Midterm Examination

Print your name: __________________________

Print your ID #: __________________________

You have 50 minutes to solve the problems. Good luck!

1. A. For $\varepsilon > 0$ compute the solution $y_\varepsilon(t)$ of

$$y'' - (2 + \varepsilon)y' + (1 + \varepsilon)y = 0, \ y(0) = 0, \ y'(0) = 1,$$

and determine the limit as $\varepsilon \to 0$.

B. For $\varepsilon > 0$ compute the solution $y_\varepsilon(t)$ of

$$y'' - 2y' + (1 + \varepsilon^2)y = 0, \ y(0) = 0, \ y'(0) = 1,$$

and determine the limit as $\varepsilon \to 0$. 

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2. A. Solve
\[
\begin{aligned}
&y'''' - 2y''' + 2y'' - 2y' + y = e^{2t}, \\
y(0) = 1, \ y'(0) = 0, \ y''(0) = 0, \ y'''(0) = 0.
\end{aligned}
\]

B. Solve
\[
\begin{aligned}
&y'''' - 2y'' + y = \sin(2t), \\
y(0) = 1, \ y'(0) = 0, \ y''(0) = 0, \ y'''(0) = 0.
\end{aligned}
\]

3. Classify every point of the following equations into ordinary, regular singular or irregular singular point. Justify your answer.

A. (i) \((x - 1)^2 \tanh(3x)y'' + (x^2 - 1)y' + (x - 1)^2 y = 0\),
    (ii) \((x - 1)^2(x + 3)y'' + (x - 4)y' + (x + 2)y = 0\),
    (iii) \(y'' + \frac{1}{(x + 1)^2}y' + \frac{1}{x + 1}y = 0\).

B. (i) \((x + 2)^2 \tanh(x)y'' + (x + 2)^2y' + (x^4 - 4)^2 y = 0\),
    (ii) \((x - 1)(x + 4)^2y'' + (x + 3)y' + (x - 2)y = 0\),
    (iii) \(y'' + \frac{1}{(x + 1)y'} + \frac{1}{(x + 1)^2}y = 0\).

4. Find the recurrence relation for the coefficients of the series solution about \(t = 0\) and the first four terms in the expansion of two linearly independent solutions of

A. \((1 + t)y'' - y = 0\)

B. \((1 - t)y'' + y = 0\)

What is the radius of convergence of the series at least?

5. Find the exponents at the singularity for

A. \((e^x - e^{-x})y'' + y' + y = 0\)

B. \((e^x - 1)y'' + \frac{1}{2} y' + y = 0\)

How does the singular solution behave at \(x = 0\)?