Midterm Examination–Solutions

1. Solve the following initial value problem:

A. $\begin{cases} y'' + \pi^2 y = \cos(\pi t) \\ y(0) = 0 \\ y'(0) = 2 \end{cases}$ **B.** $\begin{cases} y'' + ey = \sin(\sqrt{e}t) \\ y(0) = 3 \\ y'(0) = 0 \end{cases}$

<u>Solution:</u>

A. The homogeneous equation has solutions

$$y_1(t) = \sin(\pi t) \text{ and } y_2(t) = \cos(\pi t)\,,$$

since the characteristic equation $r^2 + \pi^2 = 0$ has roots $r_{1,2} = \pm\pi i$. The right-hand side is the real part of $e^{i\pi t}$ and therefore the appropriate Ansatz in this case is $Y(t) = Ate^{i\pi t}$ for the solution of

$$y'' + \pi^2 y = e^{i\pi t}\,.$$

This leads to

$$-A\pi^2 te^{i\pi t} + 2Ai\pi te^{i\pi t} + A\pi^2 te^{i\pi t} = e^{i\pi t}$$

which gives $A = \frac{1}{2\pi i}$. Taking the real part of $Y(t)$ yields $\frac{t}{2\pi}\sin(\pi t)$ as a particular solution. Finally imposing the initial conditions to the general solution

$$y(t) = c_1 y_1(t) + c_2 y_2(t) + \Re\big(Y(t)\big)$$

reveals that

$$c_2 = 0 \text{ and then that } c_1 = \frac{2}{\pi}\,.$$

B. Similar considerations to the ones above leads to the Ansatz $Y(t) = Ate^{i\sqrt{e}t}$ and to

$$-Aete^{i\sqrt{e}t} + 2Ai\sqrt{e}te^{i\sqrt{e}t} + Aete^{i\sqrt{e}t} = e^{i\sqrt{e}t}\,.$$

This time the imaginary part of $Y(t)$ is needed and therefore a particular solution is given by $-\frac{t}{2\sqrt{e}}\cos(\sqrt{e}t)$. The initial conditions then give

$$\begin{cases} c_1 y_1(0) + c_2 y_2(0) - \frac{0}{2e}\cos(\sqrt{e}0) = c_2 = 3\,, \\ \sqrt{e}c_1\cos(0) + 3\sqrt{e}\sin(0) - \frac{1}{2\sqrt{e}} = 0 \end{cases}$$

and finally $c_1 = \frac{1}{2e}$.

2. **A.** A mass of $1\,Kg$ stretches a spring $1\,m$. The mass is pulled down another $0.5\,m$ and is then released. After one complete oscillation period $10\,s$ have elapsed and an elongation of $0.4\,m$ is observed. What is the damping constant? [Use $g = 10\,m/s^2$]

B. A mass of $2\,Kg$ stretches a spring $1\,m$. The mass is pulled down another $0.5\,m$ and is then released. After one complete oscillation period $20\,s$ have elapsed and an elongation of $0.4\,m$ is observed. What is the damping constant? [Use $g = 10\,m/s^2$]

Solution:

First the spring constant k needs to be computed using $mg = k\triangle x$. This gives

$$\textbf{A.}\ k = 10 \text{ and } \textbf{B.}\ k = 20\,.$$

The equation satisfied by the mass' position is $my'' + cy' + ky = 0$ where m is given, k has just been computed and c remains unkwown so far. The general solution is given by

$$\textbf{A.}\ e^{-\frac{c}{2}t}\big[c_1\cos(\mu t) + c_2\sin(\mu t)\big] \text{ and } \textbf{B.}\ e^{-\frac{c}{4}t}\big[c_1\cos(\mu t) + c_2\sin(\mu t)\big]$$

where

$$\textbf{A.}\ \mu = \frac{1}{2}\sqrt{40 - c^2} \text{ and } \textbf{B.}\ \mu = \frac{1}{2}\sqrt{40 - c^2/4}\,.$$

The initial conditions are

$$\textbf{A.\& B.}\ y(0) = 0.5\,,\ y'(0) = 0\,.$$

Imposing the first one gives $c_1 = 0.5$ in both cases. Initially the cos and sin have values 1 and 0. After one period they clearly have the same values. Thus we don't need to c_2. From the additional information we infer that

$$\textbf{A.}\ 0.5e^{-10c/2} = 0.4 \text{ and } \textbf{B.}\ 0.5e^{-20c/4} = 0.4\,.$$

In both cases the solution is $c = -\frac{1}{5}\log(\frac{4}{5})$.

3. Compute the recursion relation and the first six coefficients in the power series expansion about $t = 0$ of two linearly independent solutions of the following equation:

A. $(1 + t^2)y'' - ty = 0$

B. $(1 - t^2)y'' + ty = 0$

<u>Solution:</u>

The origin is not a singular point for the equations and a regular series expansion $\sum_{n=0}^{\infty} a_n t^n$ is the appropriate Ansatz. Substituting in the equation leads to

$$(1 \mp t^2) \sum_{n=2}^{\infty} n(n-1)a_n t^{n-2} \pm t \sum_{n=0}^{\infty} a_n t_n = 0$$

which can be rewritten as

$$\sum_{n=0}^{\infty} \big[(n+1)(n+2)a_{n+2} \mp n(n-1)a_n \pm a_{n-1}\big] t^n = 0\,.$$

Thus the recursion relation

$$a_{n+2} = \pm \frac{n(n-1)}{(n+1)(n+2)} a_n \mp \frac{1}{(n+1)(n+2)} a_{n-1}$$

is obtained. Choosing $a_0 = 1\,,\ a_1 = 0$ and $a_0 = 0\,,\ a_1 = 1$, respectively, the power series of two linearly independent solutions can be computed. Looking at the coefficient of t^2 it follows that $a_2 = 0$ in all cases. As for a_3, set $n = 1$ in the recursion relation to obtain

$$a_3 = \pm \frac{1}{6} 0 \mp \frac{1}{6} a_0 = \mp \frac{1}{6} \text{ and } a_3 = \pm \frac{1}{6} 0 \mp \frac{1}{6} a_0 = 0$$

depending on the choice of a_0 and a_1. Then

$$a_4 = \pm \frac{2}{12} a_2 \mp \frac{1}{12} a_1 = 0 \text{ and } a_4 = \pm \frac{2}{12} a_2 \mp \frac{1}{12} a_1 = \mp \frac{1}{12}$$

and

$$a_5 = \pm \frac{6}{20} a_3 \mp \frac{1}{20} a_2 = \frac{1}{20} 0 \text{ and } a_5 = \pm \frac{6}{20} a_3 \mp \frac{1}{20} a_2 = 0$$

and finally

$$a_6 = \pm\frac{12}{30}a_4 \mp \frac{1}{30}a_3 = \mp\frac{1}{30}(\mp\frac{1}{6}) = \frac{1}{20}\,,$$

$$a_6 = \pm\frac{12}{30}a_4 \mp \frac{1}{30}a_3 = \pm\frac{12}{30}(\mp\frac{1}{12}) = -\frac{1}{30}\,.$$

4. For the following equations, classify $t_0 = 0$ into regular point (r), regular singular (rs) or irregular singular point (is). Please check your choice and justify your answer on the side or on the back of the page.

A.

Equation	r	rs	is
$t\sin(t)y'' - \cos(t)y' + e^t y = 0$			✓
$ty'' + (1-t)y' + ty = 0$		✓	
$ty'' + (e^t - 1)y' + t^2 y = 0$	✓		
$(1-e^{-2t})y'' + 5\cos(t)y' - \frac{1}{t}y = 0$		✓	
$t^2y'' - \sin^2(t)y' + (t^4 - t^2)y = 0$	✓		

B.

Equation	r	rs	is
$te^t y'' + \sin(t)y' + t^2 y = 0$	✓		
$5ty'' + (t^4 - 1)y' - 7y = 0$		✓	
$\cos(t)y'' - \frac{t}{e^t - 1}y' - e^t y = 0$	✓		
$t^2\sin(t)y'' + (e^t - e^{-t})y' + \sin(t)y = 0$			✓
$t^2y'' - t^5y' + \tan(t)y = 0$		✓	

5. Compute the general solution of the following equation:

A. $t^2y'' + 3ty' + y = t$

B. $t^2y'' + 5ty' + 4y = t^2$

Solution:

Both are Euler equations. The indicial equation is

$$\textbf{A. } r^2 + 2r + 1 = 0 \text{ and } \textbf{B. } r^2 + 4r + 4 = 0$$

and leads to the linearly independent solutions of the homogeneous equation

$$\textbf{A. } y_1(t) = \frac{1}{t}\,,\ y_2(t) = \frac{\log(t)}{t} \text{ and } \textbf{B. } y_1(t) = \frac{1}{t^2}\,,\ y_2(t) = \frac{\log(t)}{t^2}\,.$$

In order to obtain a particular solution to the inhomogeneous equation judicious guessing can be used. Remembering that the equation is equivalent to

$$\textbf{A. } y'' + 2y' + y = e^s \text{ and } \textbf{B. } y'' + 4y' + 4y = e^{2s}\,,$$

via the change of variable $s = \log(t)$. The appropriate Ansatz in this case is Ae^s and Ae^{2s}, respectively. These functions correspond to At and At^2 in the old variable, respectively. Plugging into the equation gives

$$\textbf{A. } 3At + At = t\,,\ A = \frac{1}{4} \text{ and}$$

$$\textbf{B. } 2At^2 + 10At^2 + 4At^2 = t^2\,,\ A = \frac{1}{16}$$

In conclusion the general solution is given by

$$\textbf{A. } y(t) = \frac{t}{4} + c_1\frac{1}{t} + c_2\frac{\log(t)}{t} \text{ and } \textbf{B. } y(t) = \frac{t^2}{16} + c_1\frac{1}{t^2} + c_2\frac{\log(t)}{t^2}\,.$$