# Midterm Examination I

1. Please indicate with a check mark (✓) which of the equations below are linear (L) and which are nonlinear (N). For the linear ones specify whether they are homogeneous (H) or inhomogeneous (I).

## A.

<table>
<thead>
<tr>
<th>Equation</th>
<th>L</th>
<th>N</th>
<th>H</th>
<th>I</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \tanh(y') + 5y = \cos(t) )</td>
<td>✓</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( e^{3t}y'' + \cos(t)y'' + y^2 = t^5 )</td>
<td>✓</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( (2 + \sin(t))y' + e^{\sin(t)}y = t - 1 )</td>
<td>✓</td>
<td></td>
<td>✓</td>
<td></td>
</tr>
<tr>
<td>( y'' + e^y = 8 )</td>
<td>✓</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( y'''' + t^2y'' + e^{-t}y' = \pi y )</td>
<td>✓</td>
<td>✓</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

## B.

<table>
<thead>
<tr>
<th>Equation</th>
<th>L</th>
<th>N</th>
<th>H</th>
<th>I</th>
</tr>
</thead>
<tbody>
<tr>
<td>( (3 + e^{t^2+1})y' + \sin(t)y = (t + 1)^2 )</td>
<td>✓</td>
<td></td>
<td>✓</td>
<td></td>
</tr>
<tr>
<td>( 5y' + \tanh(t) = \cos(y) )</td>
<td>✓</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( y'''' + e^y' y = 7 \sin(t) )</td>
<td>✓</td>
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<td></td>
</tr>
<tr>
<td>( y'' + \cos(y)y'' + y = t^4 )</td>
<td>✓</td>
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<td></td>
<td></td>
</tr>
<tr>
<td>( y'' + t^2y' + e^{-t}y''' = -y )</td>
<td>✓</td>
<td>✓</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

1
2. Solve the following initial value problem:

\[ \begin{align*}
\text{A. } & \begin{cases} 
\frac{dy}{dt} = -\frac{\sin(t)}{y^6} \\
y(0) = 1 
\end{cases} \\
\text{B. } & \begin{cases} 
\frac{dy}{dt} = \frac{\cos(t)}{y^4} \\
y(0) = 2 
\end{cases}
\end{align*} \]

Solution:

In both cases separation of variables is the method of choice here. It gives

\[ \begin{align*}
\text{A. } & \frac{1}{7}y^7(t) - \frac{1}{7}y^7(0) = \int_0^t y^6(\tau)y'(\tau) \, d\tau = -\int_0^t \sin(\tau) \, d\tau = \cos(t) - 1, \\
\text{B. } & \frac{1}{5}y^5(t) - \frac{1}{5}y^5(0) = \int_0^t y^4(\tau)y'(\tau) \, d\tau = \int_0^t \cos(\tau) \, d\tau = \sin(t)
\end{align*} \]

and therefore the solutions

\[ \begin{align*}
\text{A. } & y(t) = \left[7 \cos(t) - 6\right]^{1/7}, \\
\text{B. } & y(t) = \left[5 \sin(t) + 32\right]^{1/5}.
\end{align*} \]

3. Find the general solution of the following equation:

\[ \begin{align*}
\text{A. } & y' + \frac{1}{t+1}y = t \\
\text{B. } & (t+2)y' + y = t^2 + 2t
\end{align*} \]

Solution:

Here it is best to use the integrating factor method. In fact

\[ \begin{align*}
\text{A. } & [(t+1)y]' = (t+1)y' + y = (t+1)\left[y' + \frac{y}{t+1}\right] = (t+1)t, \\
\text{B. } & [(t+2)y]' = (t+2)y' + y = t^2 + 2t,
\end{align*} \]

and then

\[ \begin{align*}
\text{A. } & (t+1)y(t) - y(0) = \frac{1}{3}t^3 + \frac{1}{2}t^2 \text{ or } y(t) = \frac{1}{1+t}y_0 + \frac{\frac{1}{3}t^3 + \frac{1}{2}t^2}{1+t}, \\
\text{B. } & (t+2)y(t) - 2y(0) = \frac{1}{3}t^3 + t^2 \text{ or } y(t) = \frac{2}{t+2}y_0 + \frac{\frac{1}{3}t^3 + t^2}{t+2}.
\end{align*} \]

4. Solve the following initial value problem:
\[
\begin{align*}
A. \begin{cases} 
2y'' + 20y' + 50y = 0 \\
y(0) = 1 \\
y'(0) = 0
\end{cases} & \quad B. \begin{cases} 
3y'' - 24y' + 48y = 0 \\
y(0) = 0 \\
y'(0) = 1
\end{cases}
\end{align*}
\]

Solution:
The characteristic equation is given by
\[
A. r^2 + 10r + 25 = 0, \quad B. r^2 - 8r + 16 = 0
\]
leading to the roots
\[
A. r_{1,2} = -5, \quad B. r_{1,2} = 4,
\]
and, therefore to the fundamental solution
\[
A. y_1(t) = e^{-5t}, \quad y_2(t) = te^{-5t}, \quad B. y_1(t) = e^{4t}, \quad y_2(t) = te^{4t},
\]
It remains to find the constants \(c_1\) and \(c_2\) such that the initial conditions are satisfied for \(c_1y_1 + c_2y_2\), which leads to
\[
A. \begin{cases} 
c_1 = 1 \\
-5c_1 + c_2 = 0
\end{cases} \quad \text{and} \quad B. \begin{cases} 
c_1 = 0 \\
4c_1 + c_2 = 1
\end{cases}
\]
Finally this gives
\[
A. y(t) = e^{-5t} + 5te^{-5t}, \quad B. y(t) = te^{4t}.
\]

5. Find the general solution of the following equation:
\[
A. y'' - 3y' + 2y = e^t \\
B. y'' + 4y' + 3y = e^{-t}
\]
Solution:
First we find two linearly independent solutions of the homogeneous equation by looking at the characteristic equation
\[
A. r^2 - 3r + 2 = (r - 2)(r - 1) = 0, \quad B. r^2 + 4r + 3 = (r + 3)(r + 1) = 0,
\]
it gives the solutions
\[
A. y_1(t) = e^{2t}, \quad y_2(t) = e^t, \quad B. y_1(t) = e^{-3t}, \quad y_2(t) = e^{-t},
\]
Judicious guessing is a viable alternative for the computation of a particular solution. The appropriate Ansatz reads

\[ \text{A. } Y(t) = Ate^t, \quad \text{B. } Y(t) = Bte^{-t}, \]

since the right-hand side is also a solution of the homogeneous equation. A straightforward computation gives

\[ \text{A. } Ate^t[2 - 3 + 1] + Ae^t[-3 + 2] = e^t, \]
\[ \text{B. } Ate^{-t}[3 - 4 + 1] + Ae^{-t}[4 - 2] = e^{-t}, \]

and leads to the general solution

\[ \text{A. } y(t) = -te^t + c_1e^{2t} + c_2e^t, \quad \text{B. } y(t) = \frac{1}{2}te^{-t} + c_1e^{-3t} + c_2e^{-t}. \]