GAUGED FLOER HOMOLOGY FOR HAMILTONIAN ISOTOPIES I:
DEFINITION OF THE FLOER HOMOLOGY GROUPS

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Abstract. We construct the vortex Floer homology group \( VHF(M, \mu; H) \) for an aspherical Hamiltonian \( G \)-manifold \((M, \omega)\) with moment map \( \mu \) and a class of \( G \)-invariant Hamiltonian loop \( H_t \), following the proposal of [3]. This is a substitute for the ordinary Hamiltonian Floer homology of the symplectic quotient of \( M \). We achieve the transversality of the moduli space by the classical perturbation argument instead of the virtual technique, so the homology can be defined over \( \mathbb{Z} \) or \( \mathbb{Z}_2 \).

Contents

1. Introduction 1
2. Basic setup and outline of the construction 7
3. Asymptotic behavior of the connecting orbits 14
4. Fredholm theory 21
5. Compactness of the moduli space 28
6. Floer homology 31
Appendix A. Transversality by perturbing the almost complex structure 37
References 46

1. Introduction

1.1. Background. Floer homology, introduced by Andreas Floer (see [8], [9]), has been a great triumph of \( J \)-holomorphic curve technique invented by Gromov [17] in many areas of mathematics. Hamiltonian Floer homology gives new invariants of symplectic manifolds and its Lagrangian submanifolds and has been the most important approach towards the solution to the celebrated Arnold conjecture initiated in the theory of Hamiltonian dynamics; the Lagrangian intersection Floer homology is the basic language in defining the Fukaya category of a symplectic manifold and stating Kontsevich’s homological mirror symmetry conjecture; several Floer-type homology theory, including the instanton Floer homology ([7], [4]), Heegaard-Floer theory ([29]), Seiberg-Witten Floer homology ([22]), ECH theory ([20], [21]), has become tools of understanding lower dimensional topology.

All these different types of Floer theory, are all certain infinite dimensional Morse theory, whose constructions essentially apply Witten’s point of view ([34]). Basically, if \( f : X \to \mathbb{R} \) is certain
smooth functional on manifold $X$ (which could be infinite dimensional), then with an appropriate choice of metric on $X$, we can study the equation of negative gradient flow of $f$, of the form

$$x'(t) + \nabla f(x(t)) = 0, \quad t \in (-\infty, +\infty). \tag{1.1}$$

If some natural energy functional defined for maps from $\mathbb{R}$ to $X$ is finite for a solution to the above equation, then $x(t)$ will converges to a critical point of $f$. Assuming that all critical points of $f$ is nondegenerate, then usually we can define a Morse-type index (or relative indices) $\lambda_f : \text{Crit} f \to \mathbb{Z}$. Then for a given pair of critical points $a_-, a_+ \in \text{Crit} f$, the moduli space of solutions to the negative gradient flow equation which are asymptotic to $a_\pm$ as $t \to \pm \infty$, denoted by $M(a_-, a_+)$, has dimension equal to $\lambda_f(a_-) - \lambda_f(a_+)$, if $f$ and the metric are perturbed generically. If $\lambda_f(a_-) - \lambda_f(a_+) = 1$, because of the translation invariance of (1.1), we expect to have only finitely many geometrically different solutions connecting $a_-$ and $a_+$. In many cases (which we call the oriented case), we can also associate a sign to each such solutions.

On the other hand, we define a chain complex over $\mathbb{Z}_2$ (and over $\mathbb{Z}$ in the oriented case), spanned by critical points of $f$ and graded by the index $\lambda_f$; the boundary operator $\partial$ is defined by the (signed) counting of geometrically different trajectories of solutions to (1.1) connecting two critical points with adjacent indices. We expect a nontrivial fact that $\partial \circ \partial = 0$. So a homology group is derived.

### 1.2. Hamiltonian Floer homology and the transversality issue.

In Hamiltonian Floer theory, we have a compact symplectic manifold $(X, \omega)$ and a time-dependent Hamiltonian $H_t \in C^\infty(X)$, $t \in [0, 1]$. We can define an action functional $A_H$ on a covering space $\widetilde{LX}$ of the contractible loop space of $X$. The space $\widetilde{LX}$ consists of pairs $(x, w)$ where $x : S^1 \to X$ is a contractible loop and $w : \mathbb{D} \to X$ with $w|_{\partial \mathbb{D}} = x$; the action functional is defined as

$$A_H(x, w) = -\int_{\mathbb{D}} w^* \omega - \int_{S^1} H_t(x(t)) dt. \tag{1.2}$$

The Hamiltonian Floer homology is formally the Morse homology of the pair $\left( \widetilde{LX}, A_H \right)$.

The critical points are pairs $(x, w)$ where $x : S^1 \to X$ satisfying $x'(t) = X_{H_t}(x(t))$, where $X_{H_t}$ is the Hamiltonian vector field associated to $H_t$; these loops are 1-periodic orbits of the Hamiltonian isotopy generated by $H_t$. Then, choosing a smooth $S^1$-family of $\omega$-compatible almost complex structures $J_t$ on $X$ which induces an $L^2$-metric on the loop space of $X$, (1.1) is written as the Floer equation for a map $u$ from the infinite cylinder $\Theta = \mathbb{R} \times S^1$ to $X$, as

$$\frac{\partial u}{\partial s} + J_t \left( \frac{\partial u}{\partial t} - X_{H_t}(u) \right) = 0. \tag{1.3}$$

Here $(s, t)$ is the standard coordinates on $\Theta$. This is a perturbed Cauchy-Riemann equation, so Gromov’s theory of pseudoholomorphic curves is adopted in Floer’s theory.

There is always an issue of perturbing the equation in order to make the moduli spaces transverse, so that the ambiguity of counting of solutions doesn’t affect the resulting homology. Floer originally defined the Floer homology in the monotone case, which was soon extended by Hofer-Salamon ([19]) and Ono ([28]) to the semi-positive case. Finally by applying the “virtual technique”, Floer homology is defined for general compact symplectic manifold by Fukaya-Ono ([15]) and Liu-Tian ([23]).
1.3. Hamiltonian Floer theory in gauged $\sigma$-model. In this paper, we consider a new type of Floer homology theory proposed in [3] and motivated from Dostoglou-Salamon’s study of Atiyah-Floer conjecture (see [6]). The main analytical object is the symplectic vortex equation, which was also independently studied initially in [3] and by Ignasi Mundet in [25].

The symplectic vortex equation is a natural elliptic system appearing in the physics theory “2-dimensional gauged $\sigma$-model”. Its basic setup contains the following ingredients:

1. The target space is a triple $(M,\omega,\mu)$, where $(M,\omega)$ is a symplectic manifold with a Hamiltonian $G$-action, and $\mu$ is a moment map of the action. We also choose a $G$-invariant, $\omega$-compatible almost complex structure $J$ on $M$.

2. The domain is a triple $(P,\Sigma,\Omega)$, where $\Sigma$ is a Riemann surface, $P \to \Sigma$ is a smooth $G$-bundle, and $\Omega$ is an area form on $\Sigma$.

3. The “fields” are pairs $(A,u)$, where $A$ is a smooth $G$-connection on $P$, and $u : \Sigma \to P \times G M$ is a smooth section of the associated bundle.

Then we can write the system of equation on $(A,u)$:

$$\begin{cases} \partial_A u = 0; \\ \ast F_A + \mu(u) = 0. \end{cases} \tag{1.4}$$

Here $\partial_A u$ is the $(0,1)$-part of the covariant derivative of $u$ with respect to $A$; $F_A$ is the curvature 2-form of $A$; $\ast$ is the Hodge star operator associated to the conformal metric on $\Sigma$ with area form $\Omega$; $\mu(u)$ is the composition of $\mu$ with $u$, which, after choosing a biinvariant metric on $g$, is identified with a section of $\text{ad}P \to \Sigma$. This equation contains a symmetry under gauge transformations on $P$.

Moreover, its solutions are minimizers of the Yang-Mills-Higgs functional:

$$\mathcal{YMH}(A,u) := \frac{1}{2} \left( \|d_A u\|_{L^2}^2 + \|F_A\|_{L^2}^2 + \|\mu(u)\|_{L^2}^2 \right) \tag{1.5}$$

which generalizes the Yang-Mills functional in gauge theory and the Dirichlet energy in harmonic map theory.

Now, similar to Hamiltonian Floer theory, consider the following action functional on a covering space of the space of contractible loops in $M \times g$. Let $H : M \times S^1 \to \mathbb{R}$ be an $S^1$-family of $G$-invariant Hamiltonians; for any contractible loop $\tilde{x} := (x,f) : S^1 \to M \times g$ with a homotopy class of extensions of $x : S^1 \to M$, represented by $w : \mathbb{D} \to M$, the action functional (given first in [3]) is

$$\tilde{A}_H(x,f,w) := -\int_{\mathbb{D}} w^* \omega + \int_{S^1} (\mu(x(t)) \cdot f(t) - H_t(x(t))) \, dt. \tag{1.6}$$

The critical loops of $\tilde{A}_H$ corresponds to periodic orbits of the induced Hamiltonian on the symplectic quotient $\mathcal{M} := \mu^{-1}(0)/G$. The equation of negative gradient flows of $\tilde{A}_H$, is just the symplectic vortex equation on the trivial bundle $G \times \Theta$, with the standard area form $\Omega = ds \wedge dt$, and the connection $A$ is in temporal gauge (i.e., $A$ has no $ds$ component). If choosing an $S^1$-family of $G$-invariant, $\omega$-compatible almost complex structures $J_t$, then the equation is written as a system...
of \((u, \Psi) : \Theta \to M \times g\):

\[
\begin{align*}
\frac{\partial u}{\partial s} + J_t \left( \frac{\partial u}{\partial t} + X_\Psi(u) - Y_{H_t} \right) &= 0; \\
\frac{\partial \Psi}{\partial s} + \mu(u) &= 0.
\end{align*}
\] (1.7)

Solutions with finite energy are asymptotic to loops in \(\text{Crit} \tilde{A}_H\). Then the moduli space of such trajectories, especially those zero-dimensional ones, gives the definition of the boundary operator in the Floer chain complex, and hence the Floer homology group. We call these homology theory the vortex Floer homology. We have to use certain Novikov ring \(\Lambda\), which will be defined in Section 2, as the coefficient ring, and the vortex Floer homology will be denoted by \(VHF(M, \mu; H, J; \Lambda)\). The main part of this paper is devoted to the analysis about (1.7) and its moduli space, in order to define \(VHF(M, \mu; H, J; \Lambda)\).

1.4. Lagrange multipliers. The action functional (1.6) seems to be already complicated, not to mention its gradient flow equation (1.7). However, the action functional (1.6) is just a Lagrange multiplier of the action functional (1.2). Indeed there is a much simpler situation in the case of the Morse theory of a finite-dimensional Lagrange multiplier function, which is worth mentioning in this introduction as a model.

Suppose \(X\) is a Riemannian manifold and \(\mu : X \to \mathbb{R}\) is a smooth function, with 0 a regular value. Then consider a function \(f : X \to \mathbb{R}\) whose restriction to \(X = \mu^{-1}(0)\) is Morse. Then critical points of \(f|_X\) are the same as critical points of the Lagrange multiplier \(F : X \times \mathbb{R} \to \mathbb{R}\) defined by \(F(x, \eta) = f(x) + \eta \mu(x)\), and the Morse index as a critical point of \(f|_X\) is one less than the index as a critical point of \(F\). Then instead of considering the Morse-Smale-Witten complex of \(f|_X\), we can consider that of \(F\). In generic situation, these two chain complexes have the same homology (with a grading shifting), and a concrete correspondence can be constructed through the “adiabatic limit” (for details, see [33]).

Indeed, the vortex Floer homology proposed by Cieliebak-Gaio-Salamon and studied in this paper is an infinite-dimensional and equivariant generalization of this Lagrange multiplier technique. Therefore, the vortex Floer homology is expected to coincide with the ordinary Hamiltonian Floer homology of the symplectic quotient (the proof of this correspondence will be treated in separate work).

1.5. Advantage in achieving transversality. It seems that by considering the complicated equation (1.7) and the moduli spaces we can only recover what we have known of the Hamiltonian Floer theory of the symplectic quotient. But the trade-off is that the most crucial and sophisticated step–transversality of the moduli space–can be achieved more easily. The advantage of lifting to gauged \(\sigma\)-model is because, in many cases, \(M\) has simpler topology than \(\overline{M}\). So the issue caused by spheres with negative Chern numbers is ruled out by topological reason. This phenomenon allows us to achieve transversality of the moduli space by using the traditional “concrete perturbation” to the equation. Moreover, when using virtual technique, the Floer homology group of the symplectic quotient can only be defined over \(\mathbb{Q}\) but here it can be defined over \(\mathbb{Z}\) or \(\mathbb{Z}_2\).
1.6. Computation of the Floer homology group and adiabatic limits. The ordinary Hamiltonian Floer homology $HF(M, H)$ of a compact symplectic manifold can be shown to be canonically independent of the Hamiltonian $H$ and to be isomorphic to the singular homology of $M$. This correspondence plays a significant role in proving the Arnold conjecture. To prove this isomorphism, basically two methods have been used. One is to use a time-independent Morse function as the Hamiltonian, and try to prove that when the function is very small in $C^2$-norm, there is no “quantum contribution” when defining the boundary operator in the Floer chain complex; this was also Floer’s original argument. Another is via the Pidnikh-Salamon-Schwarz (PSS) construction, introduced in [30].

For the case of the vortex Floer homology, it has been well-expected to be isomorphic to the singular homology of the symplectic quotient $M$. To prove this isomorphism we can try the similar methods as for the ordinary Hamiltonian Floer homology (which we will discuss in Section 6.3), as well as the adiabatic limit method, which we discuss here.

Indeed, for any $\lambda > 0$, we consider a variation of (1.7)
\[
\begin{align*}
\frac{\partial u}{\partial s} + J_t \left( \frac{\partial u}{\partial t} + X_\Psi(u) - Y_{H_t} \right) &= 0; \\
\frac{\partial \Psi}{\partial s} + \lambda^2 \mu(u) &= 0.
\end{align*}
\]
(1.8)
This can be viewed as the symplectic vortex equation over the cylinder $\mathbb{R} \times S^1$ with area form replaced by $\lambda^2 ds dt$. The moduli space of solutions to the above equation also defines a Floer homology group, and by continuation method we can show that this homology is (canonically) independent of $\lambda$.

Then we would like to let $\lambda$ approach to $\infty$. By a simple energy estimate, solutions of (1.8) will “sink” into the symplectic quotient $\overline{M}$ and become Floer trajectories of the induced pair $(\overline{H}, \overline{J})$; at isolated points there will be energy blow up, and certain “affine vortices” will appear, which are finite energy solutions to the symplectic vortex equation over the complex plane $\mathbb{C}$. In the Gromov-Witten setting, the work of Gaio-Salamon [16] shows that (in special cases), the Hamiltonian-Gromov-Witten invariants with low degree insertions coincide with the Gromov-Witten invariants of the symplectic quotient, via the Kirwan map $\kappa : H^*_\mathbb{Q}(M) \to H^*(\overline{M})$. The high degree part shall be corrected, by the contribution from the affine vortices. This leads to the definition of the “quantum Kirwan map” (see [38], [35]).

In the case of vortex Floer homology, as long as we can carefully analyze the contribution of affine vortices (maybe with similar restriction on $M$ as in [16]), we could prove that $VHF(M, \mu; H)$ is isomorphic to $HF(\overline{M}; \overline{H})$, with appropriate changes of coefficients.

It is an interesting topic to consider the reversed limit $\lambda \to 0$, and it actually motivated the work of the author with S. Schecter [33], where they considered the nonequivariant, finite dimensional Morse homology. In [33] it was shown that, the Morse-Smale trajectories, as $\lambda \to 0$, will converge to certain “fast-slow” trajectories, and the counting of such trajectories defines a new chain complex, which also computes the same homology.

1.7. Gauged Floer theory for Lagrangian intersections. In Frauenfelder’s thesis and [12], he used the symplectic vortex equation on the strip $\mathbb{R} \times [0, 1]$ to define the “moment Floer homology”
for certain types of pairs of Lagrangians \((L_0, L_1)\) in \(M\). The Lagrangians are not \(G\)-invariant in general, but their intersections with \(\mu^{-1}(0)\) reduce to a pair of Lagrangians \((\overline{L}_0, \overline{L}_1)\) in the symplectic quotient \(\overline{M}\). Then by the calculation in the Morse-Bott case, he managed to prove the Arnold-Givental conjecture with certain topological assumption on \(M\).

Woodward also defined a version of gauged Floer theory in [36], where he considered a pair of Lagrangians \(L_0, L_1\) in the symplectic quotient \(\overline{M}\). They lift to a pair of \(G\)-invariant Lagrangians \(\overline{L}_0, \overline{L}_1 \subset \mu^{-1}(0) \subset M\). Then his equation for connecting orbits is the naive limit of the symplectic vortex equation on the strip \(\mathbb{R} \times [0, 1]\), by setting the area form to be zero. Since the strip is contractible, the equation is just the \(J\)-holomorphic equation on the strip, and two solutions are regarded equivalent if they differ by a constant gauge transformation. Then he applied this Floer theory to the fibres of the toric moment map for any toric orbifold and showed the relation between the nondisplacibility of toric fibres and the Hori-Vafa potential, which reproduces and extends the results of Fukaya et. al. [13] [14].

Both of the above take advantage of the simpler topology of \(M\) than the symplectic quotient, as we mentioned above, to avoid certain virtual technique. Further work are expected to relate the Hamiltonian gauged Floer theory we studied here and the Lagrangian versions, for example, by constructing the so-called “open-close map”.

1.8. **Organization and conventions of this paper.** In Section 2 we give the basic setup, including the action functional, the definition of the Floer chain complex and the equation of connecting orbits. In Section 3 we proved that each finite energy solution is asymptotic to critical loops of the action functional. In Section 4 we study the Fredholm theory of the equation of connecting orbits (modulo gauge transformations); we show that the linearized operator is a Fredholm operator whose index is equal to the difference of Conley-Zehnder indices of the two ends of the connecting orbit. In Section 5 we prove that our moduli space is compact up to breaking, if assuming the nonexistence of nontrivial holomorphic spheres. In Section 6 we summarize the previous constructions and give the definition of the vortex Floer homology (where we postpone the proof of transversality). We also prove the invariance of the homology group by using continuation method. In the final Section we give some discussions on our further work along this line.

In the appendices we provide detailed proof of several technical theorems. Most importantly, we showed that by using concrete perturbation of the almost complex structure, we can achieve transversality of the moduli space, which allows us to avoid the more sophisticated virtual technique.

We use \(\Theta\) to denote the infinite cylinder \(\mathbb{R} \times S^1\), with the axial coordinate \(s\) and angular coordinate \(t\). We denote \(\Theta_+ = [0, +\infty) \times S^1\) and \(\Theta_- = (-\infty, 0] \times S^1\).

\(G\) is a connected compact Lie group, with Lie algebra \(\mathfrak{g}\). Any \(G\)-bundle over \(\Theta\) is trivial, and we just consider the trivial bundle \(P = G \times \Theta\). Any connection \(A\) can be written as a \(\mathfrak{g}\)-valued 1-form on \(\Theta\). We always use \(\Phi\) to denote its \(ds\) component and \(\Psi\) to denote its \(dt\) component.

There is a small \(\epsilon > 0\) such that for the \(\epsilon\)-ball \(g^* \subset \mathfrak{g}^*\) centered at the origin of \(\mathfrak{g}^*\), \(U_\epsilon := \mu^{-1}(g^*)\) can be identified with \(\mu^{-1}(0) \times g^*_\epsilon\). We denote by \(\pi_\mu : U_\epsilon \to \mu^{-1}(0)\) the projection on the the first component, and by \(\pi_\mu : U_\epsilon \to \overline{M}\) the composition with the projection \(\mu^{-1}(0) \to \overline{M}\).
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2. **Basic setup and outline of the construction**

Let \((M, \omega)\) be a symplectic manifold. We assume that it is aspherical, i.e., for any smooth map \(f : S^2 \to M\), \(\int_{S^2} f^* \omega = 0\). This implies that for any \(\omega\)-compatible almost complex structure \(J\) on \(M\), there is no nonconstant \(J\)-holomorphic spheres.

Let \(G\) be a connected compact Lie group which acts on \(M\) smoothly. The infinitesimal action \(g \ni \xi \mapsto X_\xi \in \Gamma(TM)\) is an anti-homomorphism of Lie algebra. We assume the action is Hamiltonian, which means that there exists a smooth function \(\mu : M \to g^*\) satisfying

\[
\mu(gx) = \mu(x) \circ \text{Ad}_{g}^{-1}, \quad \forall \xi \in g, \quad d (\mu \cdot \xi) = \iota_{X_\xi} \omega.
\] (2.1)

Suppose we have a \(G\)-invariant, time-dependent Hamiltonian \(H \in C^\infty_c (M \times S^1)\) with compact support. For each \(t \in S^1\), the associated Hamiltonian vector field \(Y_{H_t} \in \Gamma(TM)\) is determined by

\[
\omega(Y_{H_t}, \cdot) = dH_t \in \Omega^1(M).
\] (2.2)

The flow of \(Y_{H_t}\) is a one-parameter family of diffeomorphisms

\[
\phi^t_t : M \to M, \quad \frac{d\phi^t_t(x)}{dt} = Y_{H_t}(\phi^t_t(x))
\] (2.3)

which we call a Hamiltonian path.

To achieve transversality, we put the following restriction on \(H_t\). It will be used only in the appendix.

**Hypothesis 2.1.** There exists a nonempty interval \(I \subset S^1\) such that \(H_t \equiv 0\) for \(t \in I\).

On the other hand, we need to put several assumptions to the given structures, which are still general enough to include the most important cases (e.g., toric manifolds as symplectic quotients of Euclidean spaces).

**Hypothesis 2.2.** We assume that \(\mu : M \to g^*\) is proper, \(0 \in g^*\) is a regular value of \(\mu\) and the \(G\)-action restricted to \(\mu^{-1}(0)\) is free.

With this hypothesis, \(\mu^{-1}(0)\) is a smooth submanifold of \(M\) and the symplectic quotient \(\overline{M} := \mu^{-1}(0)/G\) is a symplectic manifold, which has a canonically induced symplectic form \(\overline{\omega}\). Also, the Hamiltonian function \(H_t\) descends to a time-dependent Hamiltonian

\[
\overline{H}_t : \overline{M} \to \mathbb{R}
\] (2.4)

by the \(G\)-invariance of \(H_t\). It is easy to check that \(Y_{\overline{H}_t}\) is tangent to \(\mu^{-1}(0)\) and the projection \(\mu^{-1}(0) \to \overline{M}\) pushes \(Y_{\overline{H}_t}\) forward to \(Y_{\overline{\Pi}_t}\). Then we assume

**Hypothesis 2.3.** The induced Hamiltonian \(\overline{H}_t : \overline{M} \to \mathbb{R}\) is nondegenerate in the usual sense.

Finally, in the case when \(M\) is noncompact, we need the convexity assumption (cf. [2, Definition 2.6]).
Hypothesis 2.4. There exists a pair \((f, \mathcal{J})\), where \(f : M \to [0, +\infty)\) is a \(G\)-invariant and proper function, and \(\mathcal{J}\) is a \(G\)-invariant, \(\omega\)-compatible almost complex structure on \(M\), such that there exists a constant \(c_0 > 0\) with

\[
f(x) \geq c_0 \implies \langle \nabla \xi \nabla f(x), \xi \rangle + \langle \nabla \mathcal{J}_f \xi \nabla f(x), \mathcal{J}_x \xi \rangle \geq 0, \quad \forall \xi \in T_x M.
\]

In this paper, to achieve transversality, we need to perturb \(\mathcal{J}\) near \(\mu^{-1}(0)\) (see the appendix). The above condition is only about the behavior “near infinity”, so such perturbations don’t break the hypothesis.

2.1. Equivariant topology.

2.1.1. Equivariant spherical classes. Recall that the Borel construction of \(M\) acted by \(G\) is \(M_G := EG \times_G M\), where \(EG \to BG\) is a universal \(G\)-bundle over the classifying space \(BG\). Then the equivariant (co)homology of \(M\) is defined to be the ordinary (co)homology of \(M_G\), denoted by \(H^G_\ast(M)\) for homology and \(H^\ast_G(M)\) for cohomology.

On the other hand, for any smooth manifold \(N\), we denote by \(S^2(N)\) to be the image of the Hurwitz map \(\pi_2(N) \to H_2(N; \mathbb{Z})\), and classes in \(S^2(N)\) are called spherical classes. We define the equivariant spherical homology of \(M\) to be \(S^G_2(M) := S^2(M_G)\).

Geometrically, any generator of \(S^G_2(M)\) can be represented by the following object: a smooth principal \(G\)-bundle \(P \to S^2\) and a smooth section \(\phi : S^2 \to P \times_G M\). We denote the class of the pair \((P, \phi)\) to be \([P, \phi] \in S^G_2(M)\).

2.1.2. Equivariant symplectic form and equivariant Chern numbers. The equivariant cohomology of \(M\) can also be computed using the equivariant de Rham complex \((\Omega^\ast(M)^G, d^G)\). In \(\Omega^2(M)^G\), there is a distinguished closed form \(\omega - \mu\), called the equivariant symplectic form, which represents an equivariant cohomology class.

We are interested in the pairing \(\langle [\omega - \mu], [P, u] \rangle \in \mathbb{R}\). It can be computed in the following way. Choose any smooth connection \(A\) on \(P\). Then there exists an associated closed 2-form \(\omega_A\) on \(P \times_G M\), called the minimal coupling form. If we trivialize \(P\) locally over a subset \(U \subset S^2\), such that \(A = d + \alpha, \alpha \in \Omega^1(U, g)\) with respect to this trivialization, then \(\omega_A\) can be written as

\[
\omega_A = \pi^* \omega - d(\mu \cdot \alpha) \in \Omega^2(U \times M).
\]

Then we have

\[
\langle [\omega - \mu], [P, u] \rangle = \int_{S^2} u^* \omega_A
\]

which is independent of the choice of \(A\).

On the other hand, any \(G\)-invariant almost complex structure \(J\) on \(X\) makes \(TX\) an equivariant complex vector bundle. So we have the equivariant first Chern class

\[
c_1^G := c_1^G(TM) \in H^2_G(M; \mathbb{Z}).
\]

This is independent of the choice of \(J\).
2.1.3. Kirwan maps. The cohomological Kirwan map is a map

\[ \kappa : H^*(M; \mathbb{R}) \to H^*(\overline{M}; \mathbb{R}). \]  

(2.9)

Here we take \( \mathbb{R} \)-coefficients for simplicity. It is easy to check that

\[ \kappa([\omega - \mu]) = [\overline{\omega}] \in H_2(\overline{M}; \mathbb{R}), \quad \kappa(c^G_1) = c_1(T\overline{M}) \in H_2(\overline{M}; \mathbb{R}). \]  

(2.10)

We define

\[ N^G_2(M) = \ker[\omega - \mu] \cap \ker c^G_1 \subset S^G_2(M), \quad N^G_2(\overline{M}) = \ker[\overline{\omega}] \cap \ker c_1(T\overline{M}) \subset S_2(\overline{M}), \]  

and

\[ \Gamma := S^G_2(M)/N^G_2(M). \]  

(2.11)

2.2. The spaces of loops and equivalence classes. Let \( \tilde{\mathcal{P}} \) be the space of smooth contractible parametrized loops in \( M \times \mathfrak{g} \) and a general element of \( \tilde{\mathcal{P}} \) is denoted by

\[ \tilde{x} := (x, f) : S^1 \to M \times \mathfrak{g}. \]  

(2.12)

Let \( \tilde{\mathcal{P}} \) be a covering space of \( \mathcal{P} \), consisting of triples \( x := (x, f, [w]) \) where \( \tilde{x} = (x, f) \in \mathcal{P} \) and \([w] \) is an equivalence class of smooth extensions of \( x \) to the disk \( \mathbb{D} \). The equivalence relation is described as follows. For each pair \( w_1, w_2 : \mathbb{D} \to M \) both bounding \( x : S^1 \to M \), we have the continuous map

\[ w_{12} := w_1 \#(-w_2) : S^2 \to M \]  

(2.13)

by gluing them along the boundary \( x \). We define

\[ w_1 \sim w_2 \iff [w_{12}] = 0 \in S^G_2(M). \]  

(2.14)

Denote by \( LG := L^\infty G := C^\infty(S^1; G) \) the smooth free loop group of \( G \). Then for any point \( x_0 \in M \), we have the homomorphism

\[ l(x_0) : \pi_1(G) \to \pi_1(M, x_0) \]  

(2.15)

which is induced by mapping a loop \( t \mapsto \gamma(t) \in G \) to a loop \( t \mapsto \gamma(t)x_0 \in M \). For different \( x_1 \in M \) and a homotopy class of paths connecting \( x_0 \) and \( x_1 \), we have an isomorphism \( \pi_1(M, x_0) \simeq \pi_1(M, x_1) \); it is easy to see that \( l(x_0) \) and \( l(x_1) \) are compatible with this isomorphism. This means \( \ker l(x_0) \subset \pi_1(G) \) is independent of \( x_0 \). Then we define

\[ L_M G := \{ \gamma : S^1 \to G \mid [\gamma] \in \ker l(x_0) \subset \pi_1(G) \}. \]  

(2.16)

Let \( L_0 G \subset L_M G \) be the subgroup of contractible loops in \( G \).

It is easy to see that \( L_M G \) acts on \( \tilde{\mathcal{P}} \) (on the right) by

\[ \tilde{\mathcal{P}} \times L_M G \to \tilde{\mathcal{P}}, \quad ((x, f), h) \mapsto h^*(x, f)(t) = \left(h(t)^{-1} x(t), \text{Ad}_{h(t)}^{-1}(f(t)) + h(t)^{-1} \partial_h h(t)\right). \]  

(2.17)

Here the action on the second component can be viewed as the gauge transformation on the space of \( G \)-connections on the trivial bundle \( S^1 \times G \). (For short, we denote by \( d \log h \) the \( \mathfrak{g} \)-valued 1-form \( h^{-1} dh \), which is the pull-back by \( h \) of the left-invariant Maurer-Cartan form on \( G \).)
But $L_MG$ doesn’t act on $\tilde{\mathcal{P}}$ naturally; only the subgroup $L_0G$ does: for a contractible loop $h : S^1 \to G$, extend $h$ arbitrarily to $h : \mathbb{D} \to G$. The homotopy class of extensions is unique because $\pi_2(G) = 0$ for any connected compact Lie group ([1]). Then the class of $(h^{-1}x, h^*f, [h^{-1}w])$ in $\tilde{\mathcal{P}}$ is independent of the extension. It is easy to see that the covering map $\tilde{\mathcal{P}} \to \tilde{\mathcal{P}}$ is equivariant with respect to the inclusion $L_0G \to L_MG$. Hence it induces a covering

$$\tilde{\mathcal{P}}/L_0G \to \tilde{\mathcal{P}}/L_MG.$$  \hfill (2.19)

### 2.3. The deck transformation and the action functional.

We now define an action of $SG^2(M)$ on $\tilde{\mathcal{P}}/L_0G$. Take a class $A \in SG^2(M)$ represented by a pair $(P, u)$, where $P \to S^2$ is a principal $G$-bundle and $u : S^2 \to P \times_G M$ is a section of the associated bundle.

Consider $U_n \simeq \mathbb{C} \cup \{\infty\} \subset S^2$ as the complement of the south pole $0 \in S^2$. Take an arbitrary trivialization $\phi : P|_{U_n} \to U_n \times G$, which induces a trivialization $\phi : P \times_G M|_{U_n} \to U_n \times M$. \hfill (2.20)

Then $\phi \circ u$ is a map from $U_n$ to $M$ and there exists a loop $h : S^1 \to G$ and $x \in M$ such that

$$\lim_{r \to 0} \phi \circ u(re^{i\theta}) = h(\theta)x.$$ \hfill (2.21)

Note that the homotopy class of $h$ is independent of the trivialization $\phi$ and the choice of $x$.

Now, for any element $(x, f, [w]) \in \tilde{\mathcal{P}}$, find a smooth path $\gamma : [0, 1] \to M$ such that $\gamma(0) = w(0)$ and $\gamma(1) = x$. Then define $\gamma_h : S^1 \times [0, 1] \to M$ by $\gamma(\theta, t) = h(\theta)\gamma(t)$.

On the other hand, view $\mathbb{D} \setminus \{0\} \simeq (-\infty, 0] \times S^1$. Consider the map

$$w_h(r, \theta) = h(\theta)w(r, \theta)$$

and the “connected sum”:

$$u\#w := (\phi \circ u) \#\gamma_h \#w_h : \mathbb{D} \to M$$ \hfill (2.22)

which extends the loop

$$x_h(\theta) = h(\theta)x(\theta).$$ \hfill (2.23)

Denote $f_h := \text{Ad}_h f - \partial_h h^{-1}$. Then we define the action by

$$A\#[x, f, [w]] = [x_h, f_h, [u\#w]], \quad \forall A \in SG^2(M).$$ \hfill (2.24)

On the other hand, there exists a morphism

$$SG^2(M) \to \ker l(x_0) \subset \pi_1(G)$$ \hfill (2.25)

which sends the homotopy class of $[P, u]$ to the homotopy class of $h : S^1 \to G$ where $h$ is the one in (2.21). Then it is easy to see the following.

**Lemma 2.5.** The action (2.24) is well-defined (i.e., independent of the representatives and choices) and is the deck transformation of the covering $\tilde{\mathcal{P}}/L_0G \to \tilde{\mathcal{P}}/L_MG$. 

Now by this lemma, we denote \( \mathfrak{P} := \left( \mathfrak{P}/L_0G \right)/N_2^G \), which is again a covering of \( \mathcal{P} := \tilde{\mathcal{P}}/L_MG \), with the group of deck transformations isomorphic to \( \Gamma \). We will use \( \mathfrak{r} \) to denote an element in \( \mathfrak{P} \) or \( \mathfrak{P}/L_0G \) and use \([\mathfrak{r}]\) an element in \( \mathfrak{P} \).

We can define a 1-form \( \tilde{\mathcal{B}}_H \) on \( \tilde{\mathcal{P}} \) by

\[
T_{(x,f)} \tilde{\mathcal{P}} \ni (\xi, h) \mapsto \int_{S^1} \{ \omega \left( x(t) + X_f - Y_{H_t}(t), \xi(t) \right) + \mu(x(t)), h(t) \} \, dt \in \mathbb{R}. \tag{2.26}
\]

Its pull-back to \( \mathfrak{P} \) is exact and one of the primitives is the following action functional on \( \mathfrak{P} \):

\[
\tilde{\mathcal{A}}_H(x, f, [w]) := -\int_B w^* \omega + \int_{S^1} (\mu(x(t)) \cdot f(t) - H_t(x(t))) \, dt. \tag{2.27}
\]

The zero set of the one-form \( \tilde{\mathcal{B}}_H \) consists of pairs \((x, f)\) such that

\[
\mu(x(t)) \equiv 0, \quad \dot{x}(t) + X_{f(t)}(x(t)) - Y_{H_t}(x(t)) = 0. \tag{2.28}
\]

The critical point set of \( \tilde{\mathcal{A}}_H \) are just the preimage of \( \text{Zero} \tilde{\mathcal{B}}_H \) under the covering \( \mathfrak{P} \rightarrow \tilde{\mathcal{P}} \).

**Lemma 2.6.** \( \tilde{\mathcal{A}}_H \) is \( L_0G \)-invariant and \( \tilde{\mathcal{B}}_H \) is \( L_MG \)-invariant.

**Proof.** Take any \( h : S^1 \rightarrow G \), extend it smoothly to some \( h : \mathbb{D} \rightarrow G \). Then we see that

\[
(h^{-1}w)^* \omega = \omega \left( \partial_x(h^{-1}w), \partial_y(h^{-1}w) \right) \, dx dy = w^* \omega + d \left( \mu(h^{-1}w) \cdot d \log h \right). \tag{2.29}
\]

Also, we see that

\[
\mu(h^{-1}(t)x(t)) \cdot \left( \text{Ad}_{h(t)}^{-1} f(t) + h(t)^{-1} h'(t) \right) = \mu(x(t)) \cdot f(t) + \left( \mu(h^{-1}w) \cdot d \log h \right) \bigg|_{\partial \mathbb{D}}. \tag{2.30}
\]

By Stokes’ theorem and the \( G \)-invariance of \( H_t \), we obtain the invariance of \( \tilde{\mathcal{A}}_H \). The \( L_MG \)-invariance of \( \tilde{\mathcal{B}}_H \) follows from equivariance of the involved terms in a similar way. \( \square \)

So we have the induced action functional \( \tilde{\mathcal{A}}_H : \tilde{\mathfrak{P}}/L_0G \rightarrow \mathbb{R} \) and it satisfies the following.

**Lemma 2.7.** For any \([\mathfrak{r}] = [x, f, [w]] \in \tilde{\mathfrak{P}}/L_0G \) and any \( A \in S^G_2(M) \), we have

\[
\tilde{\mathcal{A}}_H(A \# [\mathfrak{r}]) = \tilde{\mathcal{A}}_H([\mathfrak{r}]) - \langle [\omega - \mu], A \rangle. \tag{2.31}
\]

**Proof.** Use the same notation as we define the action \( A \# [\mathfrak{r}] \), we see that

\[
\int_{\mathbb{D} \setminus \{0\}} w_h^* \omega = \int_{\mathbb{D}} w^* \omega + \int_{-\infty}^0 ds \int_{S^1} dt \omega \left( h_s \partial_s w, h_s X_{\partial_s \log h} \right) = \int_{\mathbb{D}} w^* \omega - \int_{-\infty}^0 \int_0^t d \left( \mu(w) \cdot d \log h \right) = \int_{\mathbb{D}} w^* \omega - \int_{S^1} (\mu(x(t)) - \mu(w(0))) d \log h. \tag{2.32}
\]

Also

\[
\int_{S^1 \times [0,1]} \gamma_h^* \omega = -\int_{S^1} (\mu(w(0)) - \mu(x)) d \log h. \tag{2.33}
\]

In the same way we can calculate

\[
\int_{S^1 \setminus \{0\}} (\phi \circ u)^* \omega = \langle [\omega - \mu], [P, u] \rangle - \int_{S^1} \mu(x) d \log h. \tag{2.34}
\]
So we have
\[
\tilde{A}_H(A\#[x]) = -\int_{\mathbb{D}} (w^\#w)^* \omega + \int_{S^1} \left\{ \langle \mu(h(t)x(t)), A_0 \rangle f(t) - h'(t)h(t)^{-1} \right\} dt
\]
\[
= -([\omega - \mu], A) - \int_{\mathbb{D}} w^* \omega + \int_{S^1} \langle \mu(x(t)), f(t) \rangle - H_t(x(t)) dt = -([\omega - \mu], A) + \tilde{A}_H([x]) \quad (2.35)
\]
This lemma implies that \( \tilde{A}_H \) descends to a well-defined function
\[
A_H : \mathfrak{P} \to \mathbb{R}. \quad (2.36)
\]
Our Floer theory will be formally a Morse theory of the pair \((\mathfrak{P}, A_H)\).

Before we move on to the chain complex, we see that \( A_H \) is a Lagrange multiplier function associated to the action functional \( A_{\overline{M}} \) of the induced Hamiltonian \( \overline{H} \) on the symplectic quotient \( \overline{M} \). Let \( \tilde{\mathfrak{P}_{\overline{M}}} \) be the space of contractible loops in \( \overline{M} \) and let \( \mathfrak{P}_{\overline{M}} \) be pairs \((\overline{x}, [\overline{w}] )\) where \( \overline{x} \in \tilde{\mathfrak{P}_{\overline{M}}} \) and \( \overline{w} : \mathbb{D} \to \overline{M} \) extends \( \overline{x} ; [\overline{w}] = [\overline{w}'] \) if \((-\overline{w})^\# \overline{w}'\) is annihilated by both \( \overline{\omega} \) and \( c_1(T\overline{M}) \). Then for any \((\overline{x}, [\overline{w}] )\) \( \in \mathfrak{P}_{\overline{M}} \), we can pull-back the principal \( G \)-bundle \( \mu^{-1}(0) \to \overline{M} \) to \( \mathbb{D} \). Any trivialization (or equivalently a section \( s \)) of this bundle over \( \mathbb{D} \) induces a map \( w_s : \mathbb{D} \to \mu^{-1}(0) \) whose boundary restriction, denoted by \( x : S^1 \to \mu^{-1}(0) \), lifts \( \overline{x} \). Now, if \((\overline{x}, [\overline{w}] )\) \( \in \text{Crit}_{\overline{A}_{\overline{M}}} \), i.e.
\[
0 = \overline{x}'(t) - \overline{Y}_{\overline{H}}(\overline{x}(t)) = (\overline{x}^s)_* (x'(t) - Y_{H_t}(x(t)))
\]
there exists a smooth function, \( f_s : S^1 \to \mathfrak{g} \) such that
\[
x'(t) + X_{f_s(t)}(x(t)) - Y_{H_t}(x(t)) = 0. \quad (2.38)
\]
Then this gives a map
\[
\iota : \tilde{\mathfrak{P}_{\overline{M}}} \to \tilde{\mathfrak{P}}/L_0 G
\]
\[
(\overline{x}, [\overline{w}] ) \mapsto [x_s, f_s, w_s] \quad (2.39)
\]
By the correspondence of the symplectic forms and Chern classes between upstairs and downstairs, we have

**Proposition 2.8.** The class \([x_s, f_s, w_s]\) is independent of the choice of the section \( s \) and only depends on the homotopy class of \( \overline{w} \). Moreover, it induces a map
\[
\iota : (\mathfrak{P}_{\overline{M}}, \text{Crit}_{\overline{A}_{\overline{M}}} ) \to (\mathfrak{P}, \text{Crit}_{A_H}) \quad (2.40)
\]

2.4. The Floer chain complex. For \( R = \mathbb{Z}_2, \mathbb{Z} \) or \( \mathbb{Q} \), we consider the Novikov ring of formal power series over a base ring \( R \) to be the downward completion of \( R[\Gamma] \):
\[
\Lambda_R := \Lambda^1_R := \left\{ \sum_{B \in \Gamma} \lambda_B q^B \left| \forall K > 0, \# \{ B \in \Gamma \mid \langle [\omega - \mu], B \rangle > K, \lambda_B \neq 0 \} < \infty \right\}. \quad (2.41)
\]
We define the free \( \Lambda_R \)-module generated by the set \( \text{Crit}_{A_H} \subset \mathfrak{P} \) to be \( \overline{VCF}(M, \mu; H; \Lambda_R) \). We define an equivalence relation on \( \overline{VCF}(M, \mu; H; \Lambda_R) \) by
\[
[g] \sim q^B [g'] \iff B \# [g'] = [g] \in \mathfrak{P}. \quad (2.42)
\]
Denote by \( VCF(M, \mu; H; \Lambda_R) \) the quotient \( \Lambda_R \)-module by the above equivalence relation, which is will be graded by a Conley-Zehnder type index which will be defined later in Section 4.
2.5. **Gradient flow and symplectic vortex equation.** Now we choose an $S^1$-family of $G$-invariant, $\omega$-compatible almost complex structures $J$ on $M$, we assume that

$$f(x) \geq \epsilon_0 \implies J_t(x) = \mathcal{J}(x)$$

(2.43)

where $(f, \mathcal{J})$ is the convex structure which we assume to exist in Hypothesis 2.4. Then $\omega$ and $J_t$ defines a Riemannian metric on $M$, which induces an $L^2$-metric $\tilde{g}$ on the loop space $LM$. On the other hand, we fix a biinvariant metric on the Lie algebra $g$ which induces a metric $g_2$ on $Lg$; it also identifies $g$ with $g^*$ and we use this identification everywhere in this paper without mentioning it. These choices induce a metric on $\Psi$.

Then, it is easy to see that formally, the equation for the negative gradient flow line of $A_H$ is the following equation for a pair $(u, \Psi) : \Theta \to M \times g$

$$\begin{cases}
\frac{\partial u}{\partial s} + J_t \left( \frac{\partial u}{\partial t} + X_\Psi - Y_{H_t} \right) = 0, \\
\frac{\partial \Psi}{\partial s} + \mu(u) = 0.
\end{cases}$$

(2.44)

The equation is invariant under the action of $LG$. The action is defined by

$$LG \times \text{Map}(\Theta, M \times g) \to \text{Map}(\Theta, M \times g), \quad g^*(u, \Psi)(s,t) = \left( (g(t)^{-1}u(s,t), \text{Ad}_{g(t)}^{-1}\Psi(s,t) + g(t)^{-1}\partial_t g(t) \right)$$

(2.45)

**Definition 2.9.** The energy for a flow line $\hat{u}$ (or its Yang-Mills-Higgs functional) is defined to be

$$E(\hat{u}) := E(u, \Psi) := \frac{1}{2} \|du + (X_\Psi - Y_{H_t}) \otimes dt\|_{L^2}^2 + \frac{1}{2} \|\partial \Psi\|_{L^2}^2 + \frac{1}{2} \|\mu(u)\|_{L^2}^2.$$  

(2.46)

The second equation of (2.44) is actually the symplectic vortex equation on the triple $(P, u, \Psi dt)$, where $P \to \Theta$ is the trivial $G$-bundle, $u$ is a section of $P \times_G M$ and $\Psi dt$ corresponds to the covariant derivative $d + \Psi dt$. (For a detailed introduction to the symplectic vortex equation, see [3] or [25]). This is why we name our theory the vortex Floer homology.

The connection $d + \Psi dt$ has already been put in the **temporal gauge**, i.e., it has no $ds$ component. A more general equation on pairs $(u, \alpha)$, with $\alpha = \Phi ds + \Psi dt \in \Omega^1(\Theta) \otimes g$ is

$$\begin{cases}
\frac{\partial u}{\partial s} + X_\Phi + J_t \left( \frac{\partial u}{\partial t} + X_\Psi - Y_{H_t} \right) = 0, \\
\frac{\partial \Psi}{\partial s} - \frac{\partial \Phi}{\partial t} + [\Phi, \Psi] + \mu(u) = 0.
\end{cases}$$

(2.47)

We write the object $(u, \alpha)$ in the form of a triple $\hat{u} = (u, \Phi, \Psi)$. The above equation is invariant under the action by $G_{\Theta} := C^\infty(\Theta, G)$, which is defined by

$$G_{\Theta} \times \text{Map}(\Theta, M \times g \times g) \to \text{Map}(\Theta, M \times g \times g), \quad g^{*}\left( \begin{array}{c}
u \\
\Phi \\
\Psi
\end{array} \right)(s,t) = \left( \begin{array}{c}g(s,t)^{-1}u(s,t) \\
\text{Ad}_{g(s,t)}^{-1}\Phi(s,t) + g(s,t)^{-1}\partial_s g(s,t) \\
\text{Ad}_{g(s,t)}^{-1}\Psi(s,t) + g(s,t)^{-1}\partial_t g(s,t)
\end{array} \right).$$

(2.48)
A solution $\tilde{u} = (u, \Phi, \Psi)$ to (2.47) is called a \textbf{generalized flow line}. The energy of a generalized flow line is

$$E(u, \Phi, \Psi) = \frac{1}{2} \|du + X_\Phi \otimes ds + (X_\Psi - Y_{H_t}) \otimes dt\|_{L^2} + \frac{1}{2} \|\mu(u)\|_{L^2}^2 + \frac{1}{2} \|\partial_s \Psi - \partial_t \Phi + [\Phi, \Psi]\|_{L^2}^2. \tag{2.49}$$

It is easy to see that any smooth generalized flow line is gauge equivalent via a gauge transformation in $G_\Theta$ to a smooth flow line in temporal gauge; and the energy is gauge independent.

### 2.6. Moduli space and the formal definition of the vortex Floer homology

We focus on solutions to (2.47) with finite energy. In Section 3 we will show that, any such solution is gauge equivalent to a solution $\tilde{u} = (u, \Phi, \Psi)$ such that there exists a pair $\tilde{x}_\pm = (x_\pm, f_\pm) \in \text{Zero}_{\tilde{\mathcal{B}}_H}$ with

$$\lim_{s \to \pm \infty} \Phi(s, t) = 0, \quad \lim_{s \to \pm \infty} (u(s, \cdot), \Psi(s, \cdot)) = \tilde{x}_\pm. \tag{2.50}$$

Hence for any pair $[x_\pm] \in \text{Crit}_{A_H}$, we can consider solutions which “connect” them. We denote by

$$\mathcal{M}([x_-], [x_+]; J, H) \tag{2.51}$$

the moduli space of all such solutions, modulo gauge transformation.

In the appendix we will show that, if we assume that $H_t$ vanishes for $t$ in a nonempty open subset $I \subset S^1$, and $J$ a generic, “admissible” family of almost complex structures, then the space $\mathcal{M}([x_-], [x_+]; J, H)$ is a smooth manifold, whose dimension is equal to the difference of the Conley-Zehnder indices of $[x_\pm]$. Moreover, assuming that the manifold $M$ cannot have any nonconstant pseudoholomorphic spheres, we show that $\mathcal{M}([x_-], [x_+]; J, H)$ is compact modulo breakings. Finally, there exist coherent orientations on different moduli spaces, which is similar to the case of ordinary Hamiltonian Floer theory (see [10]). Then, the signed counting of isolated gauged equivalence classes of trajectories in our case has exactly the same nature as in finite-dimensional Morse-Smale-Witten theory, which defines a boundary operator

$$\delta_J : VCF_*(M, \mu; H; \Lambda_\mathbb{Z}) \to VCF_{*-1}(M, \mu; H; \Lambda_\mathbb{Z}). \tag{2.52}$$

The vortex Floer homology is then defined

$$VHF_*(M, \mu; J, H; \Lambda_\mathbb{Z}) = H(VCF_*(M, \mu; H; \Lambda_\mathbb{Z}), \delta_J). \tag{2.53}$$

Moreover, for a different choice of the pair $(J', H')$, we can use continuation principle to prove that the chain complex $(VCF_*(M, \mu; H'; \Lambda_\mathbb{Z}), \delta_{J'})$ is chain homotopic to $(VCF_*(M, \mu; H; \Lambda_\mathbb{Z}); \delta_J)$. There is a canonical isomorphism between the homologies, and we denote the common homology group by $VHF_*(M, \mu; \Lambda_\mathbb{Z})$. The details are given in Section 6 and the appendix.

### 3. Asymptotic behavior of the connecting orbits

In this section we analyze the asymptotic behavior of solutions $\tilde{u} = (u, \Phi, \Psi)$ to (2.47) which has finite energy and for which $u(\Theta)$ has compact closure in $M$. We call such a solution a \textbf{bounded solution}. We denote the space of bounded solutions by $\tilde{\mathcal{M}}_\Theta^b$. We can also consider the equation on the half cylinder $\Theta_+$ or $\Theta_-$ and denote the spaces of bounded solutions over $\Theta_\pm$ by $\tilde{\mathcal{M}}_{\Theta_\pm}^b$.

The main theorem of this section is
Theorem 3.1. \hspace{1em} (1) Any \((u, \Phi, \Psi) \in \widetilde{M}^b_{\Theta_b} \) is gauge equivalent (via a smooth gauge transformation \(g : \Theta_b \to G\)) to a solution \((u', \Phi', \Psi') \in \widetilde{M}^b_{\Theta_b} \) such that there exist \(\tilde{x}_\pm = (x_\pm, f_\pm) \in \text{Zero} \tilde{B}_H \) and

\[
\lim_{s \to \pm \infty} (u'(s, \cdot), \Psi'(s, \cdot)) = \tilde{x}_\pm, \quad \lim_{s \to \pm \infty} \Phi(s, \cdot) = 0 \tag{3.1}
\]

uniformly for \(t \in S^1\).

(2) There exists a compact subset \(K_H \subset M \) such that for any \((u, \Phi, \Psi) \in \widetilde{M}^b_{\Theta_b} \), we have \(u(\Theta) \subset K_H \).

We will prove (1) for \(\tilde{u} \in \widetilde{M}^b_{\Theta_b} \) in temporal gauge, i.e., \(\Phi \equiv 0 \) and the case for \(\Theta_- \) is the same. Then (2) follows from a maximum principle argument, given at the end of this section. The proof is based on estimates on the energy density, which has been given by others in several different settings (see [2], [16]). The only possibly new ingredient is that we have a nonzero Hamiltonian here.

3.1. Covariant derivatives. The \(S^1\)-family of metrics \(g_t := \omega(\cdot, J_t \cdot) \) induces a metric connection \(\nabla \) on the bundle \(u^*TM\). Moreover, we define

\[
\nabla_{A,s} \xi = \nabla_s \xi + \nabla_{\xi} X_{\Phi}, \quad \nabla_{A,t} \xi = \nabla_t \xi + \nabla_{\xi} X_{\Psi}. \tag{3.2}
\]

Also, on the trivial bundle \(\Theta \times g\), define the covariant derivative

\[
\nabla_{A,s} \theta = \nabla_s \theta + [\Phi, \theta], \quad \nabla_{A,t} \theta = \nabla_t \theta + [\Psi, \theta]. \tag{3.3}
\]

We denote by \(\nabla_A \) the direct sum connection on \(u^*TM \times g\). Note that it is compatible with respect to the natural metric on this bundle.

Define the \(g\)-valued 2-form \(\rho_t \) on \(M \) by

\[
\langle \rho(\xi_1, \xi_2), \eta \rangle_t = \langle \nabla_{\xi_1} X_{\Phi}, \xi_2 \rangle_t = -\langle \nabla_{\xi_2} X_{\eta}, \xi_1 \rangle_t, \quad \xi_i \in TM, \ \eta \in g. \tag{3.4}
\]

We list several useful identities of this covariant derivative. The reader may refer to [16] for details.

Lemma 3.2. For \(\tilde{u} \) in temporal gauge, we have the following equalities

\[
\nabla_{A,s} X_{\eta} - X_{\nabla_{A,s} \eta} = \nabla_{\partial_s u} X_{\eta}, \quad \nabla_{A,t} X_{\eta} - X_{\nabla_{A,t} \eta} = \nabla_{\partial_t u + X_{\Phi}} X_{\eta}. \tag{3.5}
\]

\[
\nabla_{A,s} (\partial_s u + X_{\Phi}) - \nabla_{A,t} \partial_s u = X_{\partial_s \Psi}. \tag{3.6}
\]

\[
\nabla_{A,s} (d\mu \cdot J \xi) - d\mu \cdot J (\nabla_{A,s} \xi) = \rho(\partial_s u, \xi), \quad \nabla_{A,t} (d\mu \cdot J \xi) - d\mu \cdot J (\nabla_{A,t} \xi) = \rho(\partial_t u + X_{\Phi}, \xi). \tag{3.7}
\]

On the other hand, since our equation is perturbed by a Hamiltonian \(H\), we consider another covariant derivative which takes \(H \) into account. Let \(\phi_H^t\) be the Hamiltonian isotopy defined by \(H \), and

\[
J^H_t := (d\phi_H^t)^{-1} \circ J_t \circ d\phi_H^t, \quad t \in \mathbb{R}
\]

be the 1-parameter family of almost complex structures. They are still compatible with \(\omega\) and hence defines a family of metric \(g^H_t\), with induced inner product denoted by \(\langle \cdot, \cdot \rangle_{H,t}\). We denote the induced metric connection on \(u^*TM \) by \(\nabla^H\).
We think \((u, \Psi)\) as a map from \(\mathbb{R} \times \mathbb{R} \to M \times g\), periodic in the second variable. Then we have a well-defined connection \(D\) on \(u^*TM \to \mathbb{R} \times \mathbb{R}\), given by

\[
(D_s \xi)(s, t) = (\nabla^H_\xi)(s, t), \quad (D_t \xi)(s, t) = \nabla^H_t \xi + \nabla^H_\xi X_\Psi - \nabla^H_\xi Y_{H_t}.
\]

(3.8)

**Lemma 3.3.** \(D\) is compatible with the inner product \(\langle \cdot, \cdot \rangle_{H,t}\) on \(u^*TM\).

**Proof.** By direct calculation, we see for \(\xi, \eta \in \Gamma(u^*TM)\),

\[
\partial_s \langle \xi, \eta \rangle_{H,t} = \langle D_s \xi, \eta \rangle_{H,t} + \langle \xi, D_s \eta \rangle_{H,t},
\]

\[
\partial_t \langle \xi, \eta \rangle_{H,t} = \frac{dg}{dt} \langle \xi, \eta \rangle_{H,t} + \left( \nabla^H_t \xi + \nabla^H_\xi X_\Psi, \eta \right)_{H,t} + \left( \xi, \nabla^H_t \eta + \nabla^H_\eta X_\Psi \right)_{H,t}
\]

\[
= - \left( L_{Y_{H_t}} g_t \right) \langle \xi, \eta \rangle_{H,t} + \left( \nabla^H_t \xi + \nabla^H_\xi X_\Psi, \eta \right)_{H,t} + \left( \xi, \nabla^H_t \eta + \nabla^H_\eta X_\Psi \right)_{H,t}
\]

\[
= \langle D_t \xi, \eta \rangle_{H,t} + \langle \xi, D_t \eta \rangle_{H,t}.
\]

\(\square\)

### 3.2. Estimate of the energy density.

**Lemma 3.4.** There exist positive constants \(c_1\) and \(c_2\) depending only on \((X, \omega, J, \mu, H_t)\) and the subset \(K_u\), such that for any flow line \((u, 0, \Psi)\) in temporal gauge, we have

\[
\Delta \left( |\partial_s u|^2_{H,t} \right) \geq -c_1 |\partial_s u|^4_{H,t} - c_2.
\]

(3.9)

**Proof.** First we see that

\[
\frac{1}{2} \Delta |\partial_s u|^2_{H,t} = (\partial^2_s + \partial^2_t) |\partial_s u|^2_{H,t} = \partial_s \langle D_s \partial_s u, \partial_s u \rangle_{H,t} + \partial_t \langle D_t \partial_s u, \partial_s u \rangle_{H,t}
\]

\[
= |D_s \partial_s u|^2_{H,t} + |D_t \partial_s u|^2_{H,t} + \langle (D^2_s + D^2_t) \partial_s u, \partial_s u \rangle_{H,t}.
\]

(3.10)

We denote \(v_s = \partial_s u \in \Gamma(\mathbb{R}^2, u^*TM)\) and \(v_t = \partial_t u + X_\Psi - Y_{H_t} \in \Gamma(\mathbb{R}^2, u^*TM)\). Hence \(v_s = -J_t v_t\). Then we have the following computation

\[
D_s v_t - D_t v_s = \nabla^H_s \left( \partial_t u + X_\Psi - Y_{H_t} \right) - \nabla^H_t \partial_s u + \nabla^H_{\partial_s u} \left(-X_\Psi + Y_{H_t}\right) = X_{\partial_s \Psi} = -X_{\mu(u)}.
\]

\[
D_s v_s + D_t v_t = D_s (-J_t v_t) + D_t (J_t v_s)
\]

\[
= - \nabla^H_s (J_t v_t) + \nabla^H_t (J_t v_s) + \nabla^H_{J_t v_t} (X_\Psi - Y_{H_t})
\]

\[
= - \nabla^H_s (J_t v_t) + \nabla^H_t (J_t v_s) + [J_t v_s, X_\Psi - Y_{H_t}] + \nabla^H_{X_\Psi - Y_{H_t}} (J_t v_s)
\]

\[
= - \nabla^H_s (J_t v_t) + \nabla^H_t (J_t v_s) + \hat{J}_t v_s + [J_t v_s, X_\Psi - Y_{H_t}]
\]

\[
= - (\nabla^H_{v_s} J_t) v_t - J_t \nabla^H_{v_t} v_t + (\nabla^H_{v_t} J_t) v_s + J_t \nabla^H_{v_s} v_t + \hat{J}_t v_s + J_t [v_s, X_\Psi] + [Y_{H_t}, J_t v_s]
\]

\[
= - (\nabla^H_{v_s} J_t) v_t + (\nabla^H_{v_t} J_t) v_s - J_t [v_s, v_t] - J_t X_{\partial_s \Psi} + \hat{J}_t v_s + J_t [v_s, X_\Psi] + J_t [Y_{H_t}, v_s] + (L_{Y_{H_t}} J_t) v_s
\]

\[
= - (\nabla^H_{v_s} J_t) v_t + (\nabla^H_{v_t} J_t) v_s + (L_{Y_{H_t}} J_t) v_s + J_t X_\mu + \hat{J}_t v_s.
\]

(3.12)
On the other hand, for any \((s, t) \in \Sigma, \) any \(\xi \in T_{u(s,t)}M,\) we extend \(\xi\) and \(v_s(s, t)\) to be \(G\)-invariant vector fields locally. Then for the Riemann curvature tensor associated to \(J_s^H,\) we have

\[
R^H(v_s, X_\Psi)\xi = \nabla^H_{v_s} \nabla^H_{X_\Psi} \xi - \nabla^H_{X_\Psi} \nabla^H_{v_s} \xi - \nabla^H_{[v_s, X_\Psi]} \xi = \nabla^H_{v_s} \nabla^H_{X_\Psi} \xi - \nabla^H_{\nabla^H_{\xi} X_\Psi} \xi. \tag{3.13}
\]

Hence

\[
(D_s D_t - D_t D_s) \xi = \nabla^H_s \left( \nabla^H_t \xi + \nabla^H_t (X_\Psi - Y_{Ht}) \right) - \nabla^H_t \left( \nabla^H_s \xi - \nabla^H_s (X_\Psi - Y_{Ht}) \right) - \nabla^H_s \nabla^H_{\xi} (X_\Psi - Y_{Ht}) - \left( \frac{d}{dt} \nabla^H_s \right) \xi \tag{3.14}
\]

The third equality above uses (3.13).

Then we denote

\[
D_s^2 v_s + D_t^2 v_s = D_s (D_s v_s + D_t v_t) + (D_t D_s - D_s D_t) v_t - D_t (D_s v_t - D_t v_s) =: Q_1 + Q_2 + Q_3.
\]

By (3.12),

\[
\langle Q_1, v_s \rangle_{H,t} = \left\langle \nabla^H_s \left( - (\nabla^H_{v_s} J_t) v_t + (\nabla^H_{v_t} J_t) v_s + \left( L_{Y_{Ht}} J_t \right) v_s + J_t X_\mu + J_t v_s \right), v_s \right\rangle_{H,t} \geq -C_1 \left( |v_s|^3_{H,t} + |v_s|^2_{H,t} + |v_s|^2_{H,t} |\nabla^H_s v_s|_{H,t} + |v_s|_{H,t} |\nabla^H_s v_s|_{H,t} \right) \tag{3.15}
\]

for some \(C_1 > 0.\) Here we used the fact that \(\mu\) and \(d\mu\) are uniformly bounded because \(u(\Theta_+)\) has compact closure.

By (3.14), for some \(C_2 > 0,\) we have

\[
\langle Q_2, v_s \rangle_{H,t} = \left( -R^H(v_s, v_t + Y_{H,t}) v_t + \nabla^H_s \nabla^H_{v_t} Y_{Ht} - \nabla^H_{\nabla^H_{v_t} v_t} Y_{Ht} + \nabla^H_{v_t} X_\mu + \left( \frac{d}{dt} \nabla^H_s \right) v_t, v_s \right\rangle_{H,t} \geq -C_2 \left( |v_s|^3_{H,t} + |v_s|^2_{H,t} + |v_s|^2_{H,t} |\nabla^H_s v_s|_{H,t} \right). \tag{3.16}
\]

By (3.11), for some \(C_3 > 0,\) we have

\[
\langle Q_3, v_s \rangle_{H,t} = (D_t X_\mu, v_s)_{H,t} = \left\langle \nabla^H_t X_\mu + \nabla^H_{X_\mu} (X_\Psi - Y_{Ht}), v_s \right\rangle_{H,t} = \langle X_{d\mu \partial u} + \nabla^H_{v_t} X_\mu + [X_\mu, X_\Psi - Y_{Ht}], v_s \rangle_{H,t} \tag{3.17}
\]

\[
= \langle X_{d\mu \partial u} + \nabla^H_{v_t} X_\mu - X_{[\mu, \Psi]}, v_s \rangle = \langle X_{d\mu v_t} - X_{d\mu} X_\Psi + \nabla^H_{v_t} X_\mu - X_{[\mu, \Psi]}, v_s \rangle_{H,t} = (X_{d\mu v_t} + \nabla^H_{v_t} X_\mu, v_s)_{H,t} \geq -C_3 |v_s|^2_{H,t}.
\]
Hence for some $C_4 > 0$ and $c_1, c_2 > 0$, we have

$$\frac{1}{2} \Delta |v_s|^2_H = |D_s v_s|^2_H + |D_t v_t|^2_H + \langle Q_1 + Q_2 + Q_3, v_s \rangle_H \geq |\nabla_{s}^{q_t} v_s|^2_H + \langle Q_1 + Q_2 + Q_3, v_s \rangle_H \geq -c_1 |v_s|^4_H - c_2. \quad (3.18)$$

Now we consider the other part of the energy density. We have the following calculation:

$$\frac{1}{2} \Delta |\mu(u)|^2 = |\nabla_{A,s} \mu|^2 + |\nabla_{A,t} \mu|^2 + \langle \nabla_{A,s} \nabla_{A,s} \mu + \nabla_{A,t} \nabla_{A,t} \mu, \mu \rangle. \quad (3.19)$$

And (see [16, Lemma C.2])

$$\nabla_{A,s} \nabla_{A,s} \mu(u) + \nabla_{A,t} \nabla_{A,t} \mu(u) = \nabla_{A,s} d\mu \cdot v_s + \nabla_{A,t} d\mu \cdot v_t = \nabla_{A,t} (d\mu \cdot J v_s) - \nabla_{A,s} (d\mu \cdot J v_t) = -2 \rho(v_s, v_t) + d\mu \cdot J t (\nabla_{A,t} v_s - \nabla_{A,s} v_t) + d\mu \left( J t v_s \right) \quad (3.20)$$

$$= d\mu \cdot (J t X \mu + J t v_s) - 2 \rho(v_s, v_t).$$

Since $u(\Theta)$ has compact closure, we may assume that $\sup_{u(\Theta)} |\rho| \leq c_u$. Hence there exists $c_3, c_4 > 0$ such that

$$\Delta |\mu(u)|^2 \geq -c_3 - c_4 |v_s|^4. \quad (3.21)$$

### 3.3. Decay of energy density.

To proceed, we quote the following lemma (cf. [32, Page 12]).

**Lemma 3.5.** Let $\Omega \subset \mathbb{R}^2$ be an open subset containing the origin and $e$ is defined over $\Omega$. Suppose it satisfies

$$\Delta e \geq -A - Be^2,$$

then

$$\int_{B_r(0)} e \leq \frac{\pi}{16 B} \Rightarrow e(0) \leq \frac{8}{\pi r^2} \int_{B_r(0)} e + \frac{A r^2}{4}. \quad (3.22)$$

Apply the above lemma to the function $|v_s|^2_{H,t} + |\mu(u)|^2$ and note that different norms are actually equivalent, we see

**Proposition 3.6.** If $(u, \Psi)$ satisfies the equation (2.44) such that $u(\Theta)$ has compact closure and $E(u, \Psi) < \infty$, then the energy density

$$\left| \frac{\partial u}{\partial s} \right|^2 + \left| \frac{\partial \Psi}{\partial s} \right|^2$$

converges to 0 as $s \to \pm \infty$, uniformly in $t$. 

3.4. Approaching to periodic orbits.

**Proposition 3.7.** Any \((u,0,\Psi) \in \widetilde{\mathcal{M}}^b_{\Theta^+}\) is gauge equivalent to a solution to \((2.47)\) \((v,\Phi,\Psi)\) on \(\Theta^+\) with the following properties

1. \((v,\Psi)|_{\{s\} \times S^1}\) converges in \(C^0\) to an element of \(\text{Zero} \overline{\mathcal{B}}_H\) as \(s \to +\infty\);
2. \(\lim_{s \to +\infty} \Phi(s,\cdot) = 0\) uniformly in \(t\).

**Proof.** By the above proposition, we know that finite energy implies that \(\mu(u(s,\cdot)) \to 0\) as \(s \to +\infty\). Let \(g^s \subset g^\ast\) be the \(\epsilon\)-open ball of the origin. Then there exists a \(\epsilon\)-open ball \(U_\epsilon := \mu^{-1}(g^s) \to \mu^{-1}(0) \times g^s\). (3.23)

Then \(\mu_{|U_\epsilon}\) is just the projection of the right hand side of (3.23) onto the second factor (see for example [18]). Let \(\pi_\mu\) be the projection on to the first factor, then we define the almost complex structure

\[ J_{0,t} := \pi_\mu^* (J_t \mid_{\mu^{-1}(0)}) \]

and the vector field

\[ Y_{0,H_t} := \pi_\mu^* (Y_{H_t} \mid_{\mu^{-1}(0)}) \].

Then there exists \(K_1 > 0\) depending on \((M,\omega,J,\mu,H_t)\) such that

\[ \|J_{0,t}(x) - J_t(x)\| \leq K_1|\mu(x)|, \; \|Y_{0,H_t}(x) - Y_{H_t}(x)\| \leq K_1|\mu(x)|, \; \forall x \in U_\epsilon. \]

We denote \(\overline{\pi}_\mu := \mu^{-1}(g^s) \to \overline{M}\) the composition of \(\pi_\mu\) with the projection \(\mu^{-1}(0) \to \overline{M}\). For \((u,0,\Psi) \in \widetilde{\mathcal{M}}^b_{\Theta^+}\), we have proved that \(\mu(u)\) converges to 0 uniformly as \(s \to +\infty\). Hence for \(N := N(\epsilon)\) sufficiently large, \(u(s,\cdot)\) maps \(\Theta^N := [N, +\infty) \times S^1\) into \(U_\epsilon\). So on \(\Theta^N\), we have

\[ \partial_s u + J_{0,t} (\partial_t u + X_\Psi(u) - Y_{0,H_t}(u)) = (J_{0,t} - J_t) (\partial_t u + X_\Psi(u) - Y_{0,H_t}) + J_t (Y_{H_t} - Y_{0,H_t}). \]

(3.24)

Denoting \(\overline{\pi} := \overline{\pi}_\mu \circ u : \Theta^N \to \overline{M}\) and applying \((\overline{\pi}_\mu)_s\) to the above equality, we see on \(\Theta^N\),

\[ \|\partial_s \overline{\pi} + J_t (\partial_t \overline{\pi} - Y_{\overline{\pi}_t} (\overline{\pi}))\| \leq K_2 \epsilon \] (3.25)

for some constant \(K_2\). Here \(\overline{J}_t\) is the induced almost complex structure on \(\overline{M}\). Hence for \(s \geq N\), the family of loops \(\overline{\pi}(s,\cdot)\) in the quotient will be close (in \(C^0\)) to some 1-periodic orbits \(\gamma : S^1 \to \overline{M}\) of \(Y_{\overline{\pi}_t}\).

We take a lift \(p \in \mu^{-1}(0)\) with \(\overline{\pi}_\mu(p) = \gamma(0)\). We see that there exists a unique \(g_\gamma \in G\) such that

\[ \phi_H^1 p = g_\gamma p. \]

(3.26)

Suppose \(g_\gamma = \exp \xi_p, \; \xi_p \in \mathfrak{g}\). It is easy to see that the loop

\[ (x(t), f(t)) := (\exp(-t \xi_p) \phi_H^1 (p), \xi_p) \]

(3.27)

is an element of \(\text{Zero} \overline{\mathcal{B}}_H\). We will construct a gauge transformation \(\tilde{g}\) on \(\Theta^+\) and show that \(\tilde{g}^\ast(u,0,\Psi)\) satisfies the condition stated in this proposition.
Take a local slice of the $G$-action near $p$. That is, an embedding $i : B_{\delta}^{2n-2k} \to \mu^{-1}(0)$ where $B_{\delta}^{2n-2k}$ is the $\delta$-ball in $\mathbb{R}^{2n-2k}$ such that $i(0) = p$ and $(y, g) \mapsto g(i(y))$ is a diffeomorphism from $B_{\delta}^{2n-2k} \times G$ onto its image.

Denote $u(s, t) = (v(s, t), \xi(s, t)) \in \mu^{-1}(0) \times \mathfrak{g}^*$ with respect to the decomposition (3.23). Then for $s$ large enough, there exists a unique $g(s) \in G$ such that
\[
g(s)v(s, 0) \in i\left(B_{\delta}^{2n-2k}\right), \quad g(s)v(s, 0) \to p.
\]

Moreover, by the fact that $|\partial_s u|$ converges to zero, we see
\[
\lim_{s \to +\infty} \left|g(s)^{-1}\dot{g}(s)\right| = 0.
\]

Define $h_s(t) \in G$ by
\[
h_s(0) = 1, \quad h_s(t)^{-1}\frac{\partial h_s(t)}{\partial t} = \Psi(s, t).
\]

Then by the fact that $\lim_{s \to +\infty} |\partial_s \Psi| = 0$ we see that
\[
\lim_{s \to +\infty} |\partial_s \log h_s(t)| = 0.
\]

Thus we have
\[
d \left(g(p, g(s)h_s(1)g(s)^{-1}p)\right)
\leq d \left(g(p, \phi_H^1 g(s)v(s, 0)) + d \left(g(s)h_s(1)v(s, 0), g(s)h_s(1)g(s)^{-1}p\right)\right)
\]
\[
= d \left(g(p, \phi_H^1 g(s)v(s, 0)) + d \left(g(s)h_s(1)v(s, 0), g(s)h_s(1)v(s, 0)\right) + d \left(g(s)v(s, 0), p\right)\right)
\]
\[
=: d_1(s) + d_2(s) + d_3(s).
\]

Here $d$ is the $G$-invariant distance function induced by an invariant Riemannian metric. By (3.26) and (3.28), we have $d_1(s) + d_3(s) \to 0$. By the decay of energy density, i.e.,
\[
\lim_{s \to +\infty} \sup_t |\partial_t u + X_\Psi - Y_{H_t}| = 0,
\]

we have $d_2(s) \to 0$. Hence we have
\[
\lim_{s \to +\infty} d \left(g(p, g(s)h_s(1)g(s)^{-1}p)\right) = 0.
\]

Since the $G$-action on $\mu^{-1}(0)$ is free, we have
\[
\lim_{s \to +\infty} g(s)h_s(1)g(s)^{-1} = g_p
\]

Then by (3.33), there exists a continuous curve $\xi(s) \in \mathfrak{g}$ defined for large $s$, such that
\[
g(s)h_s(1)g(s)^{-1} = \exp \xi(s), \quad \lim_{s \to +\infty} \xi(s) = \xi_p.
\]

Then apply the gauge transformation
\[
\tilde{g}(s, t) = h_s(t)^{-1}g(s)^{-1}\exp(t\xi(s))
\]

to the pair $(u, 0, \Psi)$, we see
\[
(\tilde{g}^*u)(s, 0) = \tilde{g}(s, 0)^{-1}(u(s, 0)) = g(s)u(s, 0) \to p, \quad \tilde{g}^*(\Psi dt) = \xi(s)dt + \eta(s, t)ds
\]
Definition 3.8. Let \( \eta = \partial_s \log \tilde{g} = \tilde{g}^{-1} \frac{\partial \tilde{g}}{\partial s} \). The fact that \( \lim_{s \to +\infty} \| \eta \| = 0 \) follows from (3.29) and (3.30).

Hence
\[
\lim_{s \to +\infty} (\tilde{g}^*u)|_{(s) \times S^1} = (\exp(-t\xi_p)\phi_H^t p, \xi_p) \in \text{Zero} \tilde{B}_H.
\]

Definition 3.8. Let \( \tilde{x}_\pm := (x_\pm, f_\pm) \in \text{Zero} \tilde{B}_H \). We denote
\[
\tilde{M}(\tilde{x}_-, \tilde{x}_+):= \tilde{M}(\tilde{x}_-, \tilde{x}_+; J, H) := \left\{ (u, \Phi, \Psi) \in \tilde{M}_\Theta^b \mid \lim_{s \to \pm \infty} (u, \Phi, \Psi)|_{(s) \times S^1} = (\tilde{x}_\pm, 0) \right\}. \tag{3.37}
\]

For \( r_\pm = (\tilde{x}_\pm, [w_\pm]) \in \text{Crit} \tilde{A}_H \) which projects to \( \tilde{x}_\pm \) via \( \text{Crit} \tilde{A}_H \to \text{Zero} \tilde{B}_H \), we define
\[
\tilde{M}(r_-, r_+):= \tilde{M}(r_-, r_+; J, H) := \left\{ (u, \Phi, \Psi) \in \tilde{M}(\tilde{x}_-, \tilde{x}_+) \mid [u\# w_-] = [w_+] \right\}. \tag{3.38}
\]

Then it is easy to deduce the following energy identity for which we omit the proof.

Proposition 3.9. Let \( r_\pm \in \text{Crit} \tilde{A}_H \). Then for any \( (u, \Phi, \Psi) \in \tilde{M}(r_-, r_+) \), we have
\[
E(u, \Phi, \Psi) = \tilde{A}_H (r_-) - \tilde{A}_H (r_+). \tag{3.39}
\]

3.5. Convexity and uniform bound on flow lines. We will show in this subsection the following

Proposition 3.10. There exists a compact subset \( K_H \subset M \) such that for any \( (u, \Phi, \Psi) \in \tilde{M}_\Theta^b \), \( u(\Theta) \subset K_H \).

Proof. The proof is to use maximum principle as in [2, Subsection 2.5]. We claim this proposition is true for
\[
K_H = \text{Supp} H \cup f^{-1}([0, \epsilon_1])
\]
where
\[
\epsilon_1 = \max \left\{ \epsilon_0, \sup_{|\mu(x)| \leq 1} f(x) \right\} \tag{3.40}
\]
where \( \epsilon_0 \) is the one in Hypothesis 2.4.

Suppose the statement is not true. Then there exists a solution \( \tilde{u} = (u, \Phi, \Psi) \in \tilde{M}_\Theta^b \) which violates this condition and \((s_0, t_0) \in \Theta \) such that \( u(s_0, t_0) \notin \text{Supp} H \) and \( f(u(s_0, t_0)) > \epsilon_1 \). On the other hand, by the previous results, we know that \( \lim_{s \to \pm \infty} \mu(u(s, t)) = 0 \) so \( \lim_{s \to \pm \infty} f(u(s, t)) \leq \epsilon_1 \). Hence \( f(u) \) achieves its maximum at some point of \( \Theta \). As in the proof of [2, Lemma 2.7], we see that \( f(u) \) is subharmonic on \( u^{-1}(M \setminus K_H) \) and hence \( f(u) \) must be constant. However, this contradicts with the fact that \( \lim_{s \to \pm \infty} f(u(s, t)) \leq \epsilon_1 \). \( \square \)

4. Fredholm theory

In this section we investigate the infinitesimal deformation theory of solutions to our equation (modulo gauge). For a similar treatment of a relevant situation, the reader may refer to [4].
4.1. Banach manifolds, bundles, and local slices. First we fix two loops \( \tilde{x}_\pm \in \text{Zero} \tilde{B}_H \). For any \( k \geq 1, p > 2 \), we consider the space of \( W_{loc}^{k,p} \)-maps \( \tilde{u} := (u, \Phi, \Psi) : \Theta \rightarrow M \times g \times g \), such that \( \Phi \in W^{k,p}(\Theta, g) \) and \( (u, \Psi) \) is asymptotic to \( \tilde{x}_\pm = (x_\pm, f_\pm) \) at \( \pm \infty \) in \( W^{k,p} \)-sense. Then this is a Banach manifold, denoted by

\[
\tilde{B}^{k,p} := \tilde{B}^{k,p}(\tilde{x}_-, \tilde{x}_+). \tag{4.1}
\]

The tangent space at any element \( \tilde{u} \in \tilde{B}^{k,p} \) is the Sobolev space

\[
T_{\tilde{u}}\tilde{B}^{k,p} = W^{k,p}(\Theta, u^*TM \oplus g \oplus g). \tag{4.2}
\]

We denote by \( \tilde{\exp}^t \) the exponential map of \( M \times g \times g \), where the Riemannian metric on \( M \) is \( \omega(\cdot, J_t \cdot) \) which is \( t \)-dependent. Then the map \( \tilde{\xi} \mapsto \tilde{\exp}^t_{\tilde{\xi}} \) is a local diffeomorphism from a neighborhood of \( 0 \in T_{\tilde{u}}\tilde{B}^{k,p} \) and a neighborhood of \( \tilde{u} \) in \( \tilde{B}^{k,p} \).

Then consider a pair \( \mathfrak{r}_\pm = (x_\pm, f_\pm, [w_\pm]) \in \text{Crit} \tilde{A}_H \) with \( \tilde{x}_\pm = (x_\pm, f_\pm) \in \text{Zero} \tilde{B}_H \). We define

\[
\tilde{B}^{k,p}(\mathfrak{r}_-, \mathfrak{r}_+) := \left\{ \tilde{u} = (u, \Phi, \Psi) \in \tilde{B}^{k,p}(\tilde{x}_-, \tilde{x}_+) \mid [w_- u] = [w_+] \right\}. \tag{4.3}
\]

Let \( \mathcal{G}_0^{k+1,p} \) be the space of \( W_{loc}^{k+1,p} \)-maps \( g : \Theta \rightarrow G \) which is asymptotic to the identity of \( G \) at \( \pm \infty \). Then this is a Banach Lie group. The gauge transformation extends to a free \( \mathcal{G}_0^{k+1,p} \)-action on \( \tilde{B}^{k,p}(\tilde{x}_-, \tilde{x}_+) \) (resp. \( \tilde{B}^{k,p}(\mathfrak{r}_-, \mathfrak{r}_+) \)), because the symplectic quotient \( \mathcal{M} \) is a free quotient. Then this makes the quotient

\[
\tilde{B}^{k,p}(\tilde{x}_-, \tilde{x}_+) := \tilde{B}^{k,p}(\tilde{x}_-, \tilde{x}_+)/\mathcal{G}_0^{k+1,p} \quad \text{(resp.} \quad \tilde{B}^{k,p}(\mathfrak{r}_-, \mathfrak{r}_+) := \tilde{B}^{k,p}(\mathfrak{r}_-, \mathfrak{r}_+)/\mathcal{G}_0^{k+1,p}) \tag{4.4}
\]

a Banach manifold. Indeed, to see this we have to construct local slices of the \( \mathcal{G}_0^{k+1,p} \)-action. For any \( \tilde{u} \in \tilde{B}^{k,p} \) (whose image in \( B^{k,p} \) is denoted by \( [\tilde{u}] \)), consider the operator

\[
d_0^* : T_{[\tilde{u}]}B^{k,p} \rightarrow W^{k-1,p}(\Theta, g)
\]

\[
(\xi, \phi, \psi) \mapsto -d\mu(J_1 \xi) - \partial_\phi \phi - [\Phi, \phi] - \partial_\psi \psi - [\Psi, \psi], \tag{4.5}
\]

which is the formal adjoint of the infinitesimal \( \mathcal{G}_0^{k+1,p} \)-action. Then as in gauge theory, we have a natural identification

\[
T_{[\tilde{u}]}B^{k,p} \simeq \ker d_0^* \tag{4.6}
\]

(where the orthogonal complement is taken with respect to the \( L^2 \)-inner product) and the exponential map \( \tilde{\exp}^t \) induces a local diffeomorphism

\[
\ker d_0^* \ni \tilde{\xi} \mapsto \left[ \tilde{\exp}^t_{\tilde{\xi}} \right] \in B^{k,p}. \tag{4.7}
\]

If we have \( g_\pm \in L_0 G \), and \( \mathfrak{r}'_\pm = g_\pm^* \mathfrak{r}_\pm \in \text{Crit} \tilde{A}_H \), then the pair \( (g_-, g_+) \) extends to a smooth gauge transformation on \( \Theta \) which identifies \( \tilde{B}^{k,p}(\mathfrak{r}_-, \mathfrak{r}_+) \) with \( \tilde{B}^{k,p}(\mathfrak{r}'_-, \mathfrak{r}'_+) \). Then, with abuse of notation, if \( \mathfrak{r}_\pm \in \text{Crit} \tilde{A}_H/L_0 G \), then we can denote by \( B^{k,p}(\mathfrak{r}_-, \mathfrak{r}_+) \) to be the common quotient space. Finally, for two pairs \( [\mathfrak{r}_\pm] \in \text{Crit} \tilde{A}_H \), we define

\[
B^{k,p}([\mathfrak{r}_-], [\mathfrak{r}_+]) := \bigcup_{\mathfrak{r}_\pm \in \text{Crit} \tilde{A}_H/L_0 G, [\mathfrak{r}_\pm] = [\mathfrak{r}_\pm]} B^{k,p}(\mathfrak{r}_-, \mathfrak{r}_+), \tag{4.8}
\]

which is a discrete union of Banach manifolds.
Over $\tilde{B}^{k,p}(x_-, x_+)$, we have the smooth Banach space bundle $\tilde{E}^{k-1,p}(x_-, x_+)$, whose fibre over $\tilde{u}$ is the Sobolev space

$$\tilde{E}_u^{k-1,p} := W^{k-1,p}(u^*TM \oplus g).$$  \hspace{1cm} (4.9)

The $G^{k+1,p}_0$-action makes $\tilde{E}^{k-1,p}$ an equivariant bundle, hence descends to a Banach space bundle

$$\tilde{E}^{k-1,p}(x_-, x_+) \to B^{k,p}(x_-, x_+) \left( \text{or } \tilde{E}^{k-1,p}(\{x_-, [x_+]\}) \to B^{k,p}(\{x_-, [x_+]\}) \right).$$  \hspace{1cm} (4.10)

Moreover, the $H$-perturbed vortex equation (2.47) gives a section

$$\tilde{F} : \tilde{B}^{k,p}(x_-, x_+) \to \tilde{E}^{k-1,p}(x_-, x_+)$$  \hspace{1cm} (4.11)

which is $G^{k+1,p}$-equivariant. So it descends to a section

$$F : B^{k,p}(\{x_-, [x_+]\}) \to \tilde{E}^{k-1,p}(\{x_-, [x_+]\}).$$

Then we see that $\tilde{M}(x_-, x_+; J, H)$ is the intersection of $\tilde{F}^{-1}(0)$ with smooth objects. We define

$$M(\{x_-, [x_+]; J, H\} = F^{-1}(0),$$  \hspace{1cm} (4.12)

whose elements, by the standard regularity theory about symplectic vortex equation (see [2, Theorem 3.1]), all have smooth representatives. Therefore $M(\{x_-, [x_+]; J, H\}$ is independent of $k, p$.

The linearization of $\tilde{F}$ at $\tilde{u}$ is

$$\tilde{D}_u := d\tilde{F}_u : (\xi, \phi, \psi) \mapsto \left( \nabla_{A,s}\xi + (\nabla_\xi J_t)(\partial_t u + X_\psi - Y_{H_t}) + J_t(\nabla_{A,t}\xi - \nabla_\xi Y_{H_t}) + X_\phi + JX_\psi \right) \left( \begin{array}{c} \partial_s \psi + [\Phi, \psi] - \partial_t \phi - [\Psi, \phi] + d\mu(\xi) \\
\partial_t \end{array} \right)$$  \hspace{1cm} (4.13)

Hence the linearization of $F$, under the isomorphism (4.6), is the restriction of $\tilde{D}_u$ to $(\ker d^*_u)^\perp$.

We define the augmented linearized operator

$$D_u := \tilde{D}_u \oplus d^*_u : T_u \tilde{B}^{k,p}(x_-, x_+) \to \tilde{E}_u^{k-1,p} \oplus W^{k-1,p}(\Theta, g).$$  \hspace{1cm} (4.14)

It is a standard result that the Fredholm property of $dF_u$ is equivalent to that of $D_u$ for any representative $\tilde{u}$. Hence in the remaining of the section we will study the Fredholm property of the augmented operator.

### 4.2. Asymptotic behavior of $D_u$

Up to an $L_0G$-action we can choose representatives $x_\pm = (x_\pm, f_\pm[w_\pm])$ such that $f_\pm$ are constants $\theta_\pm \in g$. Take any $\tilde{u} = (u, \Phi, \Psi) \in \tilde{B}^{k,p}(x_-, x_+)$. 

For $\xi := (\xi, \psi, \phi) \in T_u \tilde{B}^{k,p}(x_-, x_+)$, define $\tilde{J}(\xi, \psi, \phi) = (J\xi, -\phi, \psi)$. Here $\psi$ is the variation of $\Psi$ and $\phi$ is the variation of $\Phi$. Then

$$D_u \left( \begin{array}{c} \xi \\
\psi \\
\phi \end{array} \right) = \nabla_{A,s} \left( \begin{array}{c} \xi \\
\psi \\
\phi \end{array} \right) + \tilde{J} \left( \begin{array}{c} \nabla_{A,t}\xi - \nabla_\xi Y_{H,t} \\
\nabla_{A,t}\psi \\
\nabla_{A,t}\phi \end{array} \right) + \left( \begin{array}{ccc} 0 & JL_u & L_u \\
d\mu & 0 & 0 \\
L^*_u & 0 & 0 \end{array} \right) \left( \begin{array}{c} \xi \\
\psi \\
\phi \end{array} \right) + q(s,t) \left( \begin{array}{c} \xi \\
\psi \\
\phi \end{array} \right)$$  \hspace{1cm} (4.15)
Here \( q(s, t) \) is a linear operator such that \( \lim_{s \to \pm \infty} q(s, t) = 0 \); and

\[
R_{\tilde{x}}(t) := \lim_{s \to \pm \infty} R(s, t) = \begin{pmatrix}
J_t \nabla (X_{\theta -} - Y_{H_t}) & J_t L_u & L_u \\
\frac{d\mu}{dt} & 0 & -\text{ad}\theta_+
\end{pmatrix}
\]  \hspace{1cm} (4.16)

Here \( L_u : g \to u^*TM \) is the infinitesimal action along the image of \( u \) and \( L_u^* \) is its dual.

The following result is implied by Hypothesis 2.3.

**Proposition 4.1.** \( \tilde{x} := (x, \theta) \in \text{Zero} \tilde{B}_H \), then for all \( t \)-dependent, \( G \)-invariant, \( \omega \)-compatible almost complex structure \( J_t \), the self-adjoint operator

\[
L^2 \left( S^1, x^*TM \oplus g \oplus g \right) \to L^2 \left( S^1, x^*TM \oplus g \oplus g \right)
\]

\[
\begin{pmatrix}
\xi \\
\psi \\
\phi
\end{pmatrix}
\]

\[
\mapsto
\begin{pmatrix}
\tilde{J} \begin{pmatrix}
\nabla_t X_{\theta} \\
\partial_t
\end{pmatrix} + R_{\tilde{x}}(t)
\end{pmatrix}
\begin{pmatrix}
\xi \\
\psi \\
\phi
\end{pmatrix}
\]  \hspace{1cm} (4.17)

has zero kernel.

**Proof.** Suppose \( (\xi, \psi, \phi)^T \) is in the kernel, which means

\[
J_t \nabla_\xi \xi + J_t \nabla_\xi (X_{\theta} - Y_{H_t}) + X_\phi + J_t X_\psi = 0
\]  \hspace{1cm} (4.18)

\[
-\frac{d\phi}{dt} + d\mu(\xi) - [\theta, \phi] = 0
\]  \hspace{1cm} (4.19)

\[
\frac{d\psi}{dt} + d\mu(J_t \xi) + [\theta, \psi] = 0
\]  \hspace{1cm} (4.20)

Apply \( d\mu \circ J_t \) to (4.18), we get

\[
d\mu(J_t X_\phi) = d\mu(\nabla_\xi \xi + \nabla_\xi (X_{\theta} - Y_{H_t})).
\]

Hence for any \( \eta \in g \),

\[
\frac{d}{dt} \left( d\mu(\xi), \eta \right)_\theta = \frac{d}{dt} \omega(X_\eta, \xi)
\]

\[
=\omega([Y_{H_t} - X_{\theta}, X_{\eta}], \xi) + \omega(X_\eta, \nabla_\xi \xi - \nabla_\xi (Y_{H_t} - X_{\theta}))
\]

\[
=\omega(X_{[\theta, \eta]}, \xi) + \left( d\mu(J X_\phi), \eta \right)_\theta
\]

\[
=\left( d\mu(\xi), [\theta, \eta] \right)_g + \left( d\mu(J X_\phi), \eta \right)_g
\]

\[
=\left( d\mu(J X_\phi) - [\theta, d\mu(\xi)], \eta \right)_g.
\]

Therefore,

\[
\frac{d}{dt} d\mu(\xi) = d\mu(J X_\phi) - [\theta, d\mu(\xi)].
\]  \hspace{1cm} (4.22)

Then by (4.19),

\[
d\mu(J_t X_\phi)
\]

\[
=\frac{d}{dt} d\mu(\xi) + [\theta, d\mu(\xi)]
\]

\[
=\frac{d}{dt} \left( \frac{d\phi}{dt} + [\theta, \phi] \right) + \left[ \theta, \frac{d\phi}{dt} + [\theta, \phi] \right]
\]

\[
=\phi'' + 2 [\theta, \phi'] + [\theta, [\theta, \phi]].
\]  \hspace{1cm} (4.23)
Suppose $\|\phi\|$ takes its maximum at $t = t_0 \in S^1$. Then for $t \in (t_0 - \epsilon, t_0 + \epsilon)$, define $\tilde{\phi}(t) = \text{Ad}_{e^{(t-t_0)\phi}}(\phi(t))$. Then the right hand side of (4.23) is equal to $\text{Ad}_{e^{(t-t_0)\phi}}(\tilde{\phi}'')$. Hence at $t = t_0$,

$$0 \geq \frac{1}{2} \frac{d^2}{dt^2} \|\phi\|^2 = \frac{1}{2} \frac{d^2}{dt^2} \|\tilde{\phi}\|^2 = \langle \tilde{\phi}'', \tilde{\phi} \rangle + \|\tilde{\phi}'\|^2 \tag{4.24}$$

Hence $X_{\phi} \equiv 0$, which implies $\phi \equiv 0$ and by (4.19), $\xi$ is tangent to $\mu^{-1}(0)$.

Now $x^*T\mu^{-1}(0) = E_t \oplus L_x\mathfrak{g}$, where $E_t = (L_x\mathfrak{g})^\perp \cap x^*T\mu^{-1}(0)$ and the orthogonal complement is taken with respect to the Riemannian metric $g = \omega(\cdot, J_t \cdot)$. Then with respect to this ($G$-invariant) decomposition, write

$$\xi = \xi^\perp(t) + X_{\eta(t)}.$$

Then take the $E_t$-component of (4.18), and use the nondegeneracy assumption on the induced Hamiltonian $\tilde{H}_t$ on the symplectic quotient, we see that $\xi^\perp \equiv 0$. The only equations left is

$$\begin{cases}
\nabla_t X_{\eta(t)} + \nabla_{X_{\eta(t)}}(X_{\phi} - Y_{H_t}) + X_{\psi} &= 0, \\
\psi' + [\theta, \psi] + d\mu(JX_{\eta}) &= 0.
\end{cases}$$

The first equation is equivalent to

$$\eta'' + [\theta, [\theta, \eta(t)] + \psi = 0.$$

Hence

$$\eta''(t) + 2 [\theta, \eta'(t)] + [\theta, [\theta, \eta]] = d\mu(JX_{\eta}).$$

This can be treated similarly as (4.23), using maximum principle, which shows that $\eta \equiv 0$ and hence $\psi \equiv 0$. \qed

**Corollary 4.2.** Under Hypothesis 2.3, for any $\tilde{x}_\pm \in \text{Zero} \tilde{B}_H$ and any $\tilde{u} \in \tilde{B}^{k,p}(\tilde{x}_-, \tilde{x}_+)$, the augmented linearized operator $\mathcal{D}_{\tilde{u}}$ is Fredholm for any $k \geq 1$, $p \geq 2$.

**Corollary 4.3.** There exists $\delta = \delta(\tilde{x}_\pm) > 0$ such that, for any $\tilde{u} = (u, \Phi, \Psi) \in \tilde{M}(\tilde{x}_-, \tilde{x}_+; J, H)$ and any $\tilde{\xi} \in \ker \mathcal{D}_{\tilde{u}}$, there exists $c > 0$ such that

$$\tilde{\xi}(s, t) \leq ce^{-\delta|s|}. \tag{4.25}$$

In particular, if $(u, 0, \Psi) \in \tilde{M}^{k,p}_{\tilde{u}}$, then $|\partial_s u|$ and $|\partial_s \Psi|$ decay exponentially.

**Proof.** The first part is standard, see for example [32, Lemma 2.11]. For a solution $\tilde{u} = (u, 0, \Psi)$ in temporal gauge, by the translation invariance of the equation (2.44), we see that $\tilde{\xi} = (\partial_s u, \beta_s = 0, \beta_t = \partial_t F) \in \ker \tilde{F}_\tilde{u}$. Moreover,

$$-d_0^*\tilde{\xi} = -d_0^*(\partial_s u, 0, \partial_s \Psi) = \partial_t \partial_s u + L_u^*(\partial_s u) + [\Psi, \partial_s \Psi]$$

$$= -\partial_t (\mu(u)) + d\mu(J_t \partial_s u) + [\Psi, \partial_s \Psi]$$

$$= -d\mu(\partial_t u) + d\mu(J_t \partial_s u) + [\Psi, \partial_s \Psi]$$

$$= d\mu(X_{\Phi} - Y_{H_t}) - [\Psi, \mu] = 0. \tag{4.26}$$

This implies that $\tilde{\xi} \in \ker \mathcal{D}_{\tilde{u}}$. Choose a smooth gauge transformation $g : \Theta \to G$ such that $g^*\tilde{u}$ satisfies the asymptotic condition of Proposition 3.7. Then $g^*\tilde{\xi} \in \ker \mathcal{D}_{g^*\tilde{u}}$, which decays exponentially. So does $\tilde{\xi}$. \qed
4.3. **The Conley-Zehnder indices.** In this subsection we define a grading on the set $\text{Crit} A_H$, which is analogous to the Conley-Zehnder index in usual Hamiltonian Floer theory, and we will call it the same name.

For the induced Hamiltonian system on the symplectic quotient $\tilde{M}$, we have the usual Conley-Zehnder index

$$\text{CZ} : \text{Crit} A_H \to \mathbb{Z}.$$  

We prove the following theorem

**Theorem 4.4.** There exists a function

$$\text{CZ} : \text{Crit} A_H \to \mathbb{Z}$$  

satisfying the following properties

1. For the embedding $\iota : \text{Crit} A_H \to \text{Crit} A_H$, we have
   $$\text{CZ} \circ \iota = \text{CZ};$$  

2. For any $B \in \Gamma$ and $[x] \in \text{Crit} A_H$ we have
   $$\text{CZ} (B \# [x]) = \text{CZ} ([x]) - 2c^G_1 (B).$$  

3. For $[x, y] = [x, f, [w]] \in \text{Crit} A_H$ and $\tilde{u} \in B_{k,p} ([y], [x])$ with $\tilde{u} \# [w] = [w]$, we have
   $$\text{ind} (dF_{\tilde{u}}) = \text{CZ} ([y]) - \text{CZ} ([x]).$$

We first review the notion of Conley-Zehnder index in Hamiltonian Floer homology. Let $A : [0, 1] \to \text{Sp}(2n)$ be a continuous path of symplectic matrices such that

$$A(0) = I_{2n}, \ \det (A(1) - I_{2n}) \neq 0.$$  

We can associate an integer $\text{CZ}(A)$ to $A$, called the Conley-Zehnder index. We list some properties of the Conley-Zehnder index below which will be used here (see for example [31]).

1. For any path $B : [0, 1] \to \text{Sp}(2n)$, we have $\text{CZ}(BAB^{-1}) = \text{CZ}(A)$;
2. $\text{CZ}$ is homotopy invariant;
3. If for $t > 0$, $A(t)$ has no eigenvalue on the unit circle, then $\text{CZ}(A) = 0$;
4. If $A_i : [0, 1] \to \text{Sp}(2n_i)$ for $n = 1, 2$, then $\text{CZ}(A_1 \oplus A_2) = \text{CZ}(A_1) + \text{CZ}(A_2)$;
5. If $\Phi : [0, 1] \to \text{Sp}(2n)$ is a loop with $\Phi(0) = \Phi(1) = 1d$, then
   $$\text{CZ}(\Phi A) = \text{CZ}(A) + 2\mu_M(\Phi)$$

where $\mu_M(\Phi)$ is the Maslov index of the loop $\Phi$.

With this algebraic notion, in the usual Hamiltonian Floer theory one can define the Conley-Zehnder indices for nondegenerate Hamiltonian periodic orbits. In our case, the induced Hamiltonian $H_t : \tilde{M} \to \mathbb{R}$ has the usual Conley-Zehnder index

$$\overline{\text{CZ}} : \text{Crit} A_H \to \mathbb{Z}.$$  

Then, for each $[x, f, [w]] \in \text{Crit} A_H$, the homotopy class of extensions $[w]$ induces a homotopy class of trivializations of $x^*TM$ over $S^1$. With respect to this class of trivialization, the operator (4.17) is equivalent to an operator $J_0 \partial_t + A(t)$, which defines a symplectic path. We define the
Conley-Zehnder index of $\mathfrak{r}$ to be the Conley-Zehnder index of this symplectic path. By the second and fifth axioms listed above, this index induces a well-defined function

$$\text{CZ} : \text{Crit} A_H \to \mathbb{Z}$$

which satisfies (2) and (3) of Theorem 4.4.

Now we prove (1). For any contractible periodic orbits $\mathcal{P} : S^1 \to \overline{M}$ of $Y_{\mathcal{P}}$, and any extension $\mathfrak{w} : \mathbb{D} \to \overline{M}$ of $\mathcal{P}$, we can lift the pair $(\mathcal{P}, \mathfrak{w})$ to a tuple $\mathfrak{r} = (x, f, [w]) \in \text{Crit} \tilde{A}_H$.

**Proposition 4.5.** If $\iota : \text{Crit} A_{\mathcal{P}} \to \text{Crit} A_H$ is the inclusion we described in Proposition 2.8, then

$$\text{CZ} \circ \iota = \text{CZ}.$$  

**Proof.** Since the Conley-Zehnder index is homotopy invariant, and the space of $G$-invariant $\omega$-compatible almost complex structures is connected, we will compute the Conley-Zehnder index using a special type of almost complex structures, and modify the Hamiltonian $H$.

Starting with any almost complex structure $J$ on $\overline{M}$ and a $G$-connection on $\mu^{-1}(0) \to \overline{M}$, $J$ lifts to the horizontal distribution defined by the connection. On the other hand, the biinvariant metric on $\mathfrak{g}$ gives an identification $\mathfrak{g} \simeq \mathfrak{g}^*$. We denote by $\eta^* \in \mathfrak{g}^*$ the metric dual of $\eta \in \mathfrak{g}$. Recall that we have a symplectomorphism

$$\mu^{-1}(\mathfrak{g}^*_\epsilon) \simeq \mu^{-1}(0) \times \mathfrak{g}^*_\epsilon.$$ 

For $\eta \in \mathfrak{g}$, we define $J X \eta = \eta^* \in \mathfrak{g}^*$, as a vector field on $\mu^{-1}(0) \times \mathfrak{g}^*_\epsilon$. Then this gives a $G$-invariant almost complex structure on $TM|_{\mu^{-1}(0)}$, compatible with $\omega$. Then we pullback $J$ by the projection $\mu^{-1}(0) \times \mathfrak{g}^*_\epsilon \to \mu^{-1}(0)$ and denote the pullback by $J$.

We also modify $H_t$ by requiring that $H_t(x, \eta) = H_t(x)$ for $(x, \eta) \in \mu^{-1}(0) \times \mathfrak{g}^*_\epsilon$. Then the modified $H_t$ can be continuously deformed to the original one, and it doesn’t change $\overline{H}$ hence doesn’t change $\text{Crit} A_{\mathcal{P}}$. Moreover, it is easy to check that for the modified pair $(J, H)$,

$$(L_{Y_{H_t}} J) X \eta = [L_{Y_{H_t}}, J X \eta] = 0.$$ 

Now for any $(\mathcal{P}, \mathfrak{w}) \in \text{Crit} A_{\mathcal{P}}$, we lift it to $(x, f, [w]) \in \text{Crit} \tilde{A}_H$ with $w : \mathbb{D} \to \mu^{-1}(0)$ and $f$ being a constant $\theta \in \mathfrak{g}$. Then any symplectic trivialization of $\tilde{x}^* TM \to S^1$ induces a symplectic trivialization

$$\phi : x^* TM \simeq S^1 \times \mathbb{R}^{2n-2k} \oplus (\mathfrak{g} \oplus \mathfrak{g})$$

such that $\phi(X_{\eta}, J X \zeta) = (0, \eta, \zeta)$. Then we see, with respect to $\phi$, the operator (4.17) restricted to $\mathfrak{g}^t$ is

$$\begin{pmatrix} \eta_1 \\ \psi \\ \eta_2 \\ \phi \end{pmatrix} \mapsto \tilde{J} \frac{d}{dt} \begin{pmatrix} \eta_1 \\ \psi \\ \eta_2 \\ \phi \end{pmatrix} + \begin{pmatrix} \phi - [\theta, \eta_2] \\ \eta_2 - [\theta, \phi] \\ \psi + [\theta, \eta_1] \\ \eta_1 + [\theta, \psi] \end{pmatrix} = : \begin{pmatrix} \tilde{J} \frac{d}{dt} + S \end{pmatrix} \begin{pmatrix} \eta_1 \\ \psi \\ \eta_2 \\ \phi \end{pmatrix}. $$

Here we used the property (4.38) and

$$\tilde{J} := \begin{bmatrix} 0 & -\text{Id}_{\mathfrak{g} \oplus \mathfrak{g}} \\ \text{Id}_{\mathfrak{g} \oplus \mathfrak{g}} & 0 \end{bmatrix}, \ S = \begin{bmatrix} 0 & \text{Id}_{\mathfrak{g} \oplus \mathfrak{g}} - \text{ad}_\theta \\ \text{Id}_{\mathfrak{g} \oplus \mathfrak{g}} + \text{ad}_\theta & 0 \end{bmatrix}.$$
Moreover, the operator (4.17) respect the decomposition in (4.39). Hence by the fourth axiom of Conley-Zehnder indices we listed above, we have

\[ \text{CZ} (x, \theta, [w]) = \text{CZ}(\overline{x}, \overline{w}) + \text{CZ} \left( e^{\overline{JS}t} \right). \]  

(4.42)

As we have shown in the proof of Proposition 4.1 that for any \( \theta \) the operator (4.17) is an isomorphism, we can deform \( \theta \) to zero and compute instead \( \text{CZ}(\overline{e^{JS}t}) \) for \( \overline{S}_0 = \begin{bmatrix} 0 & \text{Id}_{\mathbb{R}g} \\ \text{Id}_{\mathbb{R}g} & 0 \end{bmatrix} \) (4.43), thanks to the homotopy invariance property. Then we see that \( \overline{e^{JS}t} = \begin{bmatrix} e^{-t} & 0 \\ 0 & e^t \end{bmatrix} \) (4.44) which has no eigenvalue on the unit circle for \( t > 0 \). Therefore by the third axiom, \( \text{CZ}(\overline{e^{JS}t}) = 0. \)

\[ \square \]

5. COMPACTNESS OF THE MODULI SPACE

For a general Hamiltonian \( G \)-manifold, the failure of compactness of the moduli space of connecting orbits comes from two phenomenon. The first is the breaking of connecting orbits, which is essentially the same thing happened in finite dimensional Morse-Smale-Witten theory. The second is the blow-up of the energy density, which results in sphere bubbling. Since here we have assumed that there exists no nontrivial holomorphic sphere in \( M \), so we only have to consider the breakings.

5.1. MODULI SPACE OF STABLE CONNECTING ORBITS AND ITS TOPOLOGY. Let’s fix a pair \( [x_-] \in \text{Crit}A_H \). Denote by \( \widehat{\mathcal{M}}([x_-], [x_+]) := \mathcal{M}([x_-], [x_+]; J, H) = \mathcal{M}([x_-], [x_+]; J, H) / \mathbb{R} \) the quotient of the moduli space by the translation in the \( s \)-direction. We denote by \( \{ \overline{u} \} \) the \( \mathbb{R} \)-orbit in \( \widehat{\mathcal{M}}([x_-], [x_+]) \) of \( \overline{u} \in \mathcal{M}([x_-], [x_+]; J, H) \) and call it a trajectory from \( [x_-] \) to \( [x_+] \).

**Definition 5.1.** A broken trajectory from \( [x_-] \) to \( [x_+] \) is a collection

\[ \overline{u} := \left( \left\{ \overline{u}^{(\alpha)} \right\} \right)_{\alpha = 1, \ldots, m} := \left( \left\{ u^{(\alpha)}, \Phi^{(\alpha)}, \Psi^{(\alpha)} \right\} \right)_{\alpha = 1, \ldots, m} \]  

(5.1)

where for each \( \alpha \), \( \{ \overline{u}^{(\alpha)} \} \in \widehat{\mathcal{M}} ([x_{\alpha - 1}], [x_{\alpha}]) \) and \( E(\overline{u}^{(\alpha)}) \neq 0 \). Here \( \{ [x_{\alpha}] \}_{\alpha = 0, \ldots, m} \) is a sequence of critical points of \( A_H \) and

\[ [x_0] = [x_-], \quad [x_m] = [x_+]. \]

We regard the domain of \( \overline{u} \) as the disjoint union

\[ \bigcup_{\alpha = 1}^{m} \Theta \]

and let \( \Theta^{(\alpha)} \subset \bigcup_{\alpha = 1}^{m} \Theta \) the \( \alpha \)-th cylinder.

We denote by

\[ \widehat{\mathcal{M}}([x_-], [x_+]) \]  

(5.2)

the space of all broken trajectories from \( [x_-] \) to \( [x_+] \). Then naturally we have inclusion

\[ \widehat{\mathcal{M}} ([x_-], [x_+]) \rightarrow \overline{\mathcal{M}} ([x_-], [x_+]). \]  

(5.3)
Definition 5.2. We say that a sequence of trajectories \( \{ \tilde{u}_i \} = \{ u_i, \Phi_i, \Psi_i \} \in \tilde{\mathcal{M}}([x_-, [x_+]) \) from \([x_-] \) to \([x_+] \) converges to a broken trajectory

\[
\tilde{u} := \left\{ \tilde{u}^{(\alpha)} \right\}_{\alpha=1,...,m}
\]

if: for each \( i \), there exists sequences of numbers \( s_{i}^{(1)} < s_{i}^{(2)} < \cdots < s_{i}^{(m)} \) and gauge transformations \( g_i^{(\alpha)} \in \mathcal{G}_\Theta \) such that for each \( \alpha \)

\[
\left( g_i^{(\alpha)} \right)^* \left( s_{i}^{(\alpha)} \right)^* \left( u_i, \Phi_i, \Psi_i \right)
\]

converges to \( (u^{(\alpha)}, \Phi^{(\alpha)}, \Psi^{(\alpha)}) \) on any compact subset of \( \Theta \) and such that for any sequence of \((s_i, t_i)\) with

\[
\lim_{i \to \infty} |s_i - s_{i}^{(\alpha)}| = \infty, \forall \alpha
\]

we have

\[
\lim_{i \to \infty} e (\tilde{u}_i) (s_i, t_i) = 0.
\]

It is easy to see that this convergence is well-defined and independent of the choices of representatives of the trajectories. We can also extend this notion to sequences of broken connecting orbits. We omit that for simplicity.

The main theorem of this section is

**Theorem 5.3 (Compactness of the moduli space of stable connecting orbits).** The space \( \tilde{\mathcal{M}}([x_-, [x_+]) \) is a compact Hausdorff space with respect to the topology defined in Definition 5.2.

Indeed the proof is routine and it has been carried out in many literature for general symplectic vortex equations, for example \([25], [2], [26], [37]\). Since bubbling is ruled out, the proof is almost the same as that for finite dimensional Morse theory, while the gauge symmetry is the only additional ingredient.

5.2. Local compactness with uniform bounded energy density. For any compact subset \( K \subset \Theta \), consider a sequence of solutions \( \tilde{u}_i := (u_i, \Phi_i, \Psi_i) \) such that the image of \( u_i \) is contained in the compact subset \( K_H \subset M \) and such that

\[
\limsup_{i \to \infty} E(\tilde{u}_i) < \infty.
\]

We have

**Proposition 5.4.** There exists a subsequence (still indexed by \( i \)), a sequence of smooth gauge transformation \( g_i : K \to G \) and a solution \( \tilde{u}_\infty = (u_\infty, \Phi_\infty, \Psi_\infty) : K \to M \times g \times g \) to (2.44) on \( K \), such that the sequence \( g_i^* \tilde{u}_i \) converges to \( \tilde{u}_\infty \) uniformly with all derivatives on \( K \).

**Proof.** By the fact that \( u_i \) is contained in the compact subset \( K_H \), and the assumption that there exists no nontrivial holomorphic spheres in \( M \), we have

\[
\sup_{z \in K_i} e_{\gamma_i}(z) < \infty.
\]

Then this proposition can be proved in the standard way, as did in [2] or [26], using Uhlenbeck’s compactness theorem.

\( \square \)
5.3. **Energy quantization.** To prove the compactness of the moduli space, we need the following energy quantization property.

**Proposition 5.5.** There exists $\epsilon_0 := \epsilon_0(J, H) > 0$, such that for any connecting orbit $\tilde{u} \in \tilde{M}_G$, we have $E(\tilde{u}) \geq \epsilon_0$.

*Proof.* Suppose it is not true. Then there exists a sequence of connecting orbits, represented by solutions in temporal gauge $\tilde{v}_i := (v_i, 0, \Psi_i) \in \tilde{M}_B$, such that

$$E(\tilde{v}_i) > 0, \quad \lim_{i \to \infty} E(\tilde{v}_i) = 0. \quad (5.6)$$

We first know that there is a compact subset $K_H \subset M$ such that for every $i$, the image $v_i(\Theta)$ is contained in $K_H$. Then we must have

$$\limsup_{i \to \infty} \sup_{\Theta} (|\partial_s v_i| + |\mu(v_i)|) = 0. \quad (5.7)$$

Indeed, if the equality doesn’t hold, then we can find a subsequence which either bubbles off a nonconstant holomorphic sphere at some point $z \in \Theta$ (if the above limit is $\infty$), or (after a sequence of proper translation in $s$-direction) converges to a solution (with positive energy) on compact subsets (if the above limit is positive and finite). Either case contradicts the assumption. Therefore we conclude that for any $\epsilon > 0$, the image of $v_i$ lies in $U_\epsilon := \mu^{-1}(g_\epsilon^*)$ for $i$ sufficiently large.

Recall that we have projections $\pi_\mu : U_\epsilon \to \mu^{-1}(0)$ and $\pi_\mu : U_\epsilon \to \overline{M}$. Then for all large $i$ and any $s$, $\pi_\mu(v_i(s, \cdot))$ is $C^0$-close to a periodic orbit of $H$ in $\overline{M}$. Since those orbits are discrete (in $C^0$-topology, for example), we may fix one such orbit $\gamma \in \text{Zero} \overline{\mathcal{B}_H}$ and choose a subsequence (still indexed by $i$) such that

$$\limsup_{i \to \infty} \sup_{(s,t) \in \Theta} d(\pi_\mu(v_i(s,t)), \gamma(t)) = 0. \quad (5.8)$$

Then, use a fixed Riemannian metric on $\overline{M}$ with its exponential map $\exp$, we can write

$$\pi_\mu(v_i(s,t)) = \exp_{\gamma(t)} \xi_i(s,t)$$

where $\xi_i \in \Gamma(S^1, \gamma^*T\overline{M})$. Let $B_\epsilon(\gamma^*T\overline{M})$ be the $\epsilon$-disk of $\gamma^*T\overline{M}$. Then $\exp_{\gamma}$ pulls back $\mu^{-1}(0) \to \overline{M}$ to a $G$-bundle $Q \to B_\epsilon(\gamma^*T\overline{M})$, together with a bundle map $\tilde{\gamma} : Q \to \mu^{-1}(0)$. We can trivialize $Q$ by some

$$\phi : Q \to G \times B_\epsilon(\gamma^*T\overline{M}).$$

Now we take a lift $\tilde{x} := (x, f) \in \text{Zero} \overline{\mathcal{B}_H}$ of $\gamma$. Then we can write

$$\phi \left( \tilde{\gamma}^{-1}(x(t)) \right) = (g_0(t), \gamma(t)). \quad (5.9)$$

On the other hand, we write

$$\phi \left( \tilde{\gamma}^{-1} \pi_\mu(v_i(s,t)) \right) = (g_i(s,t), \xi_i(s,t)). \quad (5.10)$$

Take the gauge transformation $\tilde{g}_i(s,t) = g_i(s,t)g_0(t)^{-1}$. Then write

$$\tilde{u}'_i := (v'_i, \Phi'_i, \Psi'_i) := \tilde{g}_i \tilde{u}_i. \quad (5.11)$$
Then by the exponential convergence of $v_i$ as $s \to \pm \infty$, we see that $\partial_s g_i(s, t)$ decays exponentially and hence $\Phi_i'$ converges to zero as $s \to \pm \infty$. On the other hand, we see that $\phi(\tilde{\gamma}^{-1} \pi_\mu(v_i'(s, t))) = (g_0(t), \tilde{\xi}_i(s, t))$. Therefore

$$\tilde{v}_i' \in \tilde{M}(\tilde{x}, \tilde{x}).$$

(5.13)

But it is also easy to see that the homotopy class of $\tilde{v}_i'$ is trivial. Because the energy of connecting orbits only depends on its homotopy class, we see that the energy of $\tilde{v}_i'$, and hence the energy of $\tilde{v}_i$, is actually equal to zero, which contradicts with the hypothesis. \qed

5.4. **Proof of the compactness theorem.** It suffices to prove, without essential loss of generality, that for any sequence $[\tilde{u}_i] \in \mathcal{M}([x_-], [x_+]; J, H)$ represented by unbroken connecting orbits $(u_i, \Phi_i, \Psi_i) \in \tilde{M}(x_-, x_+)$, there exists a convergent subsequence. By the assumption that there exists no nontrivial holomorphic sphere in $M$, we have

$$\sup_i \left| \partial_s u_i + X_{\Phi_i}(u_i) \right| < +\infty.$$

(5.14)

Then the limit (broken) connecting orbits can be constructed by induction and the energy quantization property (Proposition 5.5) guarantees that the induction stops at finite time. The details are standard and left to the reader.

6. **Floer homology**

In this section we use the moduli spaces $\mathcal{M}([x_-], [x_+]; J, H)$ to define the vortex Floer homology group $VHF_*(M, \mu; H)$. We also discuss further works and related problems in the last three subsections.

By Corollary A.12, we can choose a generic $S^1$-family of “admissible” almost complex structures $J \in \tilde{J}^{reg}_H$ which is regular with respect to $H$. Such an object is a smooth $t$-dependent family of almost complex structures $J_t$, such that for each $t$, $J_t$ is $G$-invariant, $\omega$-compatible, and outside a neighborhood $U$ of $\mu^{-1}(0)$, $J_t \equiv \tilde{J}$; inside $U$, $J_t$ preserves a distribution $g_C$. Being regular implies that for all pairs $[x_\pm] \in \text{Crit}A_H$, the moduli space $\mathcal{M}([x_-], [x_+]; J, H)$ is a smooth manifold with

$$\dim \mathcal{M}([x_-], [x_+]; J, H) = CZ([x_-]) - CZ([x_+]).$$

(6.1)

Moreover, there is free $\mathbb{R}$-action on $\mathcal{M}([x_-], [x_+]; J, H)$ by time translation, whose orbit space is denoted by $\hat{M}([x_-], [x_+]; J, H)$. Combining the compactness theorem, we have

**Proposition 6.1.** If $J \in \tilde{J}^{reg}_H$, then

$$CZ([x_-]) - CZ([x_+]) \leq 0 \implies \mathcal{M}([x_-], [x_+]; J, H) = \emptyset;$$

(6.2)

$$CZ([x_-]) - CZ([x_+]) = 1 \implies \#\hat{M}([x_-], [x_+]; J, H) < \infty.$$
6.1. The gluing map and coherent orientation. The boundary operator of the Floer chain complex is defined by the (signed) counting of $\hat{M}([x_-,x_+];J,H)$. If we want to define the Floer homology over $\mathbb{Z}_2$, then we don’t need to orient the moduli space; otherwise, the orientation of $\hat{M}([x_-,x_+];J,H)$ can be treated in the same way as the usual Hamiltonian Floer theory, since the augmented linearized operator $D_{\mathcal{U}}$ (whose determinant is canonically isomorphic to the determinant of the actual linearization $dF_{[\mathcal{U}]}$), is of the same type of Fredholm operators considered in the abstract setting of [10]. We first give the gluing construction and then discuss the coherent orientations of the moduli spaces.

In this subsection we construct the gluing map for broken trajectories with only one breaking. The general case is similar. This construction is, in principle, the same as the standard construction in various types of Morse-Floer theory (see [32] [10]), with a gauge-theoretic flavor. The gauge symmetry makes the construction a bit more complex, since we always glue representatives, and we want to show that the gluing map is independent of the choice of the representatives.

In this subsection, we fix the choice of the admissible family $J \in \tilde{\mathcal{J}}_H^{reg}$ and omit the dependence of the moduli spaces on $J$ and $H$.

For any pair $x_+ \in \text{Crit}_A H$, we say that a solution $\tilde{u} \in \hat{M} ([x_-,x_+])$ is in $r$-temporal gauge, if its restrictions to $[r, +\infty) \times S^1$ and $(-\infty, r) \times S^1$ are in temporal gauge, for some $r > 0$. Now we fix a number $r = r_0$ and only consider solutions in $r_0$-temporal gauge.

Now we take three elements $\mathcal{R}, \eta, \zeta \in \text{Crit}_H$ with
\[
\text{CZ}(\mathcal{R}) - 1 = \text{CZ}(\eta) = \text{CZ}(\zeta) + 1. \quad (6.4)
\]
Assume $\eta = (y, \eta) : S^1 \to M \times g$. We would like to construct, for a large $R_0 > 0$, the gluing map
\[
\text{glue} : \hat{M} ([\mathcal{R}, \eta]) \times (R_0, +\infty) \times \hat{M} ([\eta, \zeta]) \to \hat{M} ([\mathcal{R}, \zeta]). \quad (6.5)
\]
Now consider two trajectories $[\tilde{u}_\pm] = [u_\pm, \Phi_\pm, \Psi_\pm], [\tilde{u}_-] \in \mathcal{M} ([\mathcal{R}, \eta]), [\tilde{u}_+] \in \mathcal{M} ([\eta, \zeta])$ with their representatives both in $r_0$-temporal gauge and $\tilde{u}_\pm$ are asymptotic to $\eta$ as $s \to +\infty$. Then there exists $R_1 > 0$ such that
\[
\pm s \geq R_1 \implies u_\pm(s, t) = \exp_{y(t)} \xi_\mp(s, t). \quad (6.6)
\]
Here $\Theta_{R_1}^+ = [R_1, +\infty) \times S^1$ and $\Theta_{R_1}^- = (-\infty, -R_1] \times S^1$ and $\xi_\pm \in W^{k,p} \left(\Theta_{R_1}^\pm, g^*TM\right)$.

Next, we take a cut-off function $\rho$ such that $s \geq 1 \implies \rho(s) = 1$, $s \leq 0 \implies \rho(s) = 0$. For each $R >> r_0$, denote $\rho_R(s) = \rho(s - R)$. We construct the “connected sum”
\[
u_R(s, t) = \begin{cases} 
  u_-(s + R, t), & s \leq \frac{R}{2} - 1 \\
  \exp_{y(t)} \left(\rho_{R/2}(s)\xi_-(s + R, t) + \rho_{R/2}(s)\xi_+(s - R, t)\right), & s \in \left[-\frac{R}{2} - 1, \frac{R}{2} + 1\right] \\
  u_+(s - R, t), & s \geq \frac{R}{2} + 1 
\end{cases} \quad (6.7)
\]
\[
(\Phi_R, \Psi_R)(s, t) = \begin{cases} 
  (\Phi_-(s + R, t), \Psi_-(s + R, t)), & s \leq -\frac{R}{2} - 1 \\
  \left(0, \rho_{R/2}(s)\Psi_-(s + R, t) + \rho_{R/2}(s)\Psi_+(s - R, t)\right), & s \in \left[-\frac{R}{2} - 1, \frac{R}{2} + 1\right] \\
  (\Phi_+(s - R, t), \Psi_+(s - R, t)), & s \geq \frac{R}{2} + 1 
\end{cases} \quad (6.8)
\]
Now it is easy to see that, if we change the choice of representatives $\tilde{u}_\pm$ which are also in $r_0$-temporal gauge, the connected sum $\tilde{u}_R := (\nu_R, \Phi_R, \Psi_R)$ doesn’t change for $s \in \left[-\frac{R}{2} - 1, \frac{R}{2} + 1\right]$ and hence
we obtain a gauge equivalent connected sum. Moreover, if we change \( \eta \) by \( \eta' \) which represent the same \([\eta] \in \text{Crit}_\mathcal{A}_H\), then we can obtain
\[
\tilde{u}'_\omega \in \hat{\mathcal{M}}(\omega, \eta'), \quad \tilde{u}'_\omega \in \hat{\mathcal{M}}(\eta', \bar{\omega})
\]  
which are also in \( r_0 \)-temporal gauge, and we obtain a connected sum \( \tilde{u}'_R \) which is gauge equivalent to \( \tilde{u}_R \).

Now we consider the augmented linearized operator \( \mathcal{D}_R := \mathcal{D}_{\tilde{u}_R} \).

**Lemma 6.2.** There exists \( c > 0 \) and \( R_0 > 0 \) such that for every \( R \geq R_0 \) and \( \tilde{\eta} \in \tilde{\mathcal{C}}^{2,p} \oplus W^{2,p}(\Theta, g) \), we have
\[
\| \mathcal{D}_R \tilde{\eta} \|_{W^{1,p}} \leq c \| \mathcal{D}_R \mathcal{D}_R \tilde{\eta} \|_{L^p}.
\]  

**Proof.** Same as the proof of [32, Proposition 3.9] □

Hence we can construct a right inverse
\[
\Omega_R := \mathcal{D}_R^* (\mathcal{D}_R \mathcal{D}_R)^{-1} : \tilde{\mathcal{C}}^{2,p} \oplus L^p(\Theta, g) \to T_{\tilde{u}_R} \mathbb{B}^{1,p}
\]  
with
\[
\| \Omega_R \| \leq c.
\]  

Now we write \( \Omega_R := (\mathcal{Q}_R, \mathcal{A}_R) \) with \( \mathcal{Q}_R : \tilde{\mathcal{C}}^{2,p} \to T_{\tilde{u}_R} \mathbb{B}^{1,p}(\omega, \bar{\omega}) \). Then actually the image of \( \mathcal{Q}_R \) lies in the kernel of \( d_0^* \) and therefore \( \mathcal{Q}_R \) is a right inverse to \( d\tilde{F}_{\tilde{u}_R} |_{\text{ker} d^*_0} \). Because our construction is natural with respect to gauge transformations, we see that \( \mathcal{Q}_R \) induces an injection
\[
\overline{\mathcal{Q}}_R : \tilde{\mathcal{C}}^{0,p}_{[\tilde{u}_R]} \to T_{[\tilde{u}_R]} \mathbb{B}^{1,p}
\]  
which is a right inverse to the linearized operator \( d\tilde{F}_{[\tilde{u}_R]} \) and which is bounded by \( c \). By the implicit function theorem, we have

**Proposition 6.3.** There exists \( R_1 > 0 \), \( \delta_1 > 0 \) such that for each \( R \geq R_1 \), there exists a unique \( \tilde{\xi} \in \text{Im} \Omega_R = \text{ker} d^*_0 \subset T_{\tilde{u}_R} \mathbb{B}^{1,p} \), \( \| \tilde{\xi} \|_{W^{1,p}} \leq \delta_1 \) such that
\[
\tilde{F}(\exp_{\tilde{u}_R} \tilde{\xi}) = 0, \quad \| \tilde{\xi} \|_{W^{1,p}} \leq 2c \| \tilde{F}(\tilde{u}_R) \|_{L^p}.
\]  

Therefore, the gluing map can be defined as
\[
\text{glue}([\tilde{u}_-], R, [\tilde{u}_+]) = \left[ \exp_{\tilde{u}_R} \tilde{\xi} \right] \in \tilde{\mathcal{M}}([\omega], [\eta]; J, H).
\]  

On the other hand, it is easy to see that the augmented linearized operators \( \mathcal{D}_{\tilde{u}} \) for all connecting orbits \( \tilde{u} \) is of “class Σ” considered in [10]. Therefore, by the main theorem of [10], there exists a “coherent orientation” with respect to the gluing construction. Choosing such a coherent orientation, then to each zero-dimensional moduli space \( \tilde{\mathcal{M}}([\omega], [\eta]; J, H) \), we can associate the counting \( \chi_J([\omega], [\eta]) \in \mathbb{Z} \), where each trajectory \([\tilde{u}]\) contributes to 1 (resp. -1) if the orientation of \([\tilde{u}]\) coincides (resp. differ from) the “flow orientation” of the solution. Then we define
\[
\delta_J : \text{VCVF}_k(M, \mu; H; \Lambda_Z) \to \text{VCVF}_{k-1}(M, \mu; H; \Lambda_Z)
\]  
\[
[x] \quad \mapsto \sum_{[\eta] \in \text{Crit}_\mathcal{A}_H} \chi_J([\omega], [\eta]) \langle [\eta] \rangle
\]  

(6.16)
As in the usual Floer theory, we have

**Theorem 6.4.** For each choice of the coherent orientation on the moduli spaces $\mathcal{M}([\mathfrak{x}], [\mathfrak{y}]; J, H)$, the operator $\delta_J$ in (6.16) defines a morphism of $\Lambda_\mathbb{Z}$-modules satisfying $\delta_J \circ \delta_J = 0$.

This makes $(VCF_k(M, \mu; H; \Lambda_\mathbb{Z}), \delta_J)$ a chain complex of $\Lambda_\mathbb{Z}$-modules, to which will be generally referred as the vortex Floer chain complex. Therefore the vortex Floer homology group is defined as the $\Lambda_\mathbb{Z}$-module

$$VHF_k(M, \mu; J, H; \Lambda_\mathbb{Z}) := \ker(\delta_J : VCF_k(M, \mu; H; \Lambda_\mathbb{Z}) \to VCF_{k-1}(M, \mu; H; \Lambda_\mathbb{Z})) \quad (6.17)$$

### 6.2. The continuation map.

Now we prove that the vortex Floer homology group defined above is independent of the choice of admissible family of almost complex structures and the time-dependent Hamiltonians, and, if we use the moduli space of (1.8) instead of (1.7) to define the Floer homology, independent of the parameter $\lambda > 0$. So far the argument has been standard, by using the continuation principle.

Let $(J^\alpha, H^\alpha, \lambda^\alpha)$ and $(J^\beta, H^\beta, \lambda^\beta)$ be two triples where $\lambda^\alpha, \lambda^\beta > 0$, $H^\alpha, H^\beta$ are $G$-invariant Hamiltonians satisfy Hypothesis 2.1 and 2.3, and $J^\alpha \in \mathcal{J}_{H^\alpha, \lambda^\alpha}^\text{reg}$, $J^\beta \in \mathcal{J}_{H^\beta, \lambda^\beta}^\text{reg}$ (for notations refer to the appendix).

We choose a cut-off function $\rho : \mathbb{R} \to [0, 1]$ with $s \leq -1 \implies \rho(s) = 1$ and $s \geq 1 \implies \rho(s) = 0$. Then we define

$$H_{s,t} = \rho(s)H_t^\alpha + (1 - \rho(s))H_t^\beta, \quad \lambda_s = \rho(s)\lambda^\alpha + (1 - \rho(s))\lambda^\beta. \quad (6.18)$$

We denote this family of Hamiltonians by $\mathcal{H}$. Now, as in the appendix, we consider the space of families of admissible almost complex structures $\mathcal{J}(J^\alpha, J^\beta)$ consisting of smooth families of almost complex structures $\mathcal{J} = (J_{s,t})_{(s,t) \in \Theta}$, such that for each $t \geq 1$,

$$|e^{i|s|}J_{s,t} - J_t^\alpha|_{C^1(\Theta_+ \times M)} < \infty, \quad |e^{i|s|}J_{s,t} - J_t^\beta|_{C^1(\Theta_+ \times M)} < \infty. \quad (6.19)$$

This is a Fréchet manifold. For any $\mathcal{J} \in \mathcal{J}(J^\alpha, J^\beta)$, we consider the following equation on $\tilde{u} = (u, \Phi, \Psi)$

$$\begin{cases}
\frac{\partial u}{\partial s} + X_\Phi(u) + J_{s,t} \left( \frac{\partial u}{\partial t} + X_\Phi(u) - Y_{H_{s,t}}(u) \right) = 0; \\
\frac{\partial \Psi}{\partial s} - \frac{\partial \Phi}{\partial t} + [\Phi, \Psi] + \lambda_s^2 \mu(u) = 0.
\end{cases} \quad (6.20)$$

For the same reason as in Section 3, any finite energy solution whose image in $M$ has compact closure is gauge equivalent to a solution which is asymptotic to $\mathfrak{r}^\alpha \in \text{Crit} \mathcal{A}_{H^\alpha}$ (resp. $\mathfrak{r}^\beta \in \text{Crit} \mathcal{A}_{H^\beta}$) as $s \to -\infty$ (resp. $s \to +\infty$). Hence for any $[\mathfrak{r}^\alpha] \in \text{Crit} \mathcal{A}_{H^\alpha}$ and $[\mathfrak{r}^\beta] \in \text{Crit} \mathcal{A}_{H^\beta}$, we can consider the moduli space of solutions

$$\mathcal{N}([\mathfrak{r}^\alpha], [\mathfrak{r}^\beta]; \mathcal{J}, \mathcal{H}, \lambda_s). \quad (6.21)$$
One thing to check in defining the continuation map is the energy bound of solutions, which implies the compactness of the moduli space. We define the energy to be

\[ E(\tilde{u}) = E(u, \Phi, \Psi) \]

\[ = \frac{1}{2} \left( |\partial_s u + X_\Phi(u)|^2_{L^2} + |\partial_t u + X_\Psi(u) - Y_{H_{s,t}}(u)|^2_{L^2} + |\lambda_s^{-1}(\partial_s \Psi - \partial_s \Phi + [\Phi, \Psi])|^2_{L^2} + |\lambda_s \mu(u)|^2_{L^2} \right) \]

where, the last equality holds only for \( \tilde{u} \) a solution to (6.20). Then we have

**Proposition 6.5.** For any solution \( \tilde{u} \) to (6.20) whose energy is finite and whose image in \( M \) has compact closure, we have

\[ E(\tilde{u}) = A_{H^\alpha}([x^\alpha]) - A_{H^\beta}([x^\beta]) - \int \frac{\partial H_{s,t}}{\partial s}(u)dsdt. \]

**Proof.** We can transform the solution \( \tilde{u} \) into temporal gauge. Then the energy density is

\[ |\partial_s u|^2 + |\lambda_s \mu(u)|^2 = \omega(\partial_s u, \partial_t u + X_\Phi - Y_{H_{s,t}}(u)) - \mu(u) \cdot \frac{\partial \Psi}{\partial s} \]

\[ = \omega(\partial_s u, \partial_t u) - \frac{\partial}{\partial s}(\mu(u) \cdot \Psi) + \frac{\partial}{\partial s}(H_{s,t}(u)) - \frac{\partial H_{s,t}}{\partial s}(u). \]

Then integrating over \( \Theta \), we obtain (6.23). \( \square \)

**Theorem 6.6.** There exists a Baire subset \( \tilde{\mathcal{J}}^{\text{reg}}_{\mathcal{J}, \lambda_s}(J^\alpha, J^\beta) \subset \tilde{\mathcal{J}}(J^\alpha, J^\beta) \), such that for any \( J \in \tilde{\mathcal{J}}^{\text{reg}}_{\mathcal{J}, \lambda_s}(J^\alpha, J^\beta) \), the moduli space \( \mathcal{N}([x^\alpha], [x^\beta]; \mathcal{J}, \mathcal{H}, \lambda_s) \) is a smooth oriented manifold with

\[ \dim \mathcal{N}([x^\alpha], [x^\beta]; \mathcal{J}, \mathcal{H}, \lambda_s) = \text{CZ}([x^\alpha]) - \text{CZ}([x^\beta]). \]

So in particular, when \( \text{CZ}([x^\alpha]) = \text{CZ}([x^\beta]) \), \( N \) is of zero dimension. The algebraic count of \( N \) gives an integer \( \chi([x^\alpha], [x^\beta]) \). Then we define the continuation map

\[ \text{cont}^\alpha : VCF_*(M, \mu, J^\alpha, H^\alpha, \lambda_s; \Lambda_2) \to VCF_*(M, \mu, J^\beta, H^\beta, \lambda_s; \Lambda_2) \]

\[ [x^\alpha] \mapsto \sum_{[x^\beta] \in \text{Crit}_{\mathcal{A}_{\mu}^{H^\beta}}} \chi([x^\alpha], [x^\beta]) [x^\beta]. \]

Now we have the similar results as in ordinary Hamiltonian Floer theory.

**Theorem 6.7.** The map \( \text{cont}^\beta \) is a chain map. The induced map on the vortex Floer homology groups is independent of the choice of the homotopy \( \mathcal{H} \), the family \( \mathcal{J} \) of almost complex structures, the cut-off function \( \rho \). In particular, \( \text{cont}^\beta \) is a chain homotopy equivalence. If \( (J^\gamma, H^\gamma) \) is another admissible pair and \( \lambda^\gamma > 0 \), then in the level of homology

\[ \text{cont}^\gamma \circ \text{cont}^\alpha = \text{cont}^\gamma. \]

**Proof.** The proof is essentially based on the construction of various gluing maps and the compactness results about \( \mathcal{N}([x^\alpha], [x^\beta]; \mathcal{J}, \mathcal{H}, \lambda_s) \) when \( \text{CZ}([x^\alpha]) - \text{CZ}([x^\beta]) = 1 \). As in the gluing map constructed in proving the property \( \delta^\gamma = 0 \), we need to specify a gauge to construct the approximate solutions. We can still use solutions in \( r \)-temporal gauge, which is a notion independent of the equation. We omit the details. \( \square \)
6.3. **Computation and Morse homology.** In this subsection we discuss the computation of the vortex Floer homology group. Before we proceed let us recall how to show that the ordinary Hamiltonian Floer homology is isomorphic to the Morse homology.

On a compact symplectic manifold $M$ we take the Hamiltonian to be the $t$-independent function $\epsilon f$, where $\epsilon$ is small and $f$ is a Morse function on the manifold $M$. Then periodic orbits of $\epsilon f$ corresponds to critical points of $f$ (which are denoted by $z_1, \ldots, z_k$), and the Conley-Zehnder index and the Morse index are related by $\text{CZ}(z_i) = n - \text{Ind}(z_i)$ where $2n = \dim M$. Then the Floer chain complex is generated by $(z_i, w_i)$, where $w_i$ is a spherical class. Then we want to show that when

$$\text{CZ}(z_i, w_i) - \text{CZ}(z_j, w_j) = 0$$

(6.27)

by taking a generic $t$-independent almost complex structure $J$ on $M$, all Floer trajectories connecting $(z_i, w_i)$ and $(z_j, w_j)$ are $t$-independent, i.e., corresponds to Morse-Smale trajectories of $\epsilon f$ with respect to the metric $\omega(\cdot, J\cdot)$. Hence the boundary operators in the Floer chain complex and the Morse-Smale-Witten chain complex coincide. So that

$$HF_\ast(M, \epsilon f; \Lambda) \cong HM^{2n-\ast}(M, \epsilon f; \mathbb{Z}) \otimes \Lambda$$

(6.28)

The main difficulty to carry out the above argument is, when $J$ is $t$-independent, one cannot easily achieve the transversality of the moduli space of Floer trajectories. Here the cylinder $\Theta$ may have a finite cover over itself

$$\pi_k(s, t) = (ks, kt)$$

(6.29)

and there might exist Floer trajectories which are multiple covers of other trajectories (when $J$ is allowed to vary with $t$, such objects don’t exist generically). The multiple covers might have higher dimensional moduli than expected, which is similar to the problem caused by the negatively covered spheres in Gromov-Witten theory. Hence to overcome this difficulty, one has to either put topological restrictions (such as semi-positivity in [28], [19]), or to use virtual technique, to say that, though the negative multiple covers have higher dimensional moduli, they contributes to zero in defining the boundary operator (see [15], [23]).

Now back to the case of vortex Floer homology. We choose a Morse function $\overline{f} : \overline{M} \to \mathbb{R}$ so that the induced Hamiltonian $\overline{H}_t := \epsilon \overline{f}$ has its periodic orbits corresponding to the critical points of $f$. Then we lift $\overline{f}$ to a function $f : M \to \mathbb{R}$ and consider the vortex Floer homology defined for the Hamiltonian $\epsilon f$. To show that the resulting homology group is isomorphic to $H_\ast(\overline{M}; \mathbb{A}_Q)$, we have to show that for $\epsilon$ small enough, the counting of Floer trajectories corresponds to the counting of negative gradient flow trajectories of a Lagrange multiplier function associated to $\epsilon f$.

If we choose to put topological restrictions to avoid virtual technique, then one difficulty is the following. To achieve transversality, we assumed that $H_t$ vanishes for certain $t \in S^1$ in the appendix. This condition doesn’t hold for $\epsilon f$, which is independent of $t$. And this time the transversality much be achieved by only using a $t$-independent almost complex structure $J$. So virtual technique is probably unavoidable in this approach.
In ordinary Hamiltonian Floer homology, another way to prove the isomorphism is the so-called Pignikhin-Salamon-Schwarz (PSS) construction, introduced in [30]. It is to consider the moduli space of “spiked disks”, which is an object interpolating between Floer trajectories and Morse trajectories. The counting of spiked disks defines a pair of chain maps between the Floer chain complex and the Morse-Smale-Witten chain complex, and using various gluing/stretching constructions one can prove that the two chain maps are homotopy inverses to each other. In the second paper of this series, we will give a PSS type construction to prove the isomorphism between $VHF_*(M,\mu;\Lambda_\mathbb{Z})$ and the Morse homology of the symplectic quotient $\overline{M}$.

**Appendix A. Transversality by perturbing the almost complex structure**

In this appendix, we treat the transversality of our moduli space of connecting orbits. For general symplectic manifold, connecting orbits may develop sphere bubbles, while the expected dimension of the moduli space of such sphere bubbles may be even larger than the expected dimension of the moduli space of connecting orbits. In this case one must use the virtual technique to say something of the structure of the compactified moduli space of connecting orbits. The boundary operator is defined by the virtual count of the number of trajectories, therefore the Floer homology is only defined over $\mathbb{Q}$ in general. Instead, in this section we restrict to the case where the virtual technique is not necessary. We remark that this special case covers most interesting examples to which we will apply our results (for example, toric manifolds as symplectic quotient of vector spaces).

First we recall the important assumption on the Hamiltonian $H_t$.

**Hypothesis A.1.** There exists a nonempty open subset $I \subset S^1$ such that $H_t(x) = 0$ for $t \in I$.

Indeed, for any $G$-invariant Hamiltonian diffeomorphism of $M$, if it is given by the time-1 map of some Hamiltonian path, then we can reparametrize the Hamiltonian path to make it vanish for $t$ lying in a small interval, while the time-1 map of the reparametrized path is the original Hamiltonian diffeomorphism. Hence this hypothesis is not an essential restriction.

**A.1. Admissible family of almost complex structures.** We know that for any $\epsilon > 0$ small enough, there exists a symplectomorphism $U = U_\epsilon := \mu^{-1}(g_\epsilon^*) \simeq \mu^{-1}(0) \times g_\epsilon^*$. Hence we have a natural projection $\pi_{\mu} : U \to \overline{M}$. There is a natural foliation on $U$, whose leaves are $Gx \times g_\epsilon^*$ with $x \in \mu^{-1}(0)$, with dimension equal to $2\dim G$. The tangent planes of this foliation is a $G$-invariant distribution on $U$, denoted by $g_\mu^C$.

Recall that we are also given an almost complex structure $J$ in Hypothesis 2.4. Now we will perturb $J$ in a specific way. This approach was similar to that in Woodward’s erratum for [36], but here we don’t have a Lagrangian submanifold.

**Definition A.2.** An admissible almost complex structure on $M$ is a $G$-invariant, $\omega$-compatible almost complex structure $J$ which preserves the distribution $g_\mu^C$ on $U$, and coincides with $J$ outside $U$. The set of all admissible almost complex structures is denoted by $\mathcal{J}(M,U,\mathfrak{J})$. We denote by $\tilde{\mathcal{J}}(M,U,\mathfrak{J})$ the space of smooth $S^1$-families of admissible almost complex structures, and define $\mathcal{J}^l(M,U,\mathfrak{J})$ and $\tilde{\mathcal{J}}^l(M,U,\mathfrak{J})$ the corresponding objects in the $C^l$-category, for $l \geq 1$. We abbreviate them by $\mathcal{J}^l$, $\tilde{\mathcal{J}}^l$ because $M,U,\mathfrak{J}$ are all fixed.
ω and any \( J \in \mathcal{J}^l \) induces a \( G \)-invariant Riemannian metric \( g_J \) on \( M \). We denote by \( T^J_M \) the orthogonal complement of \( g^U_J \) with respect to the metric \( g_J \). Then \( T^J_M \) is isomorphic to \( \pi^*_J TM \), and we have the orthogonal splitting:

\[
TU \cong \pi^*_J TM \oplus g^C_J =: T^J_M \oplus g^C_J. \tag{A.1}
\]

Now by the integrability of \( g^C_J \), we see that for any \( a, b \in \mathfrak{g} \) and any \( J \in \mathcal{J}^l \), we have

\[
[JX_a, JX_b] \in g^C_J. \tag{A.2}
\]

**Lemma A.3.** For \( l \geq 1 \), the space \( \mathcal{J}^l \) is a smooth Banach manifold. For any \( J = \{ J_t \}_{t \in S^1} \in \mathcal{J}^l \), the tangent space \( T_J \mathcal{J}^l \) is naturally identified with the space of \( G \)-invariant sections \( E : S^1 \times M \to \text{End}_{\mathbb{R}} TM \) (of class \( C^l \)), supported in the closure of \( U \), and for each \( t \in S^1 \) satisfying

i. \( J_t E_t + E_t J_t = 0 \);

ii. \( \omega(\cdot, E_t \cdot) \) is a symmetric tensor;

iii. \( g^C_J \) is invariant under \( E_t \).

Now we consider the following equation for \( \tilde{u} := (u, \Phi, \Psi) \in W^{k,p}_{loc}(\Theta, M \times \mathfrak{g} \times \mathfrak{g}) \) with \( H_t \) satisfying Hypothesis (A.1) and \( J \in \mathcal{J}^l \):

\[
\begin{cases}
\partial_s u + X_{\Phi}(u) + J_t (\partial_t u + X_{\Psi}(u) - Y_{H_t}(u)) = 0; \\
\partial_s \Psi - \partial_t \Phi + [\Phi, \Psi] + \mu(u) = 0.
\end{cases} \tag{A.3}
\]

We consider only finite energy solutions and for any pair \( r_\pm \in \text{Crit} \tilde{A}_H \), denote by \( \tilde{M}(r_\pm; J, H) \) the space of all solutions which are asymptotic to \( r_\pm \).

We can also identify any solution \( \tilde{u} \) with an object \( \tilde{v} := (v, \Phi, \Psi) \in W^{k,p}_{loc}(\mathbb{R}^2, M \times \mathfrak{g} \times \mathfrak{g}) \) by \( v(s, t) = \varphi^H_t(u(s, t)) \), and \( \Phi, \Psi \) lifts periodically in \( t \in \mathbb{R} \). It satisfies

\[
\begin{cases}
\partial_s v + X_{\Phi}(v) + J^H_t (\partial_t v + X_{\Psi}(v)) = 0; \\
\partial_s \Psi - \partial_t \Phi + [\Phi, \Psi] + \mu(u) = 0;
\end{cases} \tag{A.4}
\]

Here \( J^H_t = (\phi^H_t)_* J_t \) for \( t \in \mathbb{R} \).

For \( C^l \) almost complex structures, we have the following regularity theorem:

**Theorem A.4.** [2, Theorem 3.1] For any \( l \geq 1 \) and any solution \( \tilde{u} \in W^{k,p}_{loc}(\Theta, M \times \mathfrak{g} \times \mathfrak{g}) \) to (A.3) with \( J \in \mathcal{J}^l \), there exists a gauge transformation \( g \in G^2_{loc}(\Theta, G) \) such that \( g^* \tilde{u} \in W^{l+1,p}_{loc}(\Theta, M \times \mathfrak{g} \times \mathfrak{g}) \).

### A.2. Existence of injective points.

In this subsection we prove an important technical result, showing that for any admissible family of complex structures \( J \in \mathcal{J}^l \) and any nontrivial connecting orbit, there exist “injective” points and they form a rather large subset of \( \Theta \). We fix \( l \geq 1 \) and \( J \) in this subsection.

We first generalize the notion of injective points in [11]. For any \( C^l \)-map \( u : \Theta \to X \) with \( \lim_{s \to \pm \infty} u(s, t) = x_\pm(t) \), we define

\[
\Theta^U(u) := u^{-1}(U), \quad \Theta^U_J(u) := \Theta^U(u) \cap (\mathbb{R} \times I); \tag{A.5}
\]

\[
C^l(u) = \left\{(s, t) \in \Theta^U_J(u) \mid \partial_s u(s, t) \in g^C_J \right\}; \tag{A.6}
\]
\[ R_I (u) = \{ (s, t) \in \Theta_I^U (u) \setminus C (\tilde{u}) \mid u(s, t) \notin Gu (\mathbb{R} \setminus \{ s \}) \times \{ t \}, u(s, t) \notin Gx_{\pm} (t) \}. \]  

Note that the above sets are unchanged if we apply to \( u \) a gauge transformation.

**Lemma A.5.** For any nontrivial connecting orbit \( \tilde{u} = (u, \Phi, \Psi) \in \tilde{M} (\tilde{C}; J, I, H), \) the set \( C_I (u) \) is discrete in \( \Theta_I^U (u) \).

**Proof.** We can assume that \( \tilde{u} \) is in temporal gauge, i.e., \( \Phi \equiv 0 \). Denote \( \vec{\xi} = (\xi, 0, \partial_\nu \Psi) \) lies in the kernel of the linearized operator. So in particular, over \( \mathbb{R} \times I, \)

\[
\nabla \xi + (\nabla \xi J_t) (\partial_t u + J_X \xi) + J_t (\nabla \xi + \nabla \xi X_\Psi) = -J_t X_{\partial_\nu \Psi}. \tag{A.8}
\]

The family \( J_t \) induces a family of metrics on \( M, \) and hence a decomposition

\[ u^* TM \simeq u^* T_M^J \oplus \mathfrak{g}_U^c \tag{A.9} \]

over \( \Theta^U (u). \) Suppose \( \xi = \vec{\xi} + X_a + J_t X_b \) for two functions \( a, b : \Theta^U (u) \to \mathfrak{g} \) and \( \vec{\xi} (t) \in \Gamma \left( \Theta^U (u), u^* T_M^J \right). \) Denote \( \sigma = a + \sqrt{-1} b \in \mathfrak{g}_c \) and \( X_a + J_t X_b = X_\sigma. \) Then over \( \Theta_I^U (u), \)

\[
\nabla \xi = \nabla \xi + X_{\partial_\nu \sigma} + \nabla \xi X_\sigma + \nabla X_\sigma + \nabla X_\sigma X_\sigma \mod \mathfrak{g}_U^c; \\
(\nabla \xi J_t) (\partial_t \xi) = (\nabla \xi J_t) (\xi J_t) + (\nabla X_\sigma J_t) (J_t \xi) + (\nabla X_\sigma J_t) (J_t \xi) + (\nabla X_\sigma J_t) (J_t \xi); \\
J_t (\nabla \xi + \nabla \xi X_\Psi) = J_t \left( (\partial_t J_t) X_b + \nabla \xi + X_{\partial_\nu \sigma} + \nabla J_t + X_{\partial_\nu \sigma} \right) + \nabla \xi X_\sigma + X_{\partial_\nu \sigma} + \nabla \xi X_\sigma \mod \mathfrak{g}_U^c. \tag{A.10}
\]

(Note that \( (\partial_t J_t) X_b \in \mathfrak{g}_U^c. \) Therefore we have

\[
0 \equiv \nabla \xi + J_t \nabla \xi + \nabla X_\sigma + J_t \nabla J_t X_\sigma + (\nabla X_\sigma J_t) (J_t X_\sigma) \\
+ \left( \nabla \xi X_\sigma + (\nabla \xi J_t) (J_t \xi) + (\nabla X_\sigma J_t) J_t \xi + J_t \nabla \xi X_\sigma + J_t \nabla \xi X_\sigma \right) \\
\equiv \nabla^{0,1} \xi + C(z) \xi + J_t [J_t X_\sigma, X_\sigma] \\
\equiv \nabla^{0,1} \xi + C(z) \xi \mod \mathfrak{g}_U^c. \tag{A.11}
\]

Here \( \nabla \) is the connection on \( u^* T_M^J \) induced from the Levi-Civita connection for \( g_t = \omega (\cdot, J_t \cdot) \) by orthogonal projection, and \( C(z) \) is a zero-order operator and the last congruence follows from the integrability of \( \mathfrak{g}_U^c. \) Hence the section \( \xi \) satisfies the Cauchy-Riemann equation

\[ \nabla^{0,1} \xi + C(z) \xi = 0. \tag{A.12} \]

Here by \( \nabla \) is the connection on \( u^* T_M^J \) induced from \( \nabla \) by the splitting (A.1).

Now we apply the Carleman similarity principle of [11]. We see that the vanishing set of \( \xi \) is discrete or \( \xi \equiv 0 \) on \( \Theta_I^U (u). \) If the latter happens, then it is easy to see that \( \tilde{u} \) is a trivial connecting orbit. \( \square \)

**Lemma A.6.** Suppose we have \( (u_i, \Phi_i, \Psi_i) \in W^{1,p}_I (\Theta, M \times \mathfrak{g} \times \mathfrak{g}), i = 1, 2, \) satisfying

\[ \partial_\nu u_i + X_{\Phi_i} (u_i) + J_t (\partial_\nu u_i + X_{\Psi_i} (u_i)) = 0 \tag{A.13} \]

on an open subset \( V \subset \Theta \) with \( u_i (V) \subset U, \) such that there exists \( g \in W^{2,p}_I (V, G) \) with \( g(z) u_2 (z) = u_1 (z) \) for \( z \in V. \) Then \( g^* (u_1, \Phi_1, \Psi_1) = (u_2, \Phi_2, \Psi_2) \) on \( V. \)
Proof. We assume that $g \equiv 1$. Then $u_1 \equiv u_2 \equiv u$. By the equation
\[ \partial_* u + X_{\Phi_1}(u) + J_i (\partial_i u + X_{\Psi_i} - Y_{H_i}) = 0, \]  
(A.14)
we see that $X_{\Phi_1}(u) = X_{\Phi_2}(u)$, $X_{\Psi_1}(u) = X_{\Psi_2}(u)$. Since $u(V) \subset U$ we see that $\Phi_1 \equiv \Phi_2$, $\Psi_1 \equiv \Psi_2$. \hfill \Box

**Proposition A.7.** Let $\tilde{u}_i = (u_i, \Phi_i, \Psi_i) \in \tilde{M}^{1,p}(\tilde{x}_i; J, H)$, $i = 1, 2$ be solutions to (A.3) on $\Theta$ for some $J \in \tilde{J}^1$, such that they coincide on a small disk $B_i \subset \Theta$. Then there exists a gauge transformation $g \in G^{2,p}$ such that $g^* \tilde{u}_2 = \tilde{u}_1$.

**Proof.** We apply the approach in [5, Section 4.3.4] using the unique continuation results of [27].

1. We identify $\tilde{u}_i$ with two solutions $\tilde{v}_i := (v_i, \Phi_i, \Psi_i)$ to (A.4) over $\mathbb{R}^2$. By definition of $\tilde{v}_i$ and the hypothesis, $\tilde{v}_1$ and $\tilde{v}_2$ coincide on a small disk in $\mathbb{R}^2$. We will first prove that for all $z_0 \in \Theta$, there exists an open disk $B_{\rho_0}(z_0)$ centered at $z_0$ on which $\tilde{v}_1$ and $\tilde{v}_2$ are gauge equivalent. In fact, all such points in $\mathbb{R}^2$ form a nonempty open subset $\Omega$. If $\Omega \neq \mathbb{R}^2$, choose the largest $\rho_0 > 0$ such that $B_{\rho_0} = B_{\rho_0}(0) \subset \Omega$. Then we can patch gauge transformations together to obtain a global gauge transformation $g : \mathbb{R}^2 \rightarrow G$ such that $g^* \tilde{v}_1 = \tilde{v}_2$ over the closure of $B_{\rho_0}$ (since $B_{\rho_0}$ is simply connected). So without loss of generality, we assume that $\tilde{v}_1$ and $\tilde{v}_2$ coincide over the closure of $B_{\rho_0}$.

Then we want to show that indeed every $z_1 \in \partial B_{\rho_0}$ also lies in $\Omega$. Take a small disk $B_{\rho_1}(z_1) \subset \mathbb{R}^2$ centered at $z_1$ such that both $v_1$ and $v_2$ map $B_{\rho_1}(z_1)$ into a coordinate chart $V \subset M$, and with respect this coordinate chart, the almost complex structure $J_z(v_i(z)) : \mathbb{C}^n \rightarrow \mathbb{C}^n$ satisfies
\[ \|\text{Id} + J_z(v_i(z))I_0\| \leq \frac{1}{2}, \]  
(A.15)
where $I_0$ is the standard complex structure on $\mathbb{C}^n$.

Then we can find $z_2 \in B_{\rho_0}$ which is on the line segment between 0 and $z_1$ satisfying the following conditions.

1. There exists $\rho_2 > 0$ such that $z_1 \in B_{\rho_2}(z_2) \subset B_{\rho_1}(z_1)$;
2. The gauge transformations $g_i : B_{\rho_2}(z_2) \rightarrow G$ defined by parallel transport along the radial direction with respect to the connection $A_i$ satisfies
\[ g_i(w')^{-1}v_i(w') \in V. \]  
(A.16)
(This is because $\rho_2$ is so small so $g_i$ is very close to the identity of $G$.)

Now we will show that $\tilde{v}_1$ and $\tilde{v}_2$ are gauge equivalent over $B_{\rho_2}(z_2)$. We regard $v_1$ and $v_2$ are maps into $V \subset \mathbb{C}^n$ and by (A.16), we may assume that the connections are in radial gauge, i.e., the connection forms are $f_i d\theta$, $i = 1, 2$. Then $X_{f_i}$ is a vector field on $V$. Then we have
\[ \frac{\partial v_i}{\partial r} + J_z(v_i(z)) \left( \frac{\partial v_i}{\partial \theta} + X_{f_i}(v_i(z)) \right) = 0. \]  
(A.17)
Subtract one from another (using the linear structure of $V$), denoting $\xi(z) = v_2(z) - v_1(z)$ and $h(z) = f_2(z) - f_1(z)$, we obtain
\[
\frac{\partial \xi}{\partial r} + J_z(v_1(z)) \frac{\partial \xi}{\partial \theta} = -(J_z(v_2) - J_z(v_1)) \frac{\partial v_2}{\partial \theta} - J_z(v_2)X_{f_2}(v_2) + J_z(v_1)X_{f_1}(v_1)
\]
\[
= (J_z(v_1) - J_z(v_2)) \frac{\partial v_2}{\partial \theta} - J_z(v_1)X_{h_1}(v_1) - (J_z(v_2) - J_z(v_1))X_{f_2}(v_2) - J_z(v_2)(X_{f_2}(v_2) - X_{f_2}(v_1))
\]
\[
= : R_1(v_1, v_2, f_1, f_2).
\] (A.18)

Similarly, the difference between the vortex equations in the radial gauge gives
\[
\frac{\partial h}{\partial r} = -r(\mu(v_2) - \mu(v_1)) =: R_2(v_1, v_2, f_1, f_2).
\] (A.19)

It is easy to see that there exists a constant $K > 0$ such that
\[
|R_j(v_1, v_2, f_1, f_2)(r, \theta)| \leq K (|\xi| + |h|), \quad j = 1, 2.
\] (A.20)

Hence we have a differential inequality
\[
\left| \frac{\partial}{\partial r} \begin{pmatrix} \xi \\ h \end{pmatrix} + \begin{pmatrix} J_z(v_1) \frac{\partial \xi}{\partial \theta} \\ f \end{pmatrix} \right| \leq K (|\xi| + |h|).
\] (A.21)

We replace $\xi$ by $\zeta(z) := (\text{Id} - I_0J_z(v_1(z)))\xi(z)$ and obtain
\[
\frac{\partial \zeta}{\partial r} + I_0 \frac{\partial \zeta}{\partial \theta}
\]
\[
= -(I_0 \frac{\partial J_z(v_1)}{\partial r} - \frac{\partial J_z(v_1)}{\partial \theta}) \xi + (\text{Id} - J_z(v_1)I_0) \frac{\partial \xi}{\partial r} + (I_0 + J_z(v_1)) \frac{\partial \xi}{\partial \theta}
\] (A.22)
\[
= -(I_0 \frac{\partial J_z(v_1)}{\partial r} - \frac{\partial J_z(v_1)}{\partial \theta}) \xi + (\text{Id} - J_z(v_1)I_0) \left( \frac{\partial \xi}{\partial r} + J_z(v_1) \frac{\partial \xi}{\partial \theta} \right).
\]

If we denote by $P(\zeta, h) = \begin{pmatrix} I_0 \frac{\partial \zeta}{\partial \theta} \\ 0 \end{pmatrix}$, then $P$ is an self-adjoint differential operator on the circle $S^1$. Then we have the differential inequality for some $K' > 0$
\[
\left\| \frac{d}{dr} (\zeta, h) + P(\xi, h) \right\|_{L^2(S^1)} \leq K' (|\zeta| + |h|)_{L^2(S^1)}.
\] (A.23)

This is in the form considered in [27]. Since $(\zeta, h)$ vanishes for $r$ small, $(\zeta, h) \equiv 0$ for $r \leq \rho_2$. This implies that in radial gauge, the two solutions coincide on $B_{\rho_2}(z_2)$, which contains $z_1$. Hence $\partial B_{\rho_0} \subset \Omega$. By the compactness of $\partial B_{\rho_0}$, we obtain a disk strictly larger than $B_{\rho_0}$ which is also contained in $\Omega$. This contradicts with the definition of $\rho_0$. Therefore $\Omega = \mathbb{R}^2$.

II. Now we prove that there exists a global gauge transformation $g : \Theta \rightarrow G$ such that $g^*\tilde{u}_2 = \tilde{u}_1$. Indeed, there exists $S > 0$ such that
\[
(-\infty, -S] \times S^1 \subset \Theta^U(u_1) \cap \Theta^U(u_2).
\] (A.24)

By the property of $U$, we see that the local gauge transformations around each point $z \in (-\infty, -S] \times \mathbb{R}$ appeared in I. are unique. Hence we can patch them to obtain a global gauge transformation $g : (-\infty, -S] \times S^1 \rightarrow G$ such that $g^*\tilde{u}_2 = \tilde{u}_1$. But it is easy to see that we can extend $g$ to a longer
cylinder \((-\infty, -S + \epsilon_0] \times S^1\) with \(\epsilon_0\) independent of \(S\) (we omit the details). Hence there exists a global gauge transformation which identifies \(\tilde{u}_1\) with \(\tilde{u}_2\). \(\square\)

**Lemma A.8.** Suppose we have two solutions \((u_i, \Phi_i, \Psi_i) \in W^{2,p}(B_{r_i}, M \times g \times g)\) on \(B_{r_i} \subset \mathbb{R} \times I\), \(i = 1, 2\) to \((A.3)\), for some \(J \in \widehat{J}^l\), \(l \geq 2\), satisfying the following conditions:

1. \(u_i(B_{r_i}) \subset U\), \(u_i(0) = u_2(0)\);
2. \(u_i(B_{r_i})\) is an embedded surface which intersects each leaf of \(\mathfrak{g}_U^C\) cleanly at at most one point;
3. For each \((s, t) \in B_{r_1}\), there exists \(g \in G\) and \((s', t) \in B_{r_2}\) such that \(u_1(s, t) = g u_2(s', t)\).

Then \(r_1 \leq r_2\) and there exists a gauge transformation \(g : B_{r_1} \to G\) such that \(g^*(u_1, \Phi_1, \Psi_1) = (u_2, \Phi_2, \Psi_2)\) over \(B_{r_1}\).

**Proof.** We can find an embedded submanifold \(S \subset U\) of codimension \(\dim G\), transverse to \(G\)-orbits, such that \(u_2(B_{r_2}) \subset S\). Then we can construct a smooth map \(\pi : G \cdot S \to B_{r_2}\) such that \(\pi \circ u_1 = \text{Id}\).

The hypothesis implies that \(u_1(B_{r_1}) \subset GU_2(B_{r_2})\). Then the map \(\pi \circ u_1\) must take the form \((s, t) \mapsto (\phi(s, t), t)\), and there exists a unique \(g(s, t) \in G\) such that

\[
\tag{A.25}
u_1(s, t) = g(s, t)u_2(\phi(s, t), t).
\]

Now since \(\tilde{u}_i\) is a solution to the vortex equation, we see on \(B_{r_1}\),

\[
0 = \partial_s u_1 + X_{\Phi_i} (u_1) + J_i (\partial_t u_1 + X_{\Psi_i} (u_1)) \equiv (\partial_s u_2) \partial_s \phi + J_i (\partial_t u_2 + (\partial_s u_1) \partial_t \phi) \\
\equiv (\partial_s u_2) (\partial_s \phi - 1) + (\partial_t u_2) \partial_t \phi \quad \text{(mod } \mathfrak{g}_U^C\),
\]

But since \(u_2\) intersects with leaves of \(\mathfrak{g}_U^C\) cleanly, we see that the above congruence holds if and only if \(\partial_s \phi \equiv 1\) and \(\partial_t \phi \equiv 0\). Hence there exists \(s_0 \in \mathbb{R}\) such that \(u_1(s, t) = g(s, t)u_2(s + s_0, t)\) for each \((s, t) \in B_{r_1}\). Since \(u_1(0, 0) = u_2(0, 0)\) and the second hypothesis, \(s_0 = 0\). Hence \(r_1 \leq r_2\) and by Lemma A.6, the restriction of \(\tilde{u}_2\) to \(B_{r_1}\) is gauge equivalent to \(\tilde{u}_1\). \(\square\)

**Proposition A.9.** For any \(J \in \widehat{J}^l\) \((l \geq 2)\), and any \(\tilde{u} = (u, \Phi, \Psi) \in \tilde{M}(\mathfrak{r}_\pm; J, H)\) a nontrivial connecting orbit, the set \(R_I(u)\) is open and dense in \(\Theta_I^U(u)\).

**Proof.** I. We first prove that \(R_I(u)\) is open. If it is not the case, then there exists \((s_0, t_0) \in R_I(u)\) and a sequence \((s_i, t_i) \in \Theta_I^U(u) \setminus R_I(u)\) such that \(\lim_{i \to +\infty} (s_i, t_i) = (s_0, t_0)\). By the definition of \(R_I(u)\), we must have that for each \(i\) large enough, there exists \(g_i \in G\) and \(s'_i \in \mathbb{R} \setminus \{s_i\}\) such that

\[
\tag{A.27}u(s_i, t_i) = g_i u(s'_i, t_i).
\]

If \(s'_i\) is unbounded, then we have that (for a subsequence) \(u(s'_i, t_i) \to x_\pm(t_0)\) as \(i \to +\infty\). This contradicts with the condition that \(u(s_0, t_0) \notin G x_\pm(t_0)\). Hence \(s'_i\) must be bounded. So we may assume that \(s'_i \to s'_0 \in \mathbb{R}\). This implies that \(u(s_0, t_0) = g_0 u(s'_0, t_0)\) for some \(g_0 \in G\). By the definition of \(R_I(u)\), we must have \(s'_0 = s_0\). Hence the two different sequences \((s_i, t_i)\) and \((s'_i, t_i)\) both converge to \((s_0, t_0)\). Then \((A.27)\) implies that \(\partial_s u(s_0, t_0) \in \mathfrak{g}_U \subset \mathfrak{g}_U^C\), which contradicts with the fact that \((s_0, t_0) \in R_I(u)\).

II. Now we prove that \(R_I(u)\) is dense. Since we know that the subset \(C_I(u)\) is discrete in \(\Theta_I^U(u)\), we only have to show that every point in \(\Theta_I^U(u) \setminus C_I(u)\) can be approximated by points in \(R_I(u)\). We take a point \((s_0, t_0) \in \Theta_I^U(u) \setminus C_I(u)\). We may also assume that \(u(s_0, t_0) \notin G x_\pm(t_0)\); otherwise, for any other \(s\) close to \(s_0\), \((s, t_0)\) satisfies this condition.
Now we assume that \((s_0, t_0)\) is not in the closure of \(R_I(u)\). Then there exists \(\epsilon_0 > 0\) and a nonempty open subset \(I_0 \subset I\) with
\[
B_{\epsilon_0}(s_0, t_0) \subset \Theta(U)_{I_0}(u) \setminus R_{I_0}(u).
\] (A.28)
We may choose \(\epsilon_0\) small enough and \(S\) large enough so that
(1) For any \(|s| \geq S, |t - t_0| \leq \epsilon_0\), \(u(s, t) \notin Gu(B_{\epsilon_0}(s_0, t_0))\);
(2) For \(|t - t_0| \leq \epsilon_0\), \(u([s_0 - \epsilon_0, s_0 + \epsilon_0] \times \{t\})\) is embedded and intersects each \(G\)-orbit at at most one point.
(3) \(Gu(B_{\epsilon_0}(s_0, t_0)) \cap G (C_I(u) \cap [-S, S] \times T_0) = \emptyset\).

Now the condition (A.28) implies that for all \((s, t) \in B_{\epsilon_0}(s_0, t_0)\), there exists \(s' \in \mathbb{R} \setminus \{s\}\) such that \(u(s, t) \in Gu(s', t)\).

II. a) We claim that for each \((s, t)\), there exists only finitely many such \(s'\). Indeed, the first condition above implies that, if there are infinitely many such \(s'\), then they must have an accumulation point in \([-S, S]\), and at the accumulation point \(\partial_s u\) lies in \(\mathcal{G}_U^C\). This contradicts with the third condition. So the claim is true.

II. b) For any \((s, t) \in B_{\epsilon_0}(s_0, t_0)\), define
\[
\mathcal{K}^S((s, t)) := \{(s', t') \in [-S, S] \times I \mid s' \neq s, u(s, t) \in Gu(s', t)\}
\] (A.29)
and
\[
N := \min_{(s, t) \in B_{\epsilon_0}(s_0, t_0)} \# \mathcal{K}^S((s, t)) \geq 1.
\] (A.30)
We claim that there exists \((s^*, t^*) \in B_{\epsilon_0}(s_0, t_0)\) and \(\epsilon_1 > 0\) such that \(B_{\epsilon_1}(s^*, t^*) \subset B_{\epsilon_0}(s_0, t_0)\), and
\[
(s, t) \in B_{\epsilon_1}(s^*, t^*) \implies \# \mathcal{K}^S((s, t)) = N.
\] (A.31)
Moreover, for any \((s_\nu, t_\nu) \to (s^*, t^*)\), we have that the sequence \(\mathcal{K}^S((s_\nu, t_\nu))\) converges to \(\mathcal{K}^S((s^*, t^*))\) as subsets.

Indeed, find any \((s^*, t^*) \in B_{\epsilon_0}(s_0, t_0)\) which realizes the lower bound \(N\). If not the case, then there exists a sequence \((s_\nu, t_\nu)\) converging to \((s^*, t^*)\) but \(\# \mathcal{K}^S(s_\nu, t_\nu) > N\). If the sequence of points \(\mathcal{K}^S(s_\nu, t_\nu)\) have accumulations, then it will contradict with (3); if they don’t accumulate, then by choosing a subsequence, we see that \(\# \mathcal{K}^S(s^*, t^*) \geq N + 1\) which contradicts with the choice of \((s^*, t^*)\).

II. c) Hence we can actually assume that \((s_0, t_0) = (s^*, t^*)\) and \(\epsilon_1 = \epsilon_0\). In particular, we take \(s_1, s_2, \ldots, s_N \in [-T, T]\) be all numbers such that
\[
u(s_0, t_0) \in Gu(s_i, t_0), \quad i = 1, \ldots, N.
\]
We claim that (which is similar to that in [11, Page 262]) there exists \(\delta > 0\) and \(r > 0\) such that
\[
u(B_{2\delta}(s_0, t_0)) \subset \bigcup_{j=1}^{N} u(B_r(s_j, t_0)), \quad j \neq j' \implies B_r(s_j, t_0) \cap B_r(s_{j'}, t_0) = \emptyset.
\] (A.32)
Then we define
\[
\Sigma_j := \{(s, t) \in \overline{B}_r(s_0, t_0) \mid u(s, t) \in \text{Gcl}(u(B_r(s_j, t)))\}.
\] (A.33)
These are closed sets and \(\overline{B}_r(s_0, t_0) = \Sigma_1 \cup \cdots \cup \Sigma_N\). Then there exists \(j_0\) such that \((s_0, t_0) \in \text{Int} \Sigma_{j_0}\). Then we take a small \(\rho > 0\) such that \(B_{\rho}(s_0, t_0) \subset \text{Int} \Sigma_{j_0}\) and \(B_{\rho}(s_0, t_0) \cap B_r(s_1, t_0) = \emptyset\).
II. d) For every \((s,t) \in B_{\rho}(s_0, t_0)\), we see that \(\mathcal{K}S((s,t)) \cap B_{r}(s_{j_0}, t_0)\) contains a unique element \((s', t)\). By Lemma A.8 we see that for \(\rho \leq r\) and the two objects \(\tilde{u}(s_0 + \cdot, t_0 + \cdot)\) and \(\tilde{u}(s_{j_0} + \cdot, t_0 + \cdot)\) are gauge equivalent over \(B_{\rho}(0)\). Then by Proposition A.7, there exists a gauge transformation \(g : \Theta \to G\) such that
\[
g^*\tilde{u} = \tilde{u}(s_{j_0} - s_0 + \cdot, \cdot).
\]
Let \(\delta s = s_{j_0} - s_0 \neq 0\). Then we see that \(\tilde{u}\) and \(\tilde{u}(k\delta s + \cdot, \cdot)\) are gauge equivalent for all \(k \in \mathbb{Z}\). This implies that \(\tilde{u}(s_0, t) \in Gx_s(t)\) by the asymptotic behavior of finite energy solutions. This contradicts with our choice of \((s_0, t_0)\), which means that \(R_I(u)\) is indeed dense in \(\Theta_I^U(u)\). 

A.3. The universal moduli space over the space of admissible almost complex structures. For \(l \geq k \geq 2, p > 2\), we denote
\[
\mathcal{M}^{k,p}\left([\mathbb{x}_\pm]; \tilde{J}, H\right) := \left\{ ([\tilde{u}], J) \mid J \in \tilde{J}, \tilde{u} = [u, \Phi, \Psi] \in \mathcal{M}^{k,p}\left([\mathbb{x}_\pm]; J, H\right) \right\}.
\]

Proposition A.10. For any \(l \geq k, k \geq 1\), the universal moduli space is a \(C^{l-k}\)-Banach submanifold of \(\mathbb{B}^{k,p} \times \tilde{J}^l\).

Proof. We just need to prove that the linearized operator
\[
D : T_{[\tilde{u}]}\mathbb{B}^{k,p} \times T_J\tilde{J}^l \to \mathcal{E}^{k-1,p}_u
\] is surjective. It is equivalent to consider the augmented one, for any representative \(\tilde{u} \in \mathcal{M} ([\mathbb{x}_\pm]; J, H)\), which is
\[
\mathfrak{D} : T_{\tilde{u}}\mathbb{B}^{k,p} \times T_J\tilde{J}^l \to \mathcal{E}^{k-1,p}_u \oplus W^{k-1,p}(\Theta, g).
\]

Now, because the restriction of \(\mathfrak{D}\) to the first component, which is the augmented linearized operator \(\mathcal{D}_{\tilde{u}}\), is already Fredholm, we only need to prove that \(\mathfrak{D}\) has dense range. If not the case, then there exists a nonzero vector \(\tilde{\eta} = (\eta, \vartheta_1, \vartheta_2) \in \mathcal{E}^{k-1,p}_u \oplus W^{k-1,p}(\Theta, g)\) which lies in the \(L^2\)-orthogonal complement of the image of \(\mathfrak{D}\); therefore \(\tilde{\eta}\) also lies in the orthogonal complement of the image of \(\mathcal{D}_{\tilde{u}}\). Hence \(\tilde{\eta} \in \ker \mathfrak{D}_{\tilde{u}}^*\). By elliptic regularity associated to \(\mathfrak{D}_{\tilde{u}}^*\), we see that \(\tilde{\eta}\) is continuous. We will show that \(\tilde{\eta}\) vanishes on an nonempty open subset of \(R_I(u)\), which, by the unique continuation property of \(\mathfrak{D}_{\tilde{u}}^*\), contradicts with the fact that \(\tilde{\eta} \neq 0\).

Indeed, on \(R_I(u)\), we decompose \(\eta = \eta' + \eta''\) with respect to the splitting (A.1) (which also depends on \(t\)). However, it is easy to find \(E', E'' \in T_J\tilde{J}^l\), supported near the \(G\)-orbit of \(u(z)\) for any \(z \in R_I(u)\), such that \(E'(s, t)\) is zero on \(\mathfrak{g}_U^c\), \(E''(s, t)\) is zero on \(T_J^l\), and such that \(\mathfrak{D}(0, E')\) (resp. \(\mathfrak{D}(0, E'')\)) has positive \(L^2\)-pairing with \(\eta'\) (resp. \(\eta''\)) (the concrete way of constructing \(E'\) and \(E''\) are similar to that in [24]). Then this shows that \(\eta\) vanishes on \(R_I(u)\).

Then, looking at the first component of \(0 = \mathcal{D}_{\tilde{u}}^*(\eta, \vartheta_1, \vartheta_2)\), we see on \(R_I(u)\),
\[
J_t X_{\vartheta_1} - X_{\vartheta_2} = 0.
\]
By the definition of \(R_I(u)\), we see that \(\vartheta_1|_{R_I(u)} = \vartheta_2|_{R_I(u)} = 0\). This finishes our proof.

Now the projection
\[
\mathcal{M} \left([\mathbb{x}_\pm]; \tilde{J}^l, H\right) \to \tilde{J}^l
\]
(A.38)
is a Fredholm map of class \(C^{l-k}\). Then by Sard-Smale theorem, we obtain
Theorem A.11. For any pair $[x]$ ∈ Crit$\mathcal{A}_H$, there exists $l_0 = l_0(k, [x]) \in \mathbb{Z}_+$ such that for any $l \geq l_0$, there exists a Baire subset $\tilde{\mathcal{J}}^{\text{reg}}_H \subset \tilde{\mathcal{J}}$, such that for any $J \in \tilde{\mathcal{J}}^{\text{reg}}_H$ and any $[\tilde{u}] = [u, \Phi, \Psi] \in \mathcal{M}([x]; J, H)$, the linearized operator

$$D^{JH}_{[\tilde{u}]} : T_{[\tilde{u}]} \mathcal{B}^{k,p} \to \mathcal{E}^{k-1,p}_{[\tilde{u}]}$$

(A.39)

is surjective.

Using Taubes’ trick (see [24, Page 52]), we obtain

Corollary A.12. There exists a Baire subset $\tilde{\mathcal{J}}^{\text{reg}}_H \subset \tilde{\mathcal{J}}$ such that for any $J \in \tilde{\mathcal{J}}^{\text{reg}}_H$, the moduli space $\mathcal{M}([x]; J, H)$ is a smooth submanifold of $\mathcal{B}^{k,p}([x])$ for any $k \geq 1$ and $p > 2$.

A.4. Transversality for continuation map. It is easier to achieve transversality for the moduli space defining the continuation map, because we are allowed to have objects which depend both on $s$ and $t$. We briefly sketch the method. Suppose we have a given homotopy $H_{s,t}$ of compactly supported Hamiltonians for $(s, t) \in \Theta$ such that $H_{s,t} = H_t$ for $s \ll 0$ and $H_{s,t} = H_t^\beta$ for $s \gg 0$. Now we are given two regular families of almost complex structures $J^\alpha \in \tilde{\mathcal{J}}^{\text{reg}}_{H^\alpha}(M, U, \mathcal{J}), J^\beta \in \tilde{\mathcal{J}}^{\text{reg}}_{H^\beta}(M, U, \mathcal{J})$ for each pair $[x^\alpha] \in \text{Crit} \mathcal{A}_{H^\alpha}, [x^\beta] \in \text{Crit} \mathcal{A}_{H^\beta}$, the moduli space of solutions to the following equation

$$\begin{cases}
\frac{\partial u}{\partial s} + X_\Phi(u) + J_{s,t} \left( \frac{\partial u}{\partial t} + X_\Phi(u) - Y_{H_{s,t}}(u) \right) = 0; \\
\frac{\partial \Psi}{\partial s} - \frac{\partial \Phi}{\partial t} + [\Phi, \Psi] + \lambda^2_\mu(u) = 0
\end{cases}$$

(A.41)

satisfying the conditions that $\tilde{u} = (u, \Phi, \Psi)$ is asymptotic to $x^\alpha$ (resp. $x^\beta$) as $s \to -\infty$ (resp. $s \to +\infty$), is transverse.

Now take the function $V(s, t) = e^{s|}$ which diverges exponentially as $s \to \pm \infty$. For $l \in \mathbb{Z}_+ \cup \{\infty\}$, consider the space

$$\tilde{\mathcal{J}}^l \left( J^\alpha, J^\beta \right)$$

(A.42)

consisting of $C^l$-family of admissible almost complex structures $J_{s,t}$ (with respect to the same $U$ and $\mathcal{J}$ as before) parametrized by $(s, t) \in \Theta$, such that

$$|V(s, t)(J_{s,t} - J^\alpha)|_{C^l(\Theta_+ \times M)} < \infty, \ |V(s, t)(J_{s,t} - J^\beta)|_{C^l(\Theta_+ \times M)} < \infty.$$ (A.43)

Those $J$’s are homotopies that are asymptotic to $J^\alpha$ (resp. $J^\beta$) exponentially with their derivatives up to order $l$.

Then, for each $\mathcal{J} \in \tilde{\mathcal{J}}^l(J^\alpha, J^\beta)$, consider the moduli space

$$\mathcal{N} \left( [x^\alpha], [x^\beta]; \mathcal{J}, \mathcal{H}, \lambda_s \right) \subset \mathcal{B}^{k,p} \left( [x^\alpha], [x^\beta] \right)$$

(A.44)

of solutions to (A.41). Since all element $[\tilde{u}] \in \mathcal{M}([x^\alpha], [x^\beta]; \mathcal{J}, \mathcal{H}, \lambda_s)$ will go into $U$ as $|s| \to \infty$, this implies that every such $[\tilde{u}]$ is “irreducible” in the sense of [2]. Hence we can prove the
transversality of the universal moduli space over \( \tilde{\mathcal{F}}^l(J^\alpha, J^\beta) \) in the same way as in [2], because now the perturbation could be dependent on both \( s \) and \( t \). Then using the implicit function theorem and Taubes’ trick, it is easy to prove the following proposition.

**Proposition A.13.** There exists a Baire subset \( \tilde{\mathcal{J}}_{\mathcal{H},\lambda}^{\text{reg}} (J^\alpha, J^\beta) \subset \tilde{\mathcal{J}} (J^\alpha, J^\beta) \) such that for every \( \mathcal{J} \in \tilde{\mathcal{J}}_{\mathcal{H},\lambda}^{\text{reg}} (J^\alpha, J^\beta) \) and every pair \([x^\alpha] \in \text{Crit}_{\mathcal{A}^H_{\alpha}}, [x^\beta] \in \text{Crit}_{\mathcal{A}^H_{\beta}}\), the moduli space \( N([x^\alpha], [x^\beta]; \mathcal{J}, \mathcal{H}, \lambda_s) \) is a smooth manifold of dimension \( \text{CZ}([x^\alpha]) - \text{CZ}([x^\beta]) \).

**References**


