An Integral Equation Arising in the Study of Above Threshold Photodetachment

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Abstract
The existence of a unique global continuous solution of an integral equation arising in the study of above threshold photodetachment is established.

I. Introduction

In 1990 Faisal, Filipowicz and Rzążewski [1] proposed a model describing strong-field photodetachment processes. Indeed, they studied the case of photodetachment in a strong circularly polarized laser field using a three dimensional δ-potential mode. For further references and more detailed information on this problem we refer to Faisal, Filipowicz and Rzążewski [1] as well as to Faisal, Scanzano and Zaremba [2], and the references therein. In order to establish various properties of the observables of the model a basic integro-differential equation was derived, which reads:

$$\text{(I.1)} \quad F(\tau) = -\frac{1}{\beta}U(\tau) + \frac{1}{\sqrt{i\pi\tau}} + \frac{1}{\sqrt{i\pi}}\int_0^\tau \frac{F'(\tau-\xi)}{\sqrt{\xi}}\,d\xi + \\ - \frac{1}{2\sqrt{i\pi}}\int_0^\tau \frac{1}{\xi^{\frac{3}{2}}}\big(e^{i\frac{\gamma^2(\xi)}{\xi}} - 1\big)F(\tau-\xi)\,d\xi \ , \ \ \tau \in (0,T] \ .$$

For the interpretation of (I.1), the definition of the constant β and the functions U and γ and more details we refer to Faisal, Filipowicz and Rzążewski [1] (p. 6178). Recently Saladin and Scharf [3] showed that for certain reasons the above integro-differential equation is not completely correct. They proposed that (I.1) should be modified into the following integral equation:

$$\text{(I.2)} \quad F(x) = \alpha\int_0^x \frac{F(\tau)m(\tau)}{\sqrt{x-\tau}}\,d\tau + \beta\int_0^x R(\tau,x)F(\tau)\,d\tau + G(x) \ , \ \ x > 0 \ .$$

The constants and functions appearing in (I.2) will be specified precisely in the next section.

In this paper we do not discuss the physical background of the derivation of (I.2). For this we refer to the forthcoming paper of Saladin and Scharf [3]. Our interest in this problem, however, is to give a rigorous mathematical proof of the existence of a unique, global continuous solution $u : [0,\infty) \to \mathbf{C}$

to the equation (I.2). Substituting u in the wave function

$$\phi(r,t) = \int G(r,\tau,r_0,0)\varphi(r_0)\,d^3r_0 - \frac{2\pi}{i}\int_0^\tau G(r,\tau,0,\xi)u(\xi)\,d\xi \tag{I.3}$$

one is able to compute various observables of the original problem. For details, see Ref. 1. Here G and φ denote the Volkov's Green function and the initial condition, respectively.

To the best of our knowledge all the existing results for (I.3) are based on numerical approximation of the solution of (I.1) or (I.2). These are obtained by discretizing the integral equation involved. Our approach to equation (I.2) relies on the classical Neumann series representation of the inverse of a linear operator. Thus we do not only obtain the existence of a solution to (I.2), but also an approximation method by the classical iteration scheme. This also provides exact error estimates.

II. Mathematical formulation and existence of continuous solutions

In this section we reformulate equation (I.2) as a fixed-point equation in a space of continuous functions. To be more precise, put $J := [0,T]$ for some $T > 0$ and fix the space $E := C(J)$, consisting of all continuous functions $u : J \to \mathbf{C}$. Let $\delta > 0$. Endowed with the weighted supremum-norm $\|u\|_\delta := \sup_{x\in J} e^{-\delta x}|u(x)|$ the space E becomes a Banach space. The choice of this weighted norm allow us, as we shall see, to establish directly the existence of a global solution without any continuation argument. From the mathematical point of view the problem consists of finding a function $F \in E$ satisfying

(II.1)
$$F(x) = \alpha\int_0^x \frac{F(\tau)m(\tau)}{\sqrt{x-\tau}}\,d\tau + \beta\int_0^x R(\tau,x)F(\tau)\,d\tau + G(x)\ ,\ \ x\in[0,T]\ ,$$

where $\alpha = 4\lambda\sqrt{i\pi}$ for some $\lambda > 0$, $\beta = \frac{1}{2\pi}$ and the functions m, R and G are defined as follows:

(M1) $m(\tau) := \left(1 - \frac{i}{4\pi\lambda}b(\tau)\cdot e\right)$, $0 \le \tau \le T$. Hereby $b(\tau) = \int_0^\tau E(s)\,ds$, where $E = E(t)$ denotes a time-dependent continuous electrical field and $e \in \mathbf{R}^3$ is a unit vector.

(R1) $c^2(t,\tau) := \frac{4\mu^2}{\omega^4}\left[(\cos\omega\tau - \cos\omega t)^2 + (\omega(t-\tau) + \sin\omega\tau - \sin\omega t)^2\right]$, $(t,\tau) \in J^2$. Hereby μ and ω denote the amplitude and the frequency, respectively, of the outer field.

(R2) $k(t,\tau) := \frac{1}{t-\tau}\left(e^{i\frac{c^2(t,\tau)}{4(t-\tau)}} - 1\right)$, $0 < t,\tau \le T$.

(R3) $R(\tau,x) := \int_\tau^x \frac{k(\sigma,\tau)\,d\sigma}{[(x-\sigma)(\sigma-\tau)]^{\frac{1}{2}}}$, $0 \le \tau \le x \le T$.

(G1) $w(z) := \sum_{n=0}^\infty \frac{(iz)^n}{\Gamma(\frac{n}{2}+1)}$, where Γ is the Euler Gamma-function and $z \in \mathbf{C}$. Observe that w is the power series expansion of the modified complex error function $e^{-z^2}\mathrm{erfc}(-iz)$ (cf. Ref. 4, p. 297).

(G2) $q_1(t) := -\sqrt{i}\frac{|c(t,o)|}{2\sqrt{t}} - ai\sqrt{it}$, $q_2(t) := -\sqrt{i}\frac{|c(t,o)|}{2\sqrt{t}} + ai\sqrt{it}$, $0 < t \leq T$, where $a \in \boldsymbol{C}$.

(G3) $S(t) := \frac{N}{2|c(t,0)|}e^{i\frac{c^2(t,0)}{4t}}[w(q_2) - w(-q_1)]$, where $N \in \mathbf{R} \setminus \{0\}$ denotes a suitable normalization factor.

(G4) $G(x) := -\alpha\int_0^x \frac{S(\tau)}{\sqrt{x-\tau}}\,d\tau + \frac{1}{\pi}F(0)\int_0^x \frac{1}{\sqrt{(x-\tau)\tau}}\left(e^{i\frac{c^2(\tau,0)}{4\tau}} - 1\right)d\tau$, $x \in J$.

In order to find such a function we use the Neumann series (cf. Ref. 5, p. 191) to represent the solution of the linear equation

$$\text{(II.2)} \qquad \big(I - \Phi\big)(u) = G ,$$

where Φ is defined by

$$\text{(II.3)} \quad \Phi(u)(x) := \alpha\int_0^x \frac{u(\tau)m(\tau)}{\sqrt{x-\tau}}\,d\tau + \beta\int_0^x R(\tau,x)u(\tau)\,d\tau \ , \qquad x \in [0,T] \ , \ u \in E .$$

In a first Lemma we prove that Φ is well-defined as a self-map on E and that G is continuous. Note that in this paper const denotes a generic constant, which may differ from line to line.

Lemma 1. *Let $u \in E$. Then $\Phi(u) \in E$. Further, $G \in E$ as well.*

Proof. We divide the proof in three steps. The first two correspond to the terms of the right-hand side of (II.3) and the third to the second assertion, respectively. Let $u \in E$ and let B the Euler Beta-function, defined for $x, y \in (0,\infty)$ by

$$B(x,y) := \int_0^1 \sigma^{x-1}(1-\sigma)^{y-1}\,d\sigma .$$

Step 1. It follows by elementary calculations that

$$\Big[x \mapsto \int_0^x \frac{u(\tau)m(\tau)}{\sqrt{x-\tau}}\,d\tau\Big] \ \in \ E .$$

Step 2. We show that

$$\Big[x \mapsto \int_0^x R(\tau,x)u(\tau)\,d\tau\Big]$$

is continuous and thus bounded and uniformly continuous on J. To this end let $0 \leq x \leq y \leq T$ and consider

$$\int_0^x R(\tau,x)u(\tau)\,d\tau - \int_0^y R(\tau,y)u(\tau)\,d\tau =$$
$$= \int_0^x \big(R(\tau,x) - R(\tau,y)\big)u(\tau)\,d\tau - \int_x^y R(\tau,y)u(\tau)\,d\tau =: \mathrm{I} + \mathrm{II}$$

Note that by the power series expansion of sin and cos

$$\text{(II.4)} \qquad c^2(t,\tau) \leq \text{const}\ \omega^2(t+\tau)^2(t-\tau)^2 \ , \quad (t,\tau) \in J^2$$

and thus

$$|k(t,\tau)| = \frac{1}{t-\tau}\Big|\int_0^{\frac{c^2(t,\tau)}{4(t-\tau}} e^{i\xi}\,d\xi\Big| \le \text{const }(t+\tau)^2\,, \quad (t,\tau)\in J^2\,.$$

Hence

$$|\mathrm{II}| \le \Big|\int_x^y u(\tau)\int_\tau^y \frac{k(t,\tau)}{\big((y-t)(t-\tau)\big)^{\frac12}}\,dt\,d\tau\Big| \le$$

$$\le \text{const }\int_x^y\int_\tau^y \frac{dt\,d\tau}{\big((y-t)(t-\tau)\big)^{\frac12}}\,.$$

Setting now $\sigma := \frac{t-\tau}{y-\tau}$ we obtain

$$|\mathrm{II}| \le \text{const }\int_x^y\int_0^1 \frac{d\sigma\,d\tau}{\big(\sigma(1-\sigma)\big)^{\frac12}} = \text{const }(x-y)\,.$$

As for I we have

$$\begin{aligned}
|\,\mathrm{I}\,| &= \Big|\int_0^x u(\tau)\Big\{\int_\tau^x \frac{k(t,\tau)}{\sqrt{t-\tau}}\big(\frac{1}{\sqrt{x-t}}-\frac{1}{\sqrt{y-t}}\big)\,dt\, + \\
&\qquad -\int_x^y \frac{k(t,\tau)dt}{\big((y-t)(t-\tau)\big)^{\frac12}}\Big\}\,d\tau\Big| \\
&\le \text{const }\int_0^x\int_\tau^x \frac{1}{\sqrt{t-\tau}}\big(\frac{1}{\sqrt{x-t}}-\frac{1}{\sqrt{y-t}}\big)\,dt\,d\tau\, + \\
&\qquad + \text{const }\int_0^x\int_x^y \frac{dt\,d\tau}{\big((y-t)(t-\tau)\big)^{\frac12}} \\
&=: \mathrm{I_A} + \mathrm{I_B}\,.
\end{aligned}$$

The integrals $\mathrm{I_A}$ and $\mathrm{I_B}$ may be estimated as follows:

$$\begin{aligned}
|\mathrm{I_A}| &\le \text{const }\int_0^x\int_\tau^x \frac{1}{\sqrt{t-\tau}}\frac{\sqrt{y-t}-\sqrt{x-t}}{\sqrt{(x-t)(y-t)}}\,dt\,d\tau \\
&\le \text{const }(y-x)^{1/2-\epsilon}\int_0^x\int_\tau^x \frac{(y-x)^\epsilon\,dt\,d\tau}{\sqrt{(x-t)(t-\tau)}(y-t)^\epsilon(y-t)^{1/2-\epsilon}} \\
&\le \text{const }(y-x)^{1/2-\epsilon}\int_0^x\int_\tau^x \frac{dt\,d\tau}{\sqrt{(t-\tau)}(x-t)^{1-\epsilon}} \\
&\le \text{const }(y-x)^{1/2-\epsilon}x^{1/2+\epsilon}B(\frac12,\epsilon) \le \text{const }(y-x)^{1/2-\epsilon}.
\end{aligned}$$

and

$$\big|\mathrm{I_B}\big| \le \text{const }\int_x^y \frac{1}{\sqrt{y-t}}\int_0^x \frac{d\tau}{\sqrt{t-\tau}}\,dt \le \text{const }(y-x)^{\frac12}\,.$$

Thus, summarizing, we have

$$|\mathrm{I}| + |\mathrm{II}| \le \text{const }(y-x)^{1/2-\epsilon}\,.$$

Step 3. Consider the first term in the right-hand side of (G4). It follows from the definition of w, q_1 and q_2 that

$$w(q_2(t)) - w(-q_1(t)) = -i\sqrt{i}\frac{|c(t,0)|}{\sqrt{t}} + 2a|c(t,0)| + O(\sqrt{t})\,.$$

Hence

$$|S(t)| \le \frac{\text{const}}{\sqrt{t}}\,,\ t \in J\,. \tag{II.5}$$

Let now $y \ge x > 0$, then

$$\int_0^x \frac{S(t)}{\sqrt{x-t}}\,dt - \int_0^y \frac{S(t)}{\sqrt{y-t}}\,dt =$$
$$= \int_0^x S(t)\big(\frac{1}{\sqrt{x-t}} - \frac{1}{\sqrt{y-t}}\big)\,dt - \int_x^y \frac{S(t)}{\sqrt{y-t}}\,dt =: \mathrm{I} + \mathrm{II}\,.$$

Using (II.5) and performing the change of variables $\sigma := \frac{t-x}{y-x}$ we verify that

$$|\mathrm{II}| \le \text{const}\, x^{-1/2}(y-x)^{1/2}\,.$$

Furthermore

$$|\mathrm{I}| \le \text{const} \int_0^x \frac{1}{\sqrt{t}}\big(\frac{\sqrt{y-t}-\sqrt{x-t}}{\sqrt{y-t}\sqrt{x-t}}\big)\,dt \le$$
$$\le \text{const} \int_0^x \frac{\sqrt{y-x}}{\big(t(x-t)(y-t)\big)^{1/2}}\,dt\,.$$

By the change of variables $\sigma := \frac{t}{x}$, the last integral on the right-hand side above equals

$$\text{const}\,(y-x)^{1/2} \int_0^1 \sigma^{-1/2}(1-\sigma)^{-1/2}(y-\sigma x)^{-1/2+\epsilon}(y-\sigma x)^{-\epsilon}\,d\sigma\,. \tag{II.6}$$

Note that a positive constant M can be found such that

$$\frac{1}{y-\sigma x} \le M\frac{1}{\sigma x} \quad \text{on} \quad (0,T]. \tag{II.7}$$

Inserting (II.7) into (II.6) we obtain

$$|I| \le \text{const}\, x^{\epsilon-\frac{1}{2}}(y-x)^{\frac{1}{2}-\epsilon}B(\epsilon,\frac{1}{2})\,.$$

To prove continuity at $x = 0$, if suffices to show that $\lim_{x\to 0}\int_0^x \frac{S(t)}{\sqrt{x-t}}\,dt$ exists. By definition of $S(t)$ we have

$$S(t) = \frac{N}{2}\big(-\frac{i\sqrt{i}}{\sqrt{t}} + 2a + O(\sqrt{t})\big)e^{i\frac{c^2(t,0)}{4t}}\,.$$

This implies

$$\lim_{x\to 0}\int_0^x \frac{S(t)}{\sqrt{x-t}}dt = -\frac{N}{2}i\sqrt{i}\lim_{x\to 0}\int_0^x \frac{(e^{i\frac{c^2(t,0)}{4t}}-1)}{\sqrt{x-t}\sqrt{t}}\,dt$$
$$-\frac{N}{2}i\sqrt{i}\lim_{x\to 0}\int_0^x \frac{dt}{\sqrt{x-t}\sqrt{t}} = -\frac{N}{2}i\sqrt{i}B(\frac{1}{2},\frac{1}{2}).$$

The second term on the right hand side of (G4) still remains to be handled. Writing this term as

$$\frac{1}{\pi}F(0)\Big[\int_0^x \frac{e^{i\frac{c^2(t,0)}{4t}}-1}{\tau^{\frac{1}{2}}}\big((x-\tau)^{\frac{-1}{2}}-(y-\tau)^{\frac{-1}{2}}\big)d\tau - \int_x^y \frac{e^{i\frac{c^2(t,0)}{4t}}-1}{\tau^{\frac{1}{2}}(y-\tau)^{\frac{1}{2}}}\,d\tau\Big]\ ,$$

it follows that it can be estimated by

$$\text{const}\,(y-x)^{\frac{1}{2}}$$

for all $x, y \in [0,T]$. □

Remark 1. It follows from (II.2) and the previous Lemma that a solution u of (II.1) necessarily must satisfy the initial condition

$$u(0) = -2\lambda N\pi^{3/2}\ .$$

Remark 2. The above proof shows, that for each $u \in E$ and each $\epsilon \in (0, 1/2)$, $\Phi(u)$ is Hölder continuous of degree $1/2-\epsilon$ on each interval $[\delta, T]$, where $0 < \delta < T$.

Lemma 2. *Let δ be such that $L := 4\pi\delta^{-1/2} + \frac{5}{2}\frac{\mu^2}{\omega^2}\delta^{-1} < 1$. Then*

$$\sup_{x\in[0,T]} e^{-\delta x}|\Phi(u)(x)| \le L \sup_{x\in[0,T]} e^{-\delta x}|u(x)|\ .$$

Proof. Let $x, \sigma, \tau \in [0,T]$. Then by (R1)

$$c^2(\sigma,\tau) = \frac{4\mu^2}{\omega^4}\Big\{\big(-\int_{\omega\tau}^{\omega\sigma}\sin(\xi)d\xi\big)^2 + \big(\omega(\sigma-\tau)+\int_{\omega\tau}^{\omega\sigma}\cos(\xi)d\xi\ \big)^2\Big\}$$
$$\le \frac{20\mu^2}{\omega^2}(\sigma-\tau)^2.$$

Hence, we obtain by (R2)

$$|k(\sigma,\tau)| \le \frac{5\mu^2}{\omega^2}$$

and therefore by (R3)

$$|R(\tau,x)| \le \frac{5\mu^2}{\omega^2}B(\frac{1}{2},\frac{1}{2})\ . \tag{II.8}$$

Using (II.8) we at last verify that

$$\|\Phi(u)\|_\delta \le 4\lambda\pi\|m\|_\infty\delta^{-1/2}\|u\|_\delta + \beta\pi\frac{5\mu^2}{\omega^2}\delta^{-1}\|u\|_\delta = L\|u\|_\delta\ .$$

□

After these preparations we are now able to prove our main result which reads as follows.

Theorem 1. *Let $T > 0$. Let R and G be defined as in (R3) and (G4). Then the equation*

(II.9)
$$F(x) = 4\sqrt{i\pi}\int_0^x \frac{F(\tau)}{\sqrt{x-\tau}}\,d\tau + \frac{1}{2\pi}\int_0^x R(\tau,x)F(\tau)\,d\tau + G(x)\,,\ x \in [0,T]\,.$$

possesses a unique solution $u \in C([0,T])$. Moreover, u is locally Hölder continuous of degree $\frac{1}{2} - \epsilon$ for each $\epsilon \in (0, \frac{1}{2})$ on the interval $(0,T]$.

Proof. Fix $T > 0$. It follows from Lemma 1 and Lemma 2 that $\big(I - \Phi\big)^{-1}$ may be represented by

$$\big(I-\Phi\big)^{-1}(v) = \Big(\sum_{n=0}^{\infty}\Phi^n\Big)(v)\ ,\ v \in C([0,T])\ .$$

Hence (II.9) has a unique solution u, which is given by

$$u = \Big(\sum_{n=0}^{\infty}\Phi^n\Big)(G)\ . \tag{II.10}$$

The additional regularity for u is implied by Remark 2 . □

It is now standard to deduce from the uniqueness of the solution the following

Corollary 1. *Let R and G be defined as in (R3) and (G4). Then the equation* (I.2) *has a unique global solution $u \in C\big([0,\infty)\big)$, which is locally Hölder continuous of degree $\frac{1}{2} - \epsilon$ for each $\epsilon \in (0, \frac{1}{2})$ on the interval $(0,\infty)$.*

Equation (II.10) implies that the solution u may be approximated by

$$u_N := \Big(\sum_{n=0}^{N}\Phi^n\Big)(G)\ .$$

Corollary 2. *Let $T > 0$, $N \in \mathbf{N}$ and L be as in Lemma 2. Then the following error estimate is valid:*

$$\sup_{x\in[0,T]} e^{-\delta x}|u(x) - u_N(x)| \le \frac{L^{N+1}}{1-L}\sup_{x\in[0,T]} e^{-\delta x}|G(x)|\ .$$

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