_3(t) = te^{-2t} \).

   because of the presence of the double roots.

2. Solve the following initial value problem:
   
   A. \( \begin{cases} 
   y' = e^ty, \\
   y(0) = 1. 
   \end{cases} \)

   B. \( \begin{cases} 
   y' = e^{-t}y, \\
   y(0) = 1. 
   \end{cases} \)

   **Solution**
   
   The equations can be solved by application of the integrating factor method to give

   A. \( y(t) = \frac{1}{e}e^{e^t} \).
3. Find two linearly independent solutions of

A. \( y'' - t^3 y = 0 \)
B. \( y'' + t^3 y = 0 \)

[Explain how you get the two solutions explicitly.]

**Solution**
By a regular power series Ansatz \( y(t) = \sum_{n=0}^{\infty} a_n t^n \) we are led to the following equation:

A. \( \sum_{n=2}^{\infty} a_n n(n-1) t^{n-2} - \sum_{n=0}^{\infty} a_n t^{n+3} = 2a_2 + 6a_3 t + 12a_4 t^2 + \sum_{n=3}^{\infty} [(n+1)(n+2)a_{n+2} - a_{n-3}] t^n \)

B. \( \sum_{n=2}^{\infty} a_n n(n-1) t^{n-2} + \sum_{n=0}^{\infty} a_n t^{n+3} = 2a_2 + 6a_3 t + 12a_4 t^2 + \sum_{n=3}^{\infty} [(n+1)(n+2)a_{n+2} + a_{n-3}] t^n \)

We get that in both cases \( a_2 = a_3 = a_4 = 0 \) and the recurrence relation
\[
a_{n+2} = \pm \frac{1}{(n+1)(n+2)} a_{n-3}, \quad n \geq 3,
\]
for the remaining coefficients. Setting \( a_0 = 1 \) and \( a_1 = 0 \) for \( y_1(t) \) and \( a_0 = 0 \) and \( a_1 = 1 \) for \( y_2(t) \), respectively, and using the above information about the other coefficients produces two linearly independent solutions.

4. Solve the following equations:

A. \[
\begin{align*}
  y'' + \frac{4}{7} y' + \frac{9}{7} y &= 0, \\
  y(1) &= 0, \\
  y'(1) &= 1.
\end{align*}
\]

B. \[
\begin{align*}
  y'' - \frac{3}{7} y' + \frac{4}{7} y &= 0, \\
  y(1) &= 1, \\
  y'(1) &= 0.
\end{align*}
\]

**Solution**
The equation is of Euler type with indicial equation
\[
A. \quad r^2 + 3r + 9 = 0 \quad \text{and} \quad B. \quad r^2 - 4r + 4 = 0.
\]
which gives the two linearly independent solutions

\[ A. \ y_1(t) = t^{-3/2} \cos\left(\frac{3\sqrt{3}}{2} \log(|t|)\right), \quad y_2(t) = t^{-3/2} \sin\left(\frac{3\sqrt{3}}{2} \log(|t|)\right), \]

\[ B. \ y_2(t) = t^2, \quad y_2(t) = t^2 \log(|t|), \]

of the homogeneous equation. A linear combination of these can be used to satisfy the additional conditions and gives

\[ A. \ 0 = c_1 y_1(1) + c_2 y_2(1) = c_1 \quad \text{and then} \quad 1 = c_2 y_2'(1) = 3\sqrt{3}/2c_2 \]

and

\[ B. \ 1 = c_1 y_1(1) + c_2 y_2(1) = c_1 \quad \text{and then} \quad 0 = y_1'(1) + c_2 y_2'(1) = 2 + c_2 \]

5. Verify that \( y_1(t) \) is a solution of the given equation and compute a second linearly independent solution for

\[ y_1(t) = 3t^2 - 1, \quad (1 - t^2)y'' - 2ty' + 6y = 0. \]

\[ y_1(t) = t + 1, \quad (2t + 1)y'' - 4(t + 1)y' + 4y = 0. \]

**Solution**

This is done by reduction of order. It is easily verified that \( y_1(t) \) is a solution. The second solution can be therefore soughted in the form \( y_2(t) = v(t)y_1(t) \). Plugging this into the equation leads to

\[ A. \ (1 - t^2)[y''_1(t)v(t) + 2y'_1(t)v'(t) + y_1(t)v''(t)] + 2t[y'_1(t)v(t) + y_1(t)v'(t)] + 6y_1(t)v(t) \]
\[ = (1 - t^2)[(3t^2 - 1)v''(t) + 12tv'(t)] - 2(3t^2 - 1)v'(t) = 0 \]

\[ B. \ (2t + 1)[y''_1(t)v(t) + 2y'_1(t)v'(t) + y_1(t)v''(t)] + 4(t + 1)[y'_1(t)v(t) + y_1(t)v'(t)] + 4y_1(t)v(t) \]
\[ = (2t + 1)[(t + 1)v''(t) + 2v'(t)] - 4(t + 1)^2v'(t) = 0 \]

and eventually to:

\[ A. \ \frac{v''}{v'} = \frac{2t}{1 - t^2} - 2 \cdot \frac{6t}{3t^2 - 1}, \quad B. \ \frac{v''}{v'} = 2 + \frac{2}{2t + 1} - \frac{1}{t + 1}. \]
From this we obtain

\[
\textbf{A. } v'(t) = \frac{1 - t^2}{(3t^2 - 1)^2}, \quad \textbf{B. } v'(t) = e^{2t} \frac{2t + 1}{(t + 1)^2}.
\]

This is an acceptable answer.

6. **A.** A mass of 100 g stretches a spring 5 cm. If the mass is set in motion from its equilibrium position with a downward velocity of 10 cm/s, determine its position at time \(t\) in the absence of damping. When does the mass return to its equilibrium position for the first time?

**B.** A mass of 300 g stretches a spring 15 cm. If the mass is set in motion from its equilibrium position with a downward velocity of 5 cm/s, determine its position at time \(t\) in the absence of damping. When does the mass return to its equilibrium position for the first time?

**Solution**

Remember that the spring constant \(k\) can be obtained from the equation \(k\Delta x = mg\) knowing the displacement \(\Delta x\) at equilibrium, the mass \(m\) and gravity \(g = 10\). In both cases the \(k = 20\). Taking into account the given initial conditions, the deviation from equilibrium \(y(t)\) satisfies

\[
\textbf{A. } \begin{cases}
\frac{1}{10} y'' + 20y = 0, \\
y(0) = 0, \\
y'(0) = \frac{1}{10}.
\end{cases}
\]

\[
\textbf{B. } \begin{cases}
\frac{3}{10} y'' + 20y = 0, \\
y(0) = 0, \\
y'(0) = \frac{1}{20}.
\end{cases}
\]

The equation has solutions

\[
\textbf{A. } y_1(t) = \cos(10\sqrt{2}t), \quad y_2(t) = \sin(10\sqrt{2}t),
\]

\[
\textbf{B. } y_1(t) = \cos(10\sqrt{2/3}t), \quad y_2(t) = \sin(10\sqrt{2/3}t).
\]

Imposing the initial conditions on the general solution \(c_1y_1(t) + c_2y_2(t)\) gives in both cases that \(c_1 = 0\) and

\[
\textbf{A. } 10\sqrt{2}c_2 = \frac{1}{10}, \quad \textbf{B. } 10\sqrt{2/3}c_2 = \frac{1}{20}.
\]

7. Find all *singular* points of the given equation and determine whether each one is *regular* or *irregular.*
A. \( t^2(1-t)y'' + (t-2)y' - 3ty = 0 \)

B. \( t^2(1-t)^2y'' + 2ty' + 4y = 0 \)

**Solution**

In both cases the only singular points are located at \( t = 0 \) and \( t = 1 \).

Bringing the equations in normal form \( y'' + p(t)y' + q(t)y = 0 \) we have

A. \( tp(t) = \frac{t - 2}{t(1-t)}, \quad t^2q(t) = \frac{3t}{1-t} \)

and B. \( tp(t) = \frac{2}{(1-t)^2}, \quad t^2q(t) = \frac{4}{(1-t)^2} \),

and thus \( t = 0 \) is an irregular singular point for A and a regular singular point for B. As for \( t = 1 \) we have

A. \( (t-1)p(t) = \frac{2 - t}{t^2}, \quad (t-1)^2q(t) = \frac{3(t-1)}{t} \)

and B. \( (t-1)p(t) = -\frac{2}{t(1-t)}, \quad (t-1)^2q(t) = \frac{4}{t^2} \),

making it a regular singular point for A and an irregular one for B.

8. Solve the equation

\[
\begin{cases}
    y'' + 2y' + 2y = 5\delta(t - \pi), \\
    y(0) = 1, \\
    y'(0) = 0.
\end{cases}
\]

A. \( y(t) = e^{t}\cos(t) + e^{t}\sin(t) + 5h_0(t - \pi)e^{-(t-\pi)}\sin(t - \pi). \)

As for the other equation

\[
\begin{cases}
    y'' + 4y = 2\delta(t - 2\pi), \\
    y(0) = 0, \\
    y'(0) = 0.
\end{cases}
\]

B. \( y(t) = h_0(t - 2\pi)\sin(t - 2\pi). \)
9. Solve the initial value problem

\[ \begin{align*}
\text{A. } y' &= \begin{bmatrix} 1 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 1 \end{bmatrix} y, \ y(0) = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \\
\text{B. } y' &= \begin{bmatrix} 2 & 0 & 2 \\ 0 & 1 & 0 \\ 2 & 0 & 2 \end{bmatrix} y, \ y(0) = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}
\end{align*} \]

**Solution**

The system can be rewritten as

\[ \begin{align*}
\text{A. } &\begin{cases}
y_1' = y_1 - y_3 \\
y_2' = 2y_2 \\
y_3' = -y_1 + y_3
\end{cases} \\
\text{B. } &\begin{cases}
y_1' = 2y_1 + 2y_3 \\
y_2' = y_2 \\
y_3' = 2y_1 + 2y_3
\end{cases}
\end{align*} \]

Clearly the second equation is independent from the others and can be solved by itself to give

\[ \begin{align*}
\text{A. } y_2(t) &= e^{2t} \\
\text{B. } y_2(t) &= 2e^t
\end{align*} \]

where the initial condition was also taken into account. As for the remaining equations observe that

\[ \begin{align*}
\text{A. } & (y_1(t) + y_3(t))' = y_1'(t) + y_3'(t) = 0 \\
\text{B. } & (y_1(t) - y_3(t))' = y_1'(t) - y_3'(t) = 0
\end{align*} \]

as follows by adding and subtracting the first and the third equation, respectively. This gives

\[ \begin{align*}
\text{A. } & y_3(t) = c - y_1(t) \\
\text{B. } & y_3(t) = y_1(t) - c,
\end{align*} \]

and the system reduces to the single equation

\[ \begin{align*}
\text{A. } & y_1'(t) = -c + 2y_1(t) \\
\text{B. } & y_1(t) = 4y_1(t) - c
\end{align*} \]

which can be solved (integrating factor) to yield

\[ \begin{align*}
\text{A. } & y_1(t) = y_1(0)e^{2t} - c \int_0^t e^{2(t-\tau)} d\tau = e^{2t} + \frac{c}{2}(1 - e^{2t}) \\
\text{B. } & y_1(t) = y_1(0)e^{4t} - c \int_0^t e^{4(t-\tau)} d\tau = 2e^{4t} + \frac{c}{4}(1 - e^{4t}).
\end{align*} \]
Thus

\[ A. \quad y_3(t) = c - y_1(t) = c - \frac{c}{2} (1 + e^{2t}) - e^{2t} \]
\[ B. \quad y_3(t) = y_1(t) - c = 2e^{4t} - \frac{c}{4} (3 + e^{4t}), \]

Imposing the remaining initial condition gives

\[ A. \quad y_3(0) = c - 1 = 2 \quad B. \quad y_3(0) = 2 - c = 1, \]

and the constant \( c \) can be determined. Clearly we could have solved the problem also by computing eigenvalues and eigenvectors of the matrix \( A \) just like we learned in class.