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GRADUAL LOSS OF POSITIVITY AND HIDDEN INVARIANT CONES IN A SCALAR HEAT EQUATION

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Abstract. Invariance properties of a scalar, linear heat equation with nonlocal boundary conditions are discussed as a function of a real parameter appearing in the boundary conditions of the problem. The equation is a model for a thermostat with sensor and controller positioned at opposite ends of an interval, whence the non-locality. It is shown that the analytic semigroup associated with the evolution problem is positive if and only if the parameter is in $(-\infty, 0]$. For the corresponding elliptic problem three maximum principles are proved which hold for different parameter ranges.

1. Introduction. Consider the linear parabolic evolution problem

$$\begin{aligned}
u_t - u_{xx} &= 0\,, & (x,t) &\in (0,\pi)\times(0,\infty)\,,\\
\frac{\partial u}{\partial \nu}(0,t) + \beta\, u(\pi,t) &= 0\,, & t &\in (0,\infty)\,,\\
\frac{\partial u}{\partial \nu}(\pi,t) &= 0\,, & t &\in (0,\infty)\,,\\
u(x,0) &= u_0(x)\,, & x &\in (0,\pi)\,.
\end{aligned} \tag{1}$$

The nonlocal boundary condition $\partial_\nu u(0,t) + \beta\, u(\pi,t) = 0$ is a variation of the classical local Robin condition $\partial_\nu u(0,t) + \beta\, u(0,t) = 0$. The problem (1) can be interpreted as a model for a thermostat where the sensor measures

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the temperature at $x = \pi$ and the controller releases or extracts heat at $x = 0$ proportionally to the temperature at $x = \pi$. It is intuitively clear that only positive values of β can have a stabilizing effect on the reference state $u(x) = 0$. It may be less clear that stability is also lost when β is too large. In fact, as proved in [8], a Hopf bifurcation occurs at $\beta_0 \approx 5.6655$ and the results in [8] suggest that (1) might be termed " the parabolic oscillator ". In this paper we are interested in the invariance (positivity) properties of (1). More precisely, we study the positivity of the analytic C_0-semigroup $e^{-t\,A(\beta)}$ on $L_2(0,\pi)$ that is canonically associated with (1) (see Section 2 for details). For $\beta = 0$ we are considering the heat equation with no flux boundary conditions and it is a consequence of the parabolic maximum principle that

$$u_0 \geq 0 \qquad \Longrightarrow \qquad e^{-t\,A(\beta)} u_0 \geq 0 \ , \ t \geq 0 \,, \tag{2}$$

in the usual L_2 sense. It is also a simple application of the parabolic maximum principle to see that (2) remains true for $\beta \in (-\infty, 0]$. To give a concise formulation of our results we introduce the following closed and convex subsets (cones) of $L_2(0,\pi)$: $P := \{u \in L_2(0,\pi);\ u \geq 0\}$, $Q := \{u \in L_2(0,\pi);\ P^s u \in P \text{ and } P^a u \in P\}$, $R := \{u \in L_2(0,\pi);\ P^s u \in P\}$, where $P^s u(\cdot) := \frac{u(\cdot)+u(\pi-\cdot)}{2}$ and $P^a u(\cdot) := \frac{u(\cdot)-u(\pi-\cdot)}{2}$ are the symmetric and antisymmetric components of u, respectively.
In Section 3 we characterize the positivity of the analytic semigroup associated with (1):

Theorem. $e^{-t\,A(\beta)}(P) \subset P \ , \ t \geq 0 \,,$ iff $\beta \in (-\infty, 0]$.

The behaviour of the nonlocal problem is thus very different from its local counterpart with the classical Robin condition $\partial_\nu u(0,t) + \beta\, u(0,t) = 0$, where (2) holds for each $\beta \in \mathbb{R}$.

For the elliptic problem $A(\beta)\, u = f$ associated with (1) the main result is Theorem 5.6 where the following invariance properties (maximum principles) are proved:

Theorem. *The following assertions hold:* a) $A(\beta)^{-1}(P) \subset P$ *for* $\beta \in (0, \frac{1}{\pi}]$, b) $A(\beta)^{-1}(Q) \subset P$ *for* $\beta \in (0, \frac{2}{\pi}]$, c) $A(\beta)^{-1}(Q) \subset R$ *for* $\beta \in (0, \frac{4}{\pi}]$.

Observe that for $\beta \in (\,0, \frac{1}{\pi}\,]$ the second and third inclusions are trivial consequences of the first and for $\beta \in (\,0, \frac{2}{\pi}\,]$ the third assertion is a consequence of the second one. Thus in that sense, as β is increased, positivity is lost gradually and the nonstandard (hidden) cones Q and R come into play to formulate the weaker positivity persisting for $\beta > \frac{1}{\pi}$. What led us to consider the projections P^s and P^a in the definition of Q and R is the observation that the adjoint problem to (1) can be obtained by the linear transformation on $L_2(0,\pi)$ $Tu(\cdot) := u(\pi - \cdot)$. In Sections 4 and 5 we discuss the adjoint problem and the properties of the linear transformation above. In Section 6 we discuss the existence of positive principal eigenvalues. We find that in contrast to second order elliptic problems with local boundary conditions the existence of a positive principal eigenvalue is not related to the validity of a maximum principle. In Section 7 we briefly discuss how monotone iteration techniques can be applied to nonlinear problems when the parameter β is in the range $(0, \frac{1}{\pi}]$ where the maximum principle holds.

A problem closely related to (1) has been studied by A. Friedman and L. Jiang in [7]. A similar nonlocal boundary condition is considered together with a hysteresis functional. However, Friedman and Jiang do not discuss the spectrum of the linearization and therefore do not find a periodic solution as the result of a Hopf bifurcation. The periodic solution is constructed by a more direct approach applying Schauder's fixed point theorem. Furthermore their problem is piecewise linear. We were not aware of the paper [7] when we submitted [8] and thank M. Grinfeld and J. Ockendon for pointing out that paper to us.

In Section 8 we briefly give an intuitive interpretation of the functional analytic approach sketched in Section 2. For a more detailed introduction in extrapolation techniques the reader may also consult the introduction in [3]. Observations made in numerical experiments are also briefly reported.

We conclude this introduction with the unavoidable:

A misprint in [8]. Unfortunately, a misprint appears in [8]: Equation (7) should read $\omega(\rho,\beta) = 2i\rho(\rho\sin(\pi\rho) - \beta)$. The subsequent discussion of the zeros of ω is not affected by this misprint since the case $\rho = 0$ is easily discussed separately.

2. A sketch of the semigroup approach. In this section we briefly discuss how the linear evolution equation (1) can be reformulated and solved in the context of semigroup theory on Banach scales. We rewrite (1) more concisely as

$$\begin{array}{rcll} u_t + \mathcal{A}u & = & 0\,, & (x,t) \in \Omega \times (0,\infty)\,, \\ \mathcal{B}(\beta)u & = & 0\,, & (x,t) \in \partial\Omega \times (0,\infty)\,, \\ u(x,0) & = & u_0(x)\,, & x \in \Omega\,, \end{array} \tag{3}$$

where $\Omega := (0\,,\,\pi)$ and

$$\mathcal{A}u := -u_{xx}, \quad \mathcal{B}(\beta)u := (\mathcal{B}_1(\beta)u\,,\,\mathcal{B}_2 u\,) \,:=\, (\,\frac{\partial\, u}{\partial\,\nu}(0) \,+\, \beta u(\pi)\,,\,\frac{\partial\, u}{\partial\,\nu}(\pi)\,)\ .$$

We closely follow the approach described in [2] and [3] for solving the above evolution problem. For $(s,p) \in \mathbb{R} \times (1,\infty)$ we denote the Bessel potential spaces by $H^s_p(\Omega)$. For later convenience we also set $E_1 := H^2_2(\Omega)$, $E_{\frac{1}{2}} := H^1_2(\Omega)$, $E_0 := L_2(\Omega)$, $E_{-\frac{1}{2}} := H^1_2(\Omega)'$. The evolution equation (1) can be reformulated as an abstract Cauchy problem in the Banach space $E_{-\frac{1}{2}}$

$$\begin{aligned} \dot{u} + A_{-\frac{1}{2}}u + \beta\gamma_0'\gamma_\pi\, u \;&=\; 0\,,\ t>0\,, \\ u(0) \;&=\; u_0\,, \end{aligned} \tag{4}$$

where $\gamma_x \in \mathcal{L}(E_{\frac{1}{2}}, \mathbb{R})$, $\gamma_x u := u(x)$, for $x \in \{0, \pi\}$ and $\gamma_x' \in \mathcal{L}(\mathbb{R}, E_{-\frac{1}{2}})$ denotes the dual operator to γ_x. The above abstract Cauchy problem is obtained by reinterpreting the weak formulation of (1):

$$\begin{aligned} \int_0^\pi \dot{u}\varphi\,dx + \int_0^\pi u_x\varphi_x\,dx \,+\, \beta u(\pi)\,\varphi(0) \;&=\; 0\,,\ t>0\,, \\ u(0) \;&=\; u_0\,, \end{aligned} \tag{5}$$

where the test functions φ are chosen in $E_{\frac{1}{2}}$. We note that in (4), $A_{-\frac{1}{2}} \in \mathcal{L}(E_{\frac{1}{2}}, E_{-\frac{1}{2}})$ is the linear operator canonically induced via the L_2 duality pairing by the continuous bilinear form, the " Dirichlet form "

$$(u,v) \mapsto \int_0^\pi u_x\, v_x\, dx$$

on $E_{\frac{1}{2}} \times E_{\frac{1}{2}}$. In the following we set $A_{-\frac{1}{2}}(\beta) := A_{-\frac{1}{2}} + \beta\gamma_0'\gamma_\pi$. Since $(\mathcal{A}, \mathcal{B}(\beta))$ is not a normally elliptic boundary value problem (see [2, Section 4]) we use a perturbation argument to show that $-A_{-\frac{1}{2}}(\beta)$ is the generator of an analytic semigroup: following the notation used in [3] we denote the set of generators of analytic C_0-semigroups on $E_{-\frac{1}{2}}$ with domain $E_{\frac{1}{2}}$ by $\mathcal{H}(E_{\frac{1}{2}}, E_{-\frac{1}{2}})$.

Proposition 2.1. *For $\beta \in \mathbb{R}$, $A_{-\frac{1}{2}}(\beta) \in \mathcal{H}(E_{\frac{1}{2}}, E_{-\frac{1}{2}})$. Furthermore, $A_{-\frac{1}{2}}(\beta)^{-1} \in \mathcal{L}(E_{-\frac{1}{2}})$ for $\beta \neq 0$.*

Proof. We verify that $\gamma_0'\gamma_\pi \in \mathcal{L}(E_{\frac{1}{2}}, E_{-\frac{1}{2}})$ is a compact operator. The first assertion then follows from a standard perturbation result from semigroup theory (see e.g. [3, Chapter 1]). It is clear that we only need to prove the compactness of $\gamma_\pi \in \mathcal{L}(E_{\frac{1}{2}}, \mathbb{R})$. Since $E_{\frac{1}{2}}$ is compactly embedded in $C(\overline{\Omega})$ the compactness of γ_π follows from the factorization $E_{\frac{1}{2}} \longrightarrow C(\overline{\Omega}) \xrightarrow{\gamma_\pi} \mathbb{R}$. That $A_{-\frac{1}{2}}(\beta)$ has a bounded inverse for $\beta \neq 0$ is shown in [8] where the spectrum of $A_{-\frac{1}{2}}(\beta)$ is discussed.

Based on the above generation result interpolation-extrapolation techniques can be applied to construct a whole scale of analytic C_0-semigroups. For our purposes it is sometimes convenient to restrict the semigroup on $E_{-\frac{1}{2}}$ to E_0 so that we can formulate our invariance results in terms of the standard positive cone in E_0. The situation is best described by the following commutative diagram:

$$\begin{array}{ccccc} E_1 & \xrightarrow{A_0(\beta)} & E_0 & \xrightarrow{e^{-tA_0(\beta)}} & E_0 \\ \mathrm{d}\downarrow & & \mathrm{d}\downarrow & & \mathrm{d}\downarrow \\ E_{\frac{1}{2}} & \xrightarrow{A_{-\frac{1}{2}}(\beta)} & E_{-\frac{1}{2}} & \xrightarrow{e^{-tA_{-\frac{1}{2}}(\beta)}} & E_{-\frac{1}{2}} \end{array}$$

We refer to [2, Section 6] or [3, Chapter 5] for a complete exposition of the theory.

3. Characterization of positivity. In this section we prove our main result on the analytic C_0-semigroup on $E_{-1/2}$ generated by $-A_{-1/2}(\beta)$. Following [3, Sections 6.3 and 6.4] a *cone* C is defined as a nonempty subset of a vector space that satisfies $\mathbb{R}^+C \subset C$. A cone is *proper* if $C \cap (-C) = \{0\}$. Let $P^\#$ be the cone in $E_{-1/2}$ containing all positive functionals on $E_{1/2}$. It is given by $P^\# = \{u \in E_{-1/2} : \langle u, \phi \rangle \geq 0,\ 0 \leq \phi \in E_{1/2}\}$ and is a proper cone with nonempty interior. Then we have:

Theorem 3.1. $e^{-t\,A_{-1/2}(\beta)}(P^\#) \subset P^\#$, $t \geq 0$, *iff* $\beta \in (-\infty, 0]$. *If* $\beta \leq 0$ *and* $t > 0$, *then* $e^{-t\,A_{-1/2}(\beta)}(P^\#) \subset P$.

Proof. That $\beta \in (-\infty, 0]$ is a sufficient condition is a consequence of the parabolic maximum principle: pick an arbitrary $u_0 \in P^\#$. Since the semigroup is analytic $u(t, u_0) := e^{-tA_0(\beta)}(u) \subset E_{1/2}$ for $t > 0$. In particular, $u(t, u_0)$ is Hölder continuous for $t > 0$. By classical regularity arguments we then obtain that $u(t, u_0) \in BUC^\infty(\Omega)$ for $t > 0$. Since u is a classical solution of

$$u_t - u_{xx} = 0 \quad \text{in} \quad \Omega \times (0, \infty),$$

we can apply the parabolic maximum principle as stated in [6, Propositions 13.1 and 13.2] to conclude that $u(t, u_0) \in P$ for $t > 0$. Our assumption $\beta \leq 0$ together with [6, Proposition 13.2] is used to exclude that u changes sign at the boundary of Ω.

To show that the condition is necessary we study the resolvent of the generator $-A_{-1/2}(\beta)$ on $E_{-1/2}$: For $\lambda > 0$ consider

$$\lambda\, u + A_{-1/2}(\beta) u \quad = \quad \delta_\pi \quad , \text{ in } E_{-1/2} \tag{6}$$

where $\beta > 0$ and $\delta_\pi \in E_{-1/2}$ denotes the Dirac measure supported at the right endpoint of the interval. The solution of (6) will turn out to be negative at $x = 0$ for all sufficiently large λ. In other words we will show that

$$(\lambda + A_{-1/2}(\beta))^{-1}(P^{\#}) \not\subset P^{\#}$$

for $\lambda \in (\lambda_0, \infty)$, $\lambda_0 > 0$, i.e., $-A(\beta)$ is not resolvent positive. Consequently the semigroup $e^{-t\, A_{-1/2}(\beta)}$ is not positive for $\beta > 0$.

If the right-hand side of (6) is replaced by a function $f \in E_0$ it is easy to see that a particular solution of (6) is given by convolution with the kernel $\frac{\sinh(\pi\sqrt{\lambda})}{\sqrt{\lambda}}$. Thus its general solution is given through

$$u(x, \lambda) = \int_0^x \frac{\sinh(\sqrt{\lambda}(x - y)}{\sqrt{\lambda}} f(y)\, dy + A \cosh(\sqrt{\lambda}(\pi - x)) + B \sinh(\sqrt{\lambda}(\pi - x)),$$

where the constants A and B are fixed by imposing the boundary conditions. An easy computation leads to the following expressions for A and B:

$$B = -\int_0^\pi \frac{\cosh(\sqrt{\lambda}(\pi - y)}{\sqrt{\lambda}} f(y)\, dy, \tag{7}$$

$$A = \frac{1}{\beta + \sqrt{\lambda}\sinh(\sqrt{\lambda}\pi)} \Big[\beta \int_0^x \frac{\sinh(\sqrt{\lambda}(x - y)}{\sqrt{\lambda}} f(y)\, dy - B\sqrt{\lambda}\cosh(\sqrt{\lambda}\pi)\Big]. \tag{8}$$

Thus we obtain

$$\begin{aligned} u(0, \lambda) \quad = \quad & \frac{\frac{\beta}{\sqrt{\lambda}}\cosh(\sqrt{\lambda}\pi)}{\beta + \sqrt{\lambda}\, \sinh(\sqrt{\lambda}\, \pi)} \int_0^\pi \sinh(\sqrt{\lambda}\, (\pi - \tau))\, f(\tau)\, d\tau \qquad (9) \\ & + \frac{1 - \frac{\beta}{\sqrt{\lambda}}\sinh(\sqrt{\lambda}\pi)}{\beta + \sqrt{\lambda}\, \sinh(\sqrt{\lambda}\, \pi)} \int_0^\pi \cosh(\sqrt{\lambda}\, (\pi - \tau))\, f(\tau)\, d\tau \quad . \end{aligned}$$

Now choosing an approximating sequence $(f_n)_{n\geq 0}$ of L_2-functions for the Dirac measure δ_π, which is possible since E_0 is dense in $E_{-1/2}$, and using the continuity of the resolvent on $E_{-1/2}$ we infer from (9) that

$$\big((\lambda + A_{-1/2}(\beta))^{-1}\delta_\pi\big)(0) = \frac{1}{\beta + \sqrt{\lambda}\sinh(\sqrt{\lambda}\pi)}\Big[1 - \pi\beta\frac{\sinh(\sqrt{\lambda}\pi)}{\sqrt{\lambda}\pi}\Big] \quad (10)$$

since $(\lambda + A_{-1/2}(\beta)^{-1} f_n \to (\lambda + A_{-1/2}(\beta)^{-1}\delta_\pi$ in $E_{1/2} \hookrightarrow C[0,1]$ and

$$\int_0^\pi \sinh(\sqrt{\lambda}\,(\pi - \tau))\, f_n(\tau)\, d\tau \to \sinh(\sqrt{\lambda}\,\pi)$$
$$\int_0^\pi \cosh(\sqrt{\lambda}\,(\pi - \tau))\, f_n(\tau)\, d\tau \to \cosh(\sqrt{\lambda}\,\pi)$$

as n tends to infinity. We easily conclude that, whenever $\beta > 0$, a λ_0 can be found which makes the right-hand side of (10) negative whenever $\lambda \geq \lambda_0$. It is interesting to observe that

$$\lim_{\lambda\to 0}\big((\lambda + A_{-1/2}(\beta))^{-1}\delta_\pi\big)(0) = \frac{1}{\pi} - \beta\,.$$

It will in fact turn (see Section 5) that a maximum principle holds for $A_0(\beta)$ as long as $\beta < 1/\pi$. We refer to that section for the details.

Remark 3.2. The above result implies that $e^{-t\,A_0(\beta)}(P) \subset P$, $t \geq 0$, iff $\beta \in (-\infty, 0]$. In other words positivity is preserved along the Banach scale.

The time dependent case. For the sake of completeness we briefly discuss the case where the parameter β depends on time. The abstract results in [3, Theorem II.6.3.3] can be applied to obtain invariance properties analogous to the autonomous case. The main assumption is that $\beta(t)$ varies within the corresponding interval uniformly in time. We consider the time dependent problem:

$$\begin{array}{rcll} u_t + \mathcal{A}u &=& 0\,, & (x,t) \in \Omega \times (0,\infty)\,, \\ \mathcal{B}(\beta(t))u &=& 0\,, & (x,t) \in \partial\Omega \times (0,\infty)\,, \\ u(x,0) &=& u_0(x)\,, & x \in \Omega\,, \end{array} \quad (11)$$

where

$$\beta \in C^\rho([0,\infty), \mathbb{R}) \text{ for some } \rho \in (0,1)\,. \quad (12)$$

The abstract theory in [3, Chapter II] can now be invoked to construct a parabolic evolution operator (fundamental solution, propagator) to the above evolution problem. Furthermore the evolution operator inherits the invariance properties shared by the family of analytic semigroups generated by $-A_0(\beta(t))$, $t > 0$.

Theorem 3.3. *The evolution problem* (11) *induces a parabolic evolution operator*

$$U : J_\triangle := \{ (t,s) \in [0,\infty) \times [0,\infty) \,|\, t \geq s \} \rightarrow \mathcal{L}(E_0) \ .$$

If $\beta([0,\infty)) \subset (-\infty, 0]$, *then* $U(t,s)(P) \subset P$ *for* $(t,s) \in J_\triangle$.

Proof. The result is an application of [3, Theorem II.6.3.4].

4. The adjoint evolution problem. In this section we will define the formally adjoint problem to (1) in the usual way. The adjoint problem will reveal a symmetry which turns out to be the key for unveiling hidden invariant sets of the elliptic problem associated with (1).

Definition 4.1. We define the *formally adjoint* operators $(\mathcal{A}^\#, \mathcal{B}^\#)$ to $(\mathcal{A}, \mathcal{B})$ by

$$\mathcal{A}^\# u := -u_{xx} = \mathcal{A}u,$$

$$\mathcal{B}^\# u := (\, \mathcal{B}_1^\# u \,,\, \mathcal{B}_2^\#(\beta)\, u\,) := (\, \frac{\partial\, u}{\partial\, \nu}(0)\ ,\ \frac{\partial\, u}{\partial\, \nu}(\pi) + \beta\, u(0)\,)\ .$$

The formally adjoint evolution problem to (3) is thus given by

$$\begin{aligned} u_t + \mathcal{A}^\# u &= 0\,, & (x,t) &\in \Omega \times (0,\infty)\,, \\ \mathcal{B}^\#(\beta) u &= 0\,, & (x,t) &\in \partial\Omega \times (0,\infty)\,, \\ u(x,0) &= u_0(x)\,, & x &\in \Omega\,. \end{aligned} \tag{13}$$

Hence the adjoint problem is obtained by the change of variables $x \mapsto \pi - x$.

5. Eigenspaces and maximum principles. In this section we collect some properties of the linear transformation $T \in \mathcal{L}(E_0)$, $Tu(\cdot) := u(\pi - \cdot)$. We will then obtain various maximum principles that can be formulated in terms of the eigenspaces of T .

Lemma 5.1. *The transformation* T *has the following properties*

a) $T \in \mathrm{Lis}\,(E_0)$, $T^{-1} = T$, $T^* = T$.

b) $\sigma(T) = \{1, -1\}$, $\ker(1-T) = E_0^s$, $\ker(1+T) = E_0^a$, where

$$E_0^s := \{ \, u \in E_0 : u(\tfrac{\pi}{2} - x) = u(\tfrac{\pi}{2} + x) \text{ for } |x| \leq \tfrac{\pi}{2} \, \} \, ,$$

$$E_0^a := \{ \, u \in E_0 : u(\tfrac{\pi}{2} - x) = -u(\tfrac{\pi}{2} + x) \text{ for } |x| \leq \tfrac{\pi}{2} \, \} \, .$$

c) *E_0 is decomposed by the direct sum $E_0 = E_0^s \oplus E_0^a$. The corresponding continuous projections are given by*

$$P^s u(\cdot) := \tfrac{1}{2}(\, u(\cdot) + u(\pi - \cdot)\,) \, , \quad P^a u(\cdot) := \tfrac{1}{2}(\, u(\cdot) - u(\pi - \cdot)\,) \, .$$

Proof. a) That T is a unitary, self-adjoint, isometric isomorphism on the Hilbert space E_0 is easily verified.

b) To study the spectrum of T consider the equation

$$(\lambda - T)u = f$$

in E_0 . By part a) this is equivalent to

$$\lambda T u - u = T f \, .$$

This can be written as the system

$$\begin{bmatrix} \lambda & -1 \\ -1 & \lambda \end{bmatrix} \begin{bmatrix} u \\ Tu \end{bmatrix} = \begin{bmatrix} f \\ Tf \end{bmatrix} .$$

The spectrum of T is given by the spectrum of the above matrix and it is clear that the eigenspaces are given by the symmetric and antisymmetric functions in E_0 .

c) P^s is a continuous projection on E_0 . Hence

$$E_0 = \operatorname{range}(P^s) \oplus \ker(P^s)$$

and clearly $\operatorname{range}(P^s) = E_0^s$, $\ker(P^s) = E_0^a$.

We can now study the projection onto E_0^s and E_0^a of the following problem in E_0

$$A_0(\beta)u = f,$$

which is the abstract formulation of

$$\begin{aligned} \mathcal{A}u &= f \, , \quad x \in \Omega \, , \\ \mathcal{B}(\beta)u &= 0 \, , \quad x \in \partial\Omega \, . \end{aligned} \tag{14}$$

Proposition 5.2. For $f \in E_0$ consider the solution $u := A_0(\beta)^{-1} f \in E_1$ of (14). Then the symmetric and antisymmetric components of u

$$v := P^s u \quad \text{and} \quad w := P^a u \, ,$$

restricted to $[0, \frac{\pi}{2}]$, solve the linear system

$$\begin{aligned}
& -v_{xx} = f^s \, , && x \in (0, \frac{\pi}{2}) \, , \\
& -w_{xx} = f^a \, , && x \in (0, \frac{\pi}{2}) \, , \\
& v_x(0) = -\frac{\beta}{2}(v(0) - w(0)) \, , && w_x(0) = -\frac{\beta}{2}(v(0) - w(0)) \, , \\
& v_x(\frac{\pi}{2}) = 0 \, , \quad w(\frac{\pi}{2}) = 0 \, .
\end{aligned} \tag{15}$$

Proof. Note that the boundary condition at $x = \frac{\pi}{2}$ is a consequence of the symmetry and antisymmetry of v and w, respectively. We leave the simple verification to the reader. □

We will now derive integral representations for the solutions of (14) and (15).

Lemma 5.3. a) *The solution of* (14) *is given by*

$$u(x) = \frac{1}{\beta} \int_0^\pi f(s) ds \, - \int_x^\pi ds \int_s^\pi f(r) dr \quad .$$

b) *The solution of* (15) *is given by*

$$\begin{aligned}
v(x) &= (\frac{2}{\beta} - \frac{\pi}{2}) \int_0^{\frac{\pi}{2}} f^s(q) dq + \int_0^{\frac{\pi}{2}} dq \int_0^q f^a(r) dr + \int_0^x dq \int_q^{\frac{\pi}{2}} f^s(r) dr, \\
w(x) &= (x - \frac{\pi}{2}) \int_0^{\frac{\pi}{2}} f^s(q) dq + \int_x^{\frac{\pi}{2}} dq \int_0^q f^a(r) dr.
\end{aligned}$$

Proof. a) Integrating (14) over (x, π) and using the boundary condition at $x = \pi$ we find

$$u_x(x) = \int_x^\pi f(q) dq \, .$$

From the boundary condition we deduce

$$u(\pi) = \frac{1}{\beta} u_x(0) = \frac{1}{\beta} \int_0^\pi f(q) dq \, .$$

A second integration over (x,π) yields

$$u(\pi)-u(x)=\int_x^\pi dq\int_q^\pi f(r)dr.$$

Inserting $u(\pi)$ concludes the proof.

b) Integrating the first equation in (15) we obtain

$$\int_x^{\frac{\pi}{2}} v_{xx}(q)dq = v_x(x)-v_x(\frac{\pi}{2}) = \int_x^{\frac{\pi}{2}} f^s(q)dq.$$

In particular

$$v_x(0)=\int_x^{\frac{\pi}{2}} f^s(r)dr\ .$$

Using the boundary condition $v_x(\frac{\pi}{2})=0$ and integrating once more over $(0,x)$ we find

$$v(x)=v(0)+\int_0^x dq\int_q^{\frac{\pi}{2}} f^s(r)dr \quad . \tag{16}$$

Analogously, integrating the second equation in (15) yields

$$w_x(x)=w_x(0)-\int_0^x f^a(r)dr$$

and a second integration over $(x,\frac{\pi}{2})$ produces

$$w(x)=w_x(0)(x-\frac{\pi}{2})+\int_x^{\frac{\pi}{2}} dq\int_0^q f^a(r)dr\ \ , \tag{17}$$

where we have made use of the boundary condition $w(\frac{\pi}{2})=0$. Next we will use the boundary conditions in (15) at $x=0$, which can now be written as

$$\begin{aligned}
\int_0^{\frac{\pi}{2}} f^s(r)dr &= \frac{\beta}{2}\Big(v(0)+\frac{\pi}{2}\,w_x(0)-\int_0^{\frac{\pi}{2}} dq\int_0^q f^a(r)dr\Big)\ \ ,\\
w_x(0) &= \frac{\beta}{2}\Big(v(0)+\frac{\pi}{2}\,w_x(0)-\int_0^{\frac{\pi}{2}} dq\int_0^q f^a(r)dr\Big)\ \ .
\end{aligned}$$

Interpreting the above equations as a linear system for the unknowns $w_x(0)$ and $v(0)$ we can easily solve it and find

$$w_x(0) = \int_0^{\frac{\pi}{2}} f^s(r)dr, \quad v(0) = \left(\frac{2}{\beta} - \frac{\pi}{2}\right) \int_0^{\frac{\pi}{2}} f^s(r)dr + \int_0^{\frac{\pi}{2}} dq \int_0^q f^a(r)dr.$$

Inserting this result into (16) and (17) proves assertion b). □

Definition 5.4. We introduce the following subsets of E_0: $P := \{u \in E_0 : u(x) \geq 0$ a.e. in $\Omega\ \}$, $Q := \{u \in E_0 : P^s u$ and $P^a u \in P\}$, $R := \{u \in E_0 : P^s u \in P\}$.

We collect some easily verified properties of P, Q and R :

Remarks 5.5. a) P, Q and R are convex closed cones in E_0. The cones P and Q are proper while R is not. The following inclusions hold $Q \subset P$ and $Q \subset R$.
b) P is the standard positive, proper, generating, normal cone in E_0.

The above representations of the solution of (14) can be used to prove the following Maximum Principles :

Theorem 5.6 (Maximum Principles). *The following assertions hold*
a) $A_0(\beta)^{-1}(P) \subset P$ *for* $\beta \in (0, \frac{1}{\pi}]$.
b) $A_0(\beta)^{-1}(Q) \subset P$ *for* $\beta \in (0, \frac{2}{\pi}]$.
c) $A_0(\beta)^{-1}(Q) \subset R$ *for* $\beta \in (0, \frac{4}{\pi}]$.
Furthermore the upper bounds given for β *are sharp.*

Proof. a) Lemma 5.3 a) shows that $u(x) \geq u(0)$ for $f \geq 0$ and $x \in [0, \pi]$. From

$$u(0) = (\frac{1}{\beta} - \pi) \int_0^{\pi} f(s)ds$$

we conclude that $u(0) \geq 0$ for $\beta \in (0, \frac{1}{\pi}]$.

b) Lemma 5.3 b) yields

$$u(x) = v(x) + w(x) = (x + \frac{2}{\beta} - \pi) \int_0^{\frac{\pi}{2}} f(s)ds + \int_0^x dq \int_q^{\frac{\pi}{2}} f^s(r)dr + \int_0^x dq \int_0^q f^a(r)dr + 2 \int_x^{\frac{\pi}{2}} dq \int_0^q f^a(r)dr.$$

The assumption then implies that $u(x) \geq 0$ on $[0, \frac{\pi}{2}]$ if $(x + \frac{2}{\beta} - \pi) \geq 0$, i.e., if $\beta \in (0, \frac{2}{\pi}]$. From assertion a) we see that $\min_{x \in [0,\pi]} u(x) = u(0)$ and consequently $u \in P$ for $\beta \in (0, \frac{2}{\pi}]$.

c) By Lemma 5.3 b) we immediately see that $v(x) \geq 0$, $x \in [0, \frac{\pi}{2}]$, if $(\frac{2}{\beta} - \frac{\pi}{2}) \geq 0$, i.e., if $\beta \in (0, \frac{4}{\pi}]$. Hence $v \in P$ by its symmetry. □

We end this section with a "strong maximum principle" for Problem 14 with non homogeneous boundary conditions:

$$\begin{array}{rcll} \mathcal{A}u & = & f & , \text{ in } \Omega \,, \\ \mathcal{B}(\beta)u & = & g & , \text{ on } \partial\Omega \,, \end{array} \tag{18}$$

where $f \in E_0$ and $g := (g_1, g_2) \in \mathbb{R}^2$. We show that for $\beta \in (0, \frac{1}{\pi})$ a strong maximum principle holds for (18).

Theorem 5.7 (Strong Maximum Principle). *Assume $\beta \in (0, \frac{1}{\pi})$ and let u be the solution of (18). Then $f \geq 0$ and $(g_1, g_2) \in \mathbb{R}^2_+$ imply $u \geq 0$. If in addition $f(x) > 0$ almost everywhere on an open subset of Ω or $(g_1, g_2) \neq (0, 0)$, then $u(x) > 0$ on $\overline{\Omega}$.*

Proof. The linearity of (18) allows to decompose its solution as

$$u = u_f + u_g \,,$$

where

$$\begin{array}{rcll} \mathcal{A}\, u_f & = & f, & \text{in } \Omega \,, \\ \mathcal{B}(\beta)\, u_f & = & 0, & \text{on } \partial\Omega \,, \end{array} \tag{19}$$

$$\begin{array}{rcll} \mathcal{A}\, u_g & = & 0, & \text{in } \Omega \,, \\ \mathcal{B}(\beta)\, u_g & = & g, & \text{on } \partial\Omega \,. \end{array} \tag{20}$$

If $(g_1, g_2) = 0$, then $u = u_f$ and the assertions are a consequence of Lemma 5.3 a). If $f \equiv 0$, then noting that $u = u_g$ is an affine function we obtain by a straightforward calculation that

$$u_g(x) = g_2\, x + g_2\, (\frac{1}{\beta} - \pi) + \frac{g_1}{\beta} \,.$$

This concludes the proof. □

6. Principal eigenvalues and the Maximum Principle. In this section we discuss the existence of a *positive* principal eigenvalue in the spectrum of $A_0(\beta)$, or equivalently, the existence of positive principal eigenvalues of

$$\begin{array}{rcll} \mathcal{A}u & = & \lambda\, u \,, & x \in \Omega \,, \\ \mathcal{B}(\beta)u & = & 0 \,, & x \in \partial\Omega \,, \end{array} \tag{21}$$

and the implications on the validity of a maximum principle for $A_0(\beta)u = f$. It is a well-known result that for classical local Robin or Dirichlet boundary conditions the existence of a positive principal eigenvalue for second-order elliptic equations is equivalent to the validity of the strong maximum principle. In particular, for the local counterpart of (21):

$$\begin{aligned} \mathcal{A}u &= \lambda u\,, \quad x \in \Omega\,, \\ \mathcal{B}_1^{\mathrm{loc}}(\beta)u := u_x(0) - \beta\, u(0) &= 0\,, \\ \mathcal{B}_2^{\mathrm{loc}}u := u_x(\pi) &= 0\,, \end{aligned} \tag{22}$$

the following are equivalent (cf. [4, Theorem 2.4]):

- The least eigenvalue of (22) is positive.
- $(\mathcal{A}, \mathcal{B}^{\mathrm{loc}})$ satisfies the strong maximum principle, i.e., $(\mathcal{A}u, \mathcal{B}^{\mathrm{loc}}u) > 0 \Longrightarrow u > 0$.

Note that for each $\beta \in \mathbb{R}$ the least eigenvalue of (22) is simple and its unique normalized principal eigenfunction is strictly positive. For our non-local problem (21) we give the following result

Theorem 6.1. *For $\beta \in (0, \frac{1}{2})$ the spectral bound $s(A_0(\beta)) := inf\{Re(\lambda) : \lambda \in \sigma(A_0(\beta))\}$ is a positive principal eigenvalue, i.e., a simple eigenvalue with a unique normalized eigenfunction which is strictly positive on $\overline{\Omega}$.*

Proof. Note that the least eigenvalue of $A_0(\beta = \frac{1}{2})$ is given by $\lambda_1(\frac{1}{2}) := \rho^2$ where ρ is the first positive root of $\rho \sin(\pi\rho) - \frac{1}{2}$ (see [8, Section 3] for details). The associated normalized eigenfunction is

$$\varphi_1(\beta = \frac{1}{2})(x) := \sin(\frac{x}{2})\,,\ x \in (0, \pi)\,.$$

For $\beta = 0$ clearly the least eigenvalue is 0 and the associated normalized eigenfunction is

$$\varphi_1(\beta = 0)(x) := 1\,,\ x \in (0, \pi)\,.$$

It is easily verified that the normalized first eigenfunctions $\varphi_1(\beta) > 0$ are strictly positive as β varies in $[0, \frac{1}{2})$. The first eigenfunction picks up a nodal point as β crosses $\frac{1}{2}$. □

Remarks 6.2. a) In Theorem 5.6 we have seen that a maximum principle (in the usual sense) is only valid for $\beta \in (0, \frac{1}{\pi}]$. Hence for $\beta \in (\frac{1}{\pi}, \frac{1}{2})$ no maximum principle is available despite the existence of a positive principal eigenvalue. This shows that the characterization of the maximum principle in [4] can not be extended to the nonlocal problem $A_0(\beta)u = f$.

b) The positivity of a semigroup implies properties of the spectrum of its generator. In particular, since P is a generating and normal proper cone, the invariance

$$e^{-tA_0(\beta)}(P) \subset P \tag{23}$$

implies that $s(A_0(\beta)) \in \sigma(A_0(\beta))$. This is proved in [5, Theorem 7.4] as a consequence of the Pringsheim-Landau Lemma. The discussion of the spectrum of $A_0(\beta)$ in [8, Section 3] shows that for $\beta > \beta_r \approx 0.5792$ we have that $s(A_0(\beta)) \notin \sigma(A_0(\beta))$. Hence the general abstract result in [5] shows that (23) is not true for $\beta > \beta_r$. Using the structure of the problem we have found the sharp estimate $\beta > 0$ for the loss of the invariance (23) .

7. Monotone iterations. Having established the validity of a strong maximum principle in Section 5 we will briefly discuss the application of the method of sub- and super-solutions to nonlinear problems of the form

$$\begin{array}{rcll} \mathcal{A}u & = & f(u) & , \text{ in } \Omega , \\ \mathcal{B}(\beta)u & = & g(u) & , \text{ on } \partial\Omega . \end{array} \tag{24}$$

We will be rather brief in this section since it is quite clear that as long as a maximum principle is valid we only need to follow well-known standard arguments. We refer to the review article [1] for details. We impose the following conditions on f and $g = (g_1, g_2)$:

$$f,\, g_1, g_2 \in C^1(\mathbb{R}, \mathbb{R}) , \tag{25}$$

$$f,\, g_1, g_2 \text{ are strictly increasing.} \tag{26}$$

We define sub- and supersolutions in the usual way: we call $w \in C^2(\Omega) \cap C^1(\overline{\Omega})$ a subsolution if

$$\begin{array}{rcll} \mathcal{A}w & \leq & f(w) & , \text{ in } \Omega , \\ \mathcal{B}(\beta)w & \leq & g(w) & , \text{ on } \partial\Omega . \end{array} \tag{27}$$

The sub-solution is called strict if one inequality is strict in some $x \in \overline{\Omega}$. Super-solutions are defined by reversing the inequalities. In the next theorem we show that for $\beta \in (0, \frac{1}{\pi})$ solutions of (24) can be found by monotone iteration.

Theorem 7.1. *Assume that $\beta \in (0, \frac{1}{\pi})$ and that f and g satisfy* (25) *and* (26)*. Then if* (24) *admits a pair (φ, ψ) of ordered sub- and super-solutions the problem* (24) *possesses a least and a greatest solution $\underline{u}$ and $\overline{u}$ in the order interval $[\varphi, \psi]$.*

Proof. We verify that the ideas in [1, Section 9] can be applied. Consider the solution operators K_β of

$$\begin{aligned} \mathcal{A}u &= h \quad , \text{ in } \Omega\,, \\ \mathcal{B}(\beta)u &= 0 \quad , \text{ on } \partial\Omega\,, \end{aligned}$$

and L_β of

$$\begin{aligned} \mathcal{A}u &= 0 \quad , \text{ in } \Omega\,, \\ \mathcal{B}(\beta)u &= h \quad , \text{ on } \partial\Omega\,. \end{aligned}$$

We can reformulate (24) as the following fixed point equation in a suitable ordered Banach space: $u = S_\beta(u) := K_\beta F(u) + L_\beta G(u)$, where F and G are the superposition (Nemytskii) operators induced by f and g. By our assumption (26) these operators are increasing. The assertion now follows from the fact that S_β is increasing and completely continuous by applying a standard monotone iteration scheme. □

We conclude this section with a remark on the boundary conditions in (24).

Remark 7.2. The boundary conditions in (24) can be nonlocal, i.e. more explicitly they can chosen to be

$$\begin{aligned} \mathcal{B}_1(\beta)\, u &= \tfrac{\partial u}{\partial \nu}(0) + \beta\, u(\pi) &&= g_1(u(\xi))\,, \\ \mathcal{B}_2\, u &= \tfrac{\partial u}{\partial \nu}(\pi) &&= g_2(u(\eta))\,, \end{aligned}$$

with arbitrary ξ, $\eta \in \overline{\Omega}$.

8. Final remarks: The trace operator and its dual. The trace operator $\gamma_\pi \in \mathcal{L}(E_{\frac{1}{2}}, \mathbb{R})$ and the dual $\gamma_0' \in \mathcal{L}(\mathbb{R}, E_{-\frac{1}{2}})$ of the trace operator γ_0 appearing in the abstract formulation of (3) in $E_{-\frac{1}{2}}$:

$$\begin{aligned} \dot{u} + A_{-\frac{1}{2}}\, u &= -\,\beta\, \gamma_0'\, \gamma_\pi\, u \qquad , \ t > 0\,, \\ u(0) &= u_0\,, \end{aligned} \tag{28}$$

have a simple interpretation that we briefly discuss: for $s \in \mathbb{R}$ $\gamma_0'(s) \in E_{-\frac{1}{2}} = \mathcal{L}(E_{\frac{1}{2}}, \mathbb{R})$, and for $u \in E_{\frac{1}{2}}$ duality yields

$$\langle \gamma_0'(s)\,,\, u \rangle_{E_{-\frac{1}{2}}, E_{\frac{1}{2}}} \;=\; \langle s\,,\, \gamma_0(u) \rangle_{\mathbb{R}\,', \mathbb{R}} \;=\; s\,\gamma_0(u) \;=\; s\,u(0)\;.$$

Hence $\gamma_0'(s)$ is a Dirac measure $s\,\delta_0$ of mass s concentrated at the left boundary point $x = 0$. Note that the test functions $\mathcal{D}(\Omega)$ are not dense in $E_{\frac{1}{2}}$ and as a consequence $E_{-\frac{1}{2}} \not\subset \mathcal{D}'(\Omega)$. As an example $\gamma_0'(s) \in E_{-\frac{1}{2}}$ but $\gamma_0'(s) \notin \mathcal{D}'(\Omega)$.

The right-hand side of the abstract formulation (28) is thus a reinterpretation of the boundary conditions in (3) in the sense of (generalized) distributions. At each time t a "heat pulse" $-\beta\, u(\pi, t)\; \delta_0$ is released at the left boundary point $x = 0$. This interpretation of the boundary conditions as a source term makes the problem (3) accessible to semigroup techniques.

Considering (28) it becomes evident that it can be interpreted as a feedback control equation of the form

$$\dot{u} + A_{-\frac{1}{2}} u \;\;=\;\; -\beta\, C\, S\, u\;,\; t > 0\,,$$

where $C := \gamma_0' \in \mathcal{L}(\mathbb{R}, E_{-\frac{1}{2}})$ is the controller and $S := \gamma_\pi \in \mathcal{L}(E_{\frac{1}{2}}, \mathbb{R})$ is the sensor. In fact, (28) is a model for a rudimentary thermostat studied in [8]. For more information on feedback control problems and the related questions of controllability and observability we refer to [9].

A particular initial condition. Consider the following initial value problem in $E_{-\frac{1}{2}}$:

$$\begin{aligned} \dot{u} + A_{-\frac{1}{2}}\, u \;&=\; -\,\beta\,\gamma_0'\,\gamma_\pi\, u \qquad , \; t > 0\,, \\ u(0) \;&=\; \gamma_\pi'(1) \;=\; \delta_\pi\;\;. \end{aligned} \tag{29}$$

We have thus chosen a unit heat pulse at $x = \pi$ as the initial condition. We proved that the solution of (29) looses positivity when $\beta > 0$. The initial condition δ_π is extreme in the sense that it manipulates the controller of the thermostat to cool at $x = 0$ by heating the sensor at $x = \pi$, even though the temperature is zero almost everywhere. To observe the loss of positivity at $\beta = 0$ in numerical experiments it is thus advisable to start close to the initial condition δ_π.

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