Functional Analysis I

Course Notes by Stefan Richter
Transcribed and Annotated by Gregory $\mathbb{Z}$itelli

Polar Decomposition

Definition. An operator $W \in \mathcal{B}(\mathcal{H})$ is called a partial isometry if $\|Wx\| = \|X\|$ for all $x \in (\ker W)^{\perp}$.

Theorem. *The following are equivalent.*

(i) W *is a partial isometry.*
(ii) $P = WW^*$ *is a projection.*
(iii) W^* *is a partial isometry.*
(iv) $Q = W^*W$ *is a projection.*

Theorem (Polar Decomposition)**.** *Let* $A \in \mathcal{B}(\mathcal{H}, \mathcal{K})$*. Then there exist unique* $P \geq 0$*,* $P \in \mathcal{B}(\mathcal{H})$ *and* $W \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ *a partial isometry such that* $A = WP$*, with the initial space of* W *equal to* $\overline{\operatorname{ran} P} = (\ker P)^{\perp}$*.*

Proof. If $A \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ then $A^* \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ and $A^*A \in \mathcal{B}(\mathcal{H})$ is self adjoint. Furthermore, $\langle A^*Ax, x\rangle = \|Ax\|^2 \geq 0$, so $A^*A \geq 0$. Let $|A| = \sqrt{A^*A}$. Then $\ker A = \ker |A|$. Write $P = |A|$. Then $\mathcal{H} = \ker |A| \oplus \overline{\operatorname{ran} |A|} = \ker A \oplus \overline{\operatorname{ran} |A|}$. Let $\mathcal{D} = \ker A \oplus \operatorname{ran} |A|$, dense in $\mathcal{H}$. Define $W : \mathcal{D} \to \mathcal{K}$ by $W(y + |A|x) = Ax$, which can be shown to be well defined. The extension of W to all of $\mathcal{H}$ makes it into a partial isometry. ∎

Theorem. *Let* $T \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ *be compact, then there exists* $\{e_n\} \in \mathcal{H}$ *and* $\{f_n\} \in \mathcal{K}$*, both orthonormal, and* $s_n \geq 0$ *with* $s_n \to 0$ *such that*

$$T = \sum_{n=1}^{\infty} s_n f_n \otimes e_n$$

under norm convergence.

Proof. If T is compact, then T^*T is compact and $T^*T = \sum \lambda_n e_n \otimes e_n$ for some orthonormal set $\{e_n\} \in \mathcal{H}$ and λ_n are the eigenvalues of T^*T. Furthermore, $\lambda_n \geq 0$, so we can define $s_n = \sqrt{\lambda_n} \to 0$. Then we claim that $\sqrt{T^*T} = |T| = \sum_{n=1}^{\infty} s_n e_n \otimes e_n$. Then by the last theorem, $T = W|T|$ with W a partial isometry. Then we have that $T = \sum_{n=1}^{\infty} s_n W(e_n \otimes e_n) = \sum_{n=1}^{\infty} s_n We_n \otimes e_n$. The rest of the properties can be checked. ■

Definition. If T is compact, let $s_n(T)$ be the n^{th} largest eigenvalue of $|T|$. We say that $s_n(T)$ is the n^{th} singular number of T.

Corollary. *If T is self adjoint, $x_0 \in \mathcal{H}$, $\mathcal{D} = \{p(T)x_0\}$ is dense in $\mathcal{H}$, then there exists $\mu \in M(\sigma(T))$ such that T is unitarily equivalent to M_t on $L^2(\mu)$.*

Proposition. *Let $T \in \mathcal{B}(\mathcal{H})$ be self adjoint and cyclic. Then there exists $\mu \in M(\sigma(T))$ with $\mu \geq 0$, $\|\mu\| = 1$, and $\mu(\sigma(T)) = 1$, such that $\|f(T)x_0\|^2 = \int_{\sigma(T)} |f|^2 d\mu$ for all $f \in C(\sigma(T))$.*

Definition. Let $T \in \mathcal{B}(\mathcal{H})$ be self adjoint. We say that $x_0 \in \mathcal{H}$ is a cyclic vector for T if $\{p(T)x_0\}$ for all polynomials p is dense in $\mathcal{H}$. If T has any cyclic vectors, then T is called cyclic.

Definition. Let $T \in \mathcal{B}(\mathcal{H})$ and $S \in \mathcal{B}(\mathcal{K})$. We say that T and S are unitarily equivalent if there exists $U \in \mathcal{B}(\mathcal{H}, \mathcal{K})$, a Hilbert space isomorophism, such that $S = UTU^{-1}$.

In our new language, the proposition now says that T is unitarily equivalent to M_t on $\mathcal{B}(L^2(\mu))$ for some probability measure μ on the spectrum of T.

Corollary. *If $f \in C(\sigma(T))$ and T is self adjoint, then $f(T) = u(T) + iv(T)$ if $f = u + iv$ for real valued u, v.*

Quotient Spaces

Let $(X, \|\cdot\|)$ be a normed space, $\mathcal{M} \subseteq X$ a vector space. We define $X/\mathcal{M} = \{[x] : x \in X\}$ given by the equivalence classes $[x] = \{x + y : y \in \mathcal{M}\}$. We define a semi-norm $\|[x]\|_{X/\mathcal{M}} = \inf\{\|y\|_X : [y] = [x]\} = \inf\{\|x - z\|_X : z \in \mathcal{M}\}$.

Lemma. *If $\mathcal{M}$ is a closed subspace of X, then the quotient semi-norm $\|\cdot\|_{X/\mathcal{M}}$ is a norm on $X/\mathcal{M}$ and the following all hold.*

(i) Let $Q : X \to X/\mathcal{M}$ be the natural inclusion, then $\|Q(x)\|_{X/\mathcal{M}}$.
(ii) If X is a Banach space, then so is $X/\mathcal{M}$.
(iii) A set $W \subseteq X/\mathcal{M}$ is open in $X/\mathcal{M}$ if and only if $Q^{-1}(W)$ is open in X.
(iv) Q maps open sets to open sets.

If $X = \mathcal{H}$ is a Hilbert space and $\mathcal{M} \subseteq \mathcal{H}$ is a subspace, then we actually have that $\mathcal{H}/\mathcal{M} \cong \mathcal{M}^{\perp}$.

Proposition. *Let X be a normed space, $\mathcal{M}, \mathcal{N} \subseteq X$ closed subspaces. Suppose that the dimension of $\mathcal{N}$ is finite, then $\mathcal{M} + \mathcal{N}$ is closed.*

Proof. Consider $Q : X \to X/\mathcal{M}$. Then the dimension of $Q(\mathcal{N})$ is finite as well, making it closed in $X/\mathcal{M}$. Then $Q^{-1}(Q(\mathcal{N}))$ is closed in X, but this is precisely $\mathcal{M} + \mathcal{N}$. ■

Definition. Suppose that $X = \prod_{\alpha \in A} X_\alpha = \{\{x_\alpha\}_{\alpha \in A} : x_\alpha \in X_\alpha\}$ is the full product space over vector spaces X_α. Then for $1 \leq p < \infty$, we define $\bigoplus_p X_\alpha = \left\{x \in X : \sum_{\alpha \in A} \|x_\alpha\|_\alpha^p < \infty\right\}$ and $\bigoplus_\infty X_\alpha = \{x \in X : \sup_\alpha \|x_\alpha\|_\alpha < \infty\}$.

Note that if each X_α is a Banach space, then $\bigoplus_p X_\alpha$ and $\bigoplus_\infty X_\alpha$ are Banach spaces.

Hahn-Banach Theorem

Definition. A hyperplane in a vector space X is a subspace $\mathcal{M}$ where $X/\mathcal{M}$ has dimension equal to one.

From general results about functionals on a normed vector space, it follows that hyperplanes are either closed or dense.

Definition. We say that a function $p : X \to \mathbb{R}$ is sublinear if

(i) $p(x + y) \leq p(x) + p(y)$ for all $x, y \in X$.
(ii) $p(tx) = tp(x)$ for all $t > 0$, $x \in X$.

Theorem (Hahn-Banach Theorem)**.** *Let X be a vector space over $\mathbb{R}$, $g : X \to \mathbb{R}$ sublinear. Let $\mathcal{M} \subseteq X$ be a subspace, with $f : \mathcal{M} \to \mathbb{R}$ linear and $f(x) \leq q(x)$ on $\mathcal{M}$. Then there exists $F : X \to \mathbb{R}$, linear, such that F agrees with f on $\mathcal{M}$ and $F \leq g$ on all of X.*

Reflexive Spaces

Corollary. *Let X be normed, $x \in X$, then $\|x\| = \sup\{|f(x)| : f \in X^*, \|f\|_* = 1\}$ and the supremum is attained.*

Proof. If $\|f\|_* \leq 1$, then $|f(x)| \leq \|f\|_*\|x\| \leq \|x\|$, which implies that the supremum is less than or equal to $\|x\|$. Let $\mathcal{M} = \{\beta x : \beta \in \mathbb{F}\} \subseteq X$, and define $F : \mathcal{M} \to \mathbb{F}$ by $F(\beta x) = \beta\|x\|$. Then $\|F\| = 1$ and $|F(\beta x)| = |\beta|\|x\| = \|\beta x\|$. By the Hahn-Banach theorem, there exists $f_0 \in X^*$ such that $\|f_0\| = \|F\| = 1$ and $f_0|_{\mathcal{M}} = F$, with $f_0(x) = F(x) = \|x\|$. ■

Note that when $f \in X^*$, we define $\|f\|_* = \sup\{|f(x)| : \|x\| \leq 1, x \in X\}$, but this supremum may not be attained. If $1 < p < \infty$ and $X = L^p(\mu)$, then $X^* \cong L^q(\mu)$ and $L^q(\mu)^* \cong L^p(\mu)$ and the supremum is attained from this duality.

More generally, we consider the canonical inclusion $X \to X^{**}$ by $f \mapsto j(x) \in C^{**}$ where $j(x)(f) = f(x)$. Note that $j(x)$ is linear and bounded, and that $\|j(x)\| \leq \|x\|$. By the corollary, there exists $f \in X^*$ such that $\|f\| \leq 1$ and $|j(x)(f)| = |f(x)| = \|x\|$, which further implies that $\|j(x)\| = \|x\|$.

Definition. We say that a space X is reflexive if the canonical inclusion j is onto.

Note that only Banach spaces can be reflexive, since X^{**} is automatically Banach and reflexivity means there is a Banach space isomorphism between X and X^{**}.

Corollary. *Let X be normed, $\mathcal{M} \subseteq X$ a vector subspace, $x_0 \in X \backslash \mathcal{M}$. If d is the distance between x_0 and $\mathcal{M}$, with $d > 0$, then there exists $f \in X^*$ such that $f(x_0) = 1$, $\|f\| = \frac{1}{d}$, and $f = 0$ on $\mathcal{M}$.*

Proof. We have that $\overline{\mathcal{M}}$ is a closed subspace of X, with $X/\overline{\mathcal{M}}$ a normed space. Let $Q : X \to X/\overline{\mathcal{M}}$ be the projection, then $d = \|[x_0]\|$. Then there exists $f_0 \in (X/\overline{\mathcal{M}})^*$ such that $\|f_0\| \leq 1$ and $f_0([x_0]) = \|[x_0]\|$. Let $f = f_0 \circ Q$, linear. Then $f(x_0) = d$. Now simply scale f by d, and we have a function such that $\|f(x_0)\| = 1$, $\|f\| \leq \frac{1}{d}$, and $f = 0$ on $\mathcal{M}$.

Now we know that there exists a sequence $y_n \in \mathcal{M}$ such that $\|x_0 - y_n\| \to d$. Then define $z_n = \frac{x_0 - y_n}{\|x_0 - y_n\|}$ such that $\|z_n\| = 1$ and $|f(z_n)| \to \frac{1}{d}$. Therefore, we have that $\|f\| = \frac{1}{d}$. ∎

Corollary. *Let X be normed, $\mathcal{M} \subseteq X$ a vector subspace. Then $\overline{\mathcal{M}}$ is the intersection of all closed hyperplanes that contain $\mathcal{M}$, i.e. $\overline{\mathcal{M}} = \bigcap\{\ker f : f \in X^*,\ \mathcal{M} \subseteq \ker f\}$.*

Definition. Let X be normed, $S \subseteq X$ a subset. Then we say that $S^\perp = \{f \in X^* : \forall x \in S,\ f(x) = 0\}$, called the annihilator of S.

Note that $S^\perp$ is always a closed subspace of X^*.

Theorem. *Let X be normed, $\mathcal{M} \subseteq X$ a vector subspace, then*

$$\mathcal{M}^* \cong X^*/\mathcal{M}^\perp$$

More precisely, $\rho : X^*/\mathcal{M}^\perp \to \mathcal{M}^*$ with $\rho([f]) = f|_{\mathcal{M}}$ is well defined and is an isometric isomorphism onto $\mathcal{M}^*$.

Theorem (Baire Category Theorem)**.** *Let (X, d) be a complete metric space. If V_n are dense and open in X for $n \in \mathbb{N}$, then $\bigcap_n V_n$ is dense in X.*

Corollary. *Let X be a complete metric space. If F_n is a sequence of closed sets with $\bigcup_{n=1}^{\infty} F_n = X$, then there exists some $N \in \mathbb{N}$ such that the interior of F_N is nonempty.*

Proof. Suppose that the interior of every F_n is empty. Let $V_n = X \backslash F_n$, open. The lack of an interior means that each V_n is dense, and so $\bigcap_n V_n \neq \varnothing$, so that $\bigcup_n F_n \neq X$. ■

The application for us is that if X is a normed space with a countable (infinite) Hamel basis, then it is impossible for X to be complete. Specifically, we cannot put a norm on the polynomials which makes them complete.

Definition. We say that a set W in a metric space X is nowhere dense if the interior of its closure is empty.

Definition. We say that a set A in a metric space X is of first category in X if A is a countable union of nowhere dense sets. A set which is not of the firs category is of the second category.

The Baire Category theorem now says that complete metric spaces are of the second category.

Open Mapping Theorem

Theorem (Open Mapping Theorem)**.** *Let X and Y be Banach spaces, with $T \in \mathcal{B}(X, Y)$. If T is onto, then T is open, namely $T(G)$ is open for all G open.*

Corollary. *Let X and Y be Banach spaces, $T : X \to Y$ linear. Then T is one-to-one and has closed range if and only if T is bounded below.*

Theorem. *If X and Y are Banach spaces and $T : X \to Y$ is linear, then T is bounded if and only if it has a closed graph.*

As an example, let μ be a finite measure on $(X, \mathcal{M})$, with $1 \leq p < \infty$. Suppose that $\varphi f \in L^p(\mu)$ for all $f \in L^p(\mu)$. Then $\varphi \in L^\infty(\mu)$, proved using the closed graph theorem on M_φ

Definition. Let X be a Banach space with subspace $\mathcal{M}$. We say that $\mathcal{M}$ is complemented in X if there is a closed subspace $\mathcal{N}$ of X such that $\mathcal{M} \cap \mathcal{N} = \{0\}$ and $\mathcal{M} + \mathcal{N} = X$.

Theorem.

(i) If $\mathcal{M}$ and $\mathcal{N}$ are complementary subspaces of X, then there exists $E \in \mathcal{B}(X)$ with $E^2 = E$ such that $\mathcal{M} = \operatorname{ran} E$ and $\mathcal{N} = \ker E$.

(ii) If $E \in \mathcal{B}(X)$, $E^2 = E$, then $\operatorname{ran} E$ is closed and is complementary to $\ker E$.

As additional facts, if $\mathcal{M} \subseteq X$ s a finite dimensional subspace, then it is complemented. Similarly, if $\mathcal{M}$ is a subspace with $X/\mathcal{M}$ finite dimensional, then $\mathcal{M}$ is complemented. The Hardy spaces H^p are complemented in L^p for $1 < p < \infty$, but not for H^1.

Recall that we had the Fourier transform $F : L^2(\mathbb{T}) \to \ell^2$ given by $F(f) = \hat{f}$. We can also define the Fourier transform for objects in L^1, such that $F : L^1 \to \ell^\infty$ is bounded and linear.

Theorem (Riemann-Lebesgue). *If $f \in L^1(\mathbb{T})$, then $\hat{f}(n) \to 0$ as $n \to \pm\infty$.*

Proof. We say that $p(e^{it}) = \sum_{n=-N}^{N} a_n e^{int}$ is a trigonometric polynomial. The trigonometric polynomials are dense in $C(\mathbb{T})$, and $C(\mathbb{T})$ is dense in $L^1(\mathbb{T})$. Consequently, these polynomials are dense in $L^1(\mathbb{T})$ under the $L^1(\mathbb{T})$ norm.

Let $f \in L^1(\mathbb{T})$, and choose $\epsilon > 0$. Then there is a trigonometric polynomial p such that $\|f - p\| < \epsilon$. Since p is just a polynomial, there exists N such that $\widehat{p}(n) = 0$ for all $|n| > N$. Then $|\widehat{f}(n)| \leq |\widehat{f}(n) - \widehat{p}(n)| + |\widehat{p}(n)| \leq \|f - p\| < \epsilon$, so that $\widehat{f}(n) \to 0$ as $|n| \to \infty$. ■

Consequently, $F : L^1 \to c_0 \subseteq \ell^\infty$. We could then ask, is F onto? Suppose that it is, then there exists $c > 0$ such that $\|F(f)\|_\infty \geq c\|f\|$. If we take $f_n(e^{it}) = \sum_{k=-n}^{n} e^{ikt}$, then $\|F(f_n)\|_\infty = 1$ but $\|f_n\| \to \infty$. Therefore, F cannot be onto ℓ^∞.

Subnormal Operators

Theorem. *Let T be a self-adjoint operator with cyclic vector x_0 with $\|x_0\| = 1$. Then there exists $\mu \geq 0$ and $\|\mu\| = 1$ on $\sigma(T) \subseteq \mathbb{R}$ such that $\|p(T)x_0\|^2 = \int_{\sigma(T)} |p|^2 d\mu$. Furthermore, T is unitarily equivalent to M_t on $L^2(\mu)$.*

If T is self adjoint and $\mathcal{M}$ is an invariant subspace, then $\mathcal{M}$ is necessarily reducing. Furthermore, $T_1 = T|_{\mathcal{M}}$ will be self adjoint.

Definition. An operator T has a $*$-cyclic vector x_0 if $\{p(T,T^*)x_0\}$ is dense in $\mathcal{H}$.

Theorem. *If T is normal with a $*$-cyclic vector x_0, where $\|x_0\| = 1$, then there exists $\mu \geq 0$ and $\|\mu\| = 1$ on $\sigma(T) \subseteq \mathbb{C}$ such that $\|p(T,T^*)x_0\|^2 = \int_{\sigma(T)} |p(z,\overline{z})|^2 d\mu$. Furthermore, T is unitarily equivalent to M_z on $L^2(\mu)$.*

Definition. We say that $S \in \mathcal{B}(\mathcal{H})$ is subnormal if there exists a Hilbert space $\mathcal{K}$ with $\mathcal{H} \subseteq \mathcal{K}$ and a normal operator $N \in \mathcal{B}(\mathcal{K})$ such that $S = N|_{\mathcal{H}}$.

Theorem. *If S is subnormal and has cyclic vector x_0 with $\|x_0\| = 1$, then there exists a measure μ on $K \subseteq \mathbb{C}$ compact, with $\|\mu\| = 1$, such that $\|p(S)x_0\|^2 = \int_K |p|^2 d\mu$ and S is unitarily equivalent to M_z on $P^2(\mu) = \overline{\{p : p(z) = \sum_{i=1}^n a_i z^i\}}$ in $L^2(\mu)$. Furthermore, $\{p(S)x_0\}$ is dense in $\mathcal{H}$.*

Principle of Uniform Boundedness

Theorem. *Let $\mathcal{A} \subseteq \mathcal{B}(X,Y)$ where X is a Banach space. Suppose for all $x \in X$, $\sup\{\|Ax\|_Y : A \in \mathcal{A}\} < \infty$. Then there exists $M > 0$ such that $\sup\{\|A\| : A \in \mathcal{A}\} \leq M < \infty$.*

Proof. For $x \in X$, set $\varphi(x) = \sup\{\|Ax\| : A \in \mathcal{A}\}$ and $E_n = \{x \in X : \varphi(x) \leq n\}$. Then $X = \bigcup_{n=1}^{\infty} E_n$, and E_n is closed in X. Let $x_k \in E_n$ such that $x_k \to x$ in X. Then for al $A \in \mathcal{A}$ we have that $\|Ax_k\| \leq \varphi(x_k) \leq n$, so that $Ax_k \to Ax$. This implies that $\|Ax\| \leq n$, so that $x \in E_n$. By the Baire category theorem, there exists some n_0 such that the interior of E_n is nonzero.

Take $x_0 \in E_n$ and $\epsilon > 0$ so that $B(x_0, \epsilon) \subseteq E_n$. Let $y \in X$ so that $\|y\| < 1$, then $x_0 + \epsilon y \in B(x_0, \epsilon)$ so that $\|A(x_0 - \epsilon y)\| \leq n_0$ for all $A \in \mathcal{A}$. But $y = \frac{1}{\epsilon}(\epsilon y + x_0) - \frac{1}{\epsilon}x_0$, and so $\|Ay\| \leq \frac{1}{\epsilon}n_0 + \frac{1}{\epsilon}\|Ax_0\| = M$. ■

Corollary (Banach-Steinhaus Theorem). *If X ia a Banach space, and if $A_n \in \mathcal{B}(X,Y)$, and if for all $x \in X$ there exists some $y \in Y$ such that $A_n x \to y$, then there exists $A \in \mathcal{B}(X,Y)$ such that $A_n x \to Ax$ for all $x \in X$.*

Now consider the issue of Fourier series convergence. When the Fourier series is used on L^2, the convergence of the series is understood to be in the L^2 norm, but this is distinct from almost everywhere convergence. In fact, on L^1 it was proved by Kolmogorov that there exists a function that diverges almost everywhere. It was later shown by Carleson (for L^2) and Hunt (for L^p with $p > 1$) that the Fourier series does converge almost everywhere.

A similarly question is that if $f \in C(\mathbb{T})$, then does the Fourier series of f converge everywhere on $\mathbb{T}$? The answer will turn out to be no. Suppose $f \in C(\mathbb{T})$ and $\sup_N |S_N(f)(1)| < \infty$. Recall that

$$S_N(f)(1) = \int_{-\pi}^{\pi} f(e^{it}) \sum_{n=-N}^{N} e^{-int} \frac{dt}{2\pi} = \int_{-\pi}^{\pi} f(e^{it}) D_N(t) \frac{dt}{2\pi}$$

Where $D_n(t) = \frac{\sin\left(\left(n+\frac{1}{2}\right)t\right)}{\sin\frac{t}{2}}$, so that $\int |D_n| dt \to \infty$. We also have that $|S_n(f)(1)| \leq \|f\|_\infty \|D_n\|_{L^1}$, and so $S_N \in \mathcal{B}(C(\mathbb{T}), \mathbb{C})$. By the principle of uniform boundedness, $\|S_N\|_{C(\mathbb{T})^*} \leq M$ for all N. Furthermore, $\|S_n\|_{C(\mathbb{T})^*} \leq \|D_N\|_{L^1}$. We need a $g_n \in C(\mathbb{T})$ with $\|g_n\| \leq 1$ such that $S_N(g_n) \to \int |D_n| \frac{dt}{2\pi}$.

Taking f to be 1 if $D_N(e^{it}) \geq 0$ and -1 if $D_n(e^{it}) < 0$, we have that $f D_N = |D_N|$. By Lusin's theorem, there exists $g_n \in C(\mathbb{T})$ such that $|g_n(e^{it})| \leq \sup |f(e^{it})| = 1$, with $g_n \to f$ almost everywhere. Then by dominated convergence, $\int g_n D_N \to \int f D_N = \int |D_N|$, which diverges.

Recall the Hardy space $H^1(\mathbb{T})$, which is the subspace of $L^1(\mathbb{T})$ that has only nonnegative Fourier coefficients. The space $H^1(\mathbb{T})$ is closed in $L^1(\mathbb{T})$. We claim that $H^1(\mathbb{T})$ is not complemented in $L^1(\mathbb{T})$. We let P be the Szego projection from $L^1(\mathbb{T})$ onto $H^1(\mathbb{T})$, which truncates the negative Fourier terms. For $r < 1$ we define $f_r(e^{it}) = \sum_{n=-\infty}^{\infty} r^{|n|} e^{int} = \frac{1}{1-re^{it}} + \frac{1}{1-re^{-it}} - 1 = \frac{1-r^2}{|1-re^{it}|^2}$. We call this the Poisson kernel. Note that $\|f_r\|_1 = 1$ for all $r < 1$.

We want to show that $H^1(\mathbb{T})$ is not complemented because there is no bounded projection onto it. We do this by showing that the Szego projection is unbounded. This follows directly from Fatou's lemma,

$$\infty = \int \frac{1}{|1-e^{it}|} \frac{dt}{2\pi} \leq \liminf_{r \to 1} \int \frac{1}{|1-re^{it}|} \frac{dt}{2\pi} = \liminf \|Pf_r\|_1$$

Topological Vector Spaces

Definition. Let X be a vector space with a topology. We say that X is a topological vector space if the operations of $(x, y) \mapsto x + y$ and $(\alpha, x) \mapsto \alpha x$ are continuous with respect to the topologies induced on $X \times X$ and $\mathbb{F} \times X$.

A seminorm on X is a function which acts like a norm, with the exception that nonzero elements of X may return zero. For instance, the function $p(x_1, x_2) = |x_1|$ is a semi-norm on $\mathbb{R}^2$. Semi-norms can typically be extended to actual norms by adding in additional terms that differentiate between elements that would normally return zero.

Let $\mathcal{P}$ be a collection of semi-norms on a vector space X. Define a topology $\tau(\mathcal{P})$ on X by considering $e = \Big\{\{x \in X : p(x - x_0) < \epsilon\} : \epsilon > 0, x_0 \in X, p \in \mathcal{P}\Big\}$ and using them as a sub-base.

Proposition. *A net $x_\alpha \to x$ converges in $\tau(\mathcal{P})$ if and only if $p(x_\alpha - x) \to 0$ for all $p \in \mathcal{P}$.*

Proof. Suppose $x_\alpha \to x$ in $\tau(\mathcal{P})$. Then for any $\epsilon > 0$ and any $p \in \mathcal{P}$ there exists α_0 such that for all $\alpha \leq \alpha_0$ we have that $p(x_\alpha - x) < \epsilon$, so that $p(x_\alpha - x) \to 0$.

Next, suppose that $p(x_\alpha - x) \to 0$ for all $p \in \mathcal{P}$. Let U be open in $\tau(\mathcal{P})$, with $x \in U$. Then there exists $p_1, \ldots, p_n$, $x_1, \ldots, x_n$, and $\epsilon_1, \ldots, \epsilon_n$ such that $p_i(x_i - x) < \epsilon_i$. By the triangle inequality, $|p_i(x_\alpha - x_i) - p_i(x - x_i)| \leq p_i(x_\alpha - x) \to 0$. Then there exists α_0 such that for all i, $p_i(x_\alpha - x_i) < \epsilon_i$ for all $\alpha \geq \alpha_i$. Then $x_\alpha \in U$ for all $\alpha \geq \alpha_0$. ∎

Proposition. *The topology $\tau(\mathcal{P})$ on the space X makes it a topological vector space.*

Definition. A topological vector space is called locally convex if its topology is generated by a family of semi-norms $\mathcal{P}$ such that $\bigcap_{p \in \mathcal{P}} \{x : p(x) = 0\} = \{0\}$.

Note that locally convex spaces are always Hausdorff. The primary example of such a construction out of semi-norms are supremal norms on open sets. If $K \subseteq \mathbb{R}^n$ is a compact set, there is a natural norm on $C(K)$ given by $\|f\| = \sup_{x \in K} |f(x)|$. On an open set $\Omega \subseteq \mathbb{R}^n$, however, it is not immediately obvious how to define a natural topology. If $\Omega \subseteq \mathbb{C}$ and we are interested in holomorphic functions on

Ω, then we can define semi-norms as the supremum over increasing compact sets $K \subseteq \Omega$, and then the induced topology is that if uniform convergence on compact subsets.

If X is a normed space, then for each $f \in X^*$ we can define the semi-norm $p_(x) = |f(x)|$. The topology induced by these semi-norms is called the weak topology on X, and is understood as convergence $x_k \to x$ when $f(x_k) \to f(x)$ for all $f \in X^*$ (or more generally for nets x_α).

Proposition. *Let X be a topological vector space, p a semi-norm on X, then the following are equivalent.*

(i) p is continuous.
(ii) The set $\{x \in X : p(x) < 1\}$ is open in X.
(iii) Zero is in the interior of $\{x \in X : p(x) < 1\}$.
(iv) Zero is in the interior of $\{x \in X : p(x) \leq 1\}$.
(v) p is continuous at zero.
(vi) There exists a continuous semi-norm q on X such that $p(x) \leq q(x)$ for all $x \in X$.

As an example, let $\mathcal{H}$ be a separable Hilbert space with orthonormal basis $\{e_n\}$, and consider the set $S = \{\sqrt{n} e_n : n \in \mathbb{N}\}$. The set S is closed in $\mathcal{H}$ in the norm topology, as demonstrated: if $x_k \in S$ is such that $x_k \to x$, then there exists k_0 such that $k > k_0$ implies that $x_k = \sqrt{n} e_n$ for some fixed n, and $x = \sqrt{n} e_n \in S$. Strangely enough, zero is in the weak closure of S but there is no sequence $x_k \in S$ that converges to zero weakly.

Suppose that zero is not in the weak closure of S, then there exists some weakly open U containing zero such that $U \cap S = \varnothing$. This further implies that there exist $y_1, \ldots, y_j \in \mathcal{H}$ and $\epsilon_1, \ldots, \epsilon_j > 0$ such that $0 \in \bigcap_{i=1}^j U_{y_i}(\epsilon_i) \subseteq U$, where $U_{y_i}(\epsilon_i) = \{x \in \mathcal{H} : |\langle x, y_i \rangle| < \epsilon_i\}$. Then for all n there exists an i so that $|\langle e_n, y_i \rangle|^2 \geq \frac{\epsilon_i^2}{n}$. Then if $\delta = \min \epsilon_i^2 > 0$, we have tat $\sum_{i=1}^j \|y_i\|^2 \geq \sum_{n=1}^\infty \frac{\delta}{n} = \infty$.

Consequently, the idea of elements converging to form a topology does not necessarily apply to weak topologies. On the other hand, we will see that convex subsets of normed spaces are closed in norm if and only if they are weakly closed.

Theorem. *If $f : X \to \mathbb{F}$ is linear, then it is continuous in the weak topology on X if and only if $f \in X^*$.*

In particular, the weak topology on a vector space is the smallest topology such that the continuous linear functionals are still continuous.

Theorem. *Let X be a topological vector space, with $f : X \to \mathbb{F}$ linear. Then the following are equivalent:*

(i) *f is continuous.*
(ii) *f is continuous at zero.*
(iii) *f is continuous at some point.*
(iv) *$\ker f$ is closed.*
(v) *$\ker f$ is not dense in X.*
(vi) *$x \mapsto |f(x)|$ defines a continuous semi-norm on X.*

If X is a locally convex space defined by a family of semi-norms, then these are all equivalent to the existence of polynomials $p_1, \dots, p_n$ and $\alpha_1, \dots, \alpha_n > 0$ such that $|f(x)| \leq \sum_{i=1}^{n} \alpha_i p_i(x)$.

Recall the Minkowski functional $g(x) = \inf\{t > 0 : x \in tG\}$ for an open, convex set $G \subseteq X$ containing zero in a topological vector space.

Theorem. *The Minkowski functional is sublinear and continuous at zero, and $G = \{x \in X : g(x) < 1\}$.*

Corollary. *Let X be a topological vector space, G open, nonempty, convex, and $0 \notin G$. Then there exists a closed hyperplane $\mathcal{M} \subseteq X$ such that $\mathcal{M} \cap G = \varnothing$.*

Theorem. *Let X be a topological vector space, and G an open, nonempty, convex subset. If Y is an affine subspace such that $Y \cap G = \varnothing$, then there exists an affine closed hyperplane $\mathcal{M}$ such that $Y \subseteq \mathcal{M}$.*

As an example, let $\mathcal{H}$ be a separable Hilbert space with orthonormal basis $\{e_n\}$, and consider the set $S = \{\sqrt{n}e_n : n \in \mathbb{N}\}$. The set S is closed in $\mathcal{H}$ in the norm topology, as demonstrated: if $x_k \in S$ is such that $x_k \to x$, then there exists k_0 such that $k > k_0$ implies that $x_k = \sqrt{n}e_n$ for some fixed n, and $x = \sqrt{n}e_n \in S$. Strangely enough, zero is in the weak closure of S but there is no sequence $x_k \in S$ that converges to zero weakly.

Suppose that zero is not in the weak closure of S, then there exists some weakly open U containing zero such that $U \cap S = \varnothing$. This further implies that there exist

$y_1, \ldots, y_j \in \mathcal{H}$ and $\epsilon_1, \ldots, \epsilon_j > 0$ such that $0 \in \bigcap_{i=1}^{j} U_{y_i}(\epsilon_i) \subseteq U$, where $U_{y_i}(\epsilon_i) = \{x \in \mathcal{H} : |\langle x, y_i \rangle| < \epsilon_i\}$. Then for all n there exists an i so that $|\langle e_n, y_i \rangle|^2 \geq \frac{\epsilon_i^2}{n}$. Then if $\delta = \min \epsilon_i^2 > 0$, we have tat $\sum_{i=1}^{j} \|y_i\|^2 \geq \sum_{n=1}^{\infty} \frac{\delta}{n} = \infty$.

Consequently, the idea of elements converging to form a topology does not necessarily apply to weak topologies. On the other hand, we will see that convex subsets of normed spaces are closed in norm if and only if they are weakly closed.

Theorem. *If $f : X \to \mathbb{F}$ is linear, then it is continuous in the weak topology on X if and only if $f \in X^*$.*

In particular, the weak topology on a vector space is the smallest topology such that the continuous linear functionals are still continuous.

Theorem. *Let X be a topological vector space, with $f : X \to \mathbb{F}$ linear. Then the following are equivalent:*

(i) f *is continuous.*
(ii) f *is continuous at zero.*
(iii) f *is continuous at some point.*
(iv) $\ker f$ *is closed.*
(v) $\ker f$ *is not dense in* X*.*
(vi) $x \mapsto |f(x)|$ *defines a continuous semi-norm on* X*.*

If X is a locally convex space defined by a family of semi-norms, then these are all equivalent to the existence of polynomials $p_1, \ldots, p_n$ and $\alpha_1, \ldots, \alpha_n > 0$ such that $|f(x)| \leq \sum_{i=1}^{n} \alpha_i p_i(x)$.

Recall the Minkowski functional $g(x) = \inf\{t > 0 : x \in tG\}$ for an open, convex set $G \subseteq X$ containing zero in a topological vector space.

Theorem. *The Minkowski functional is sublinear and continuous at zero, and $G = \{x \in X : g(x) < 1\}$.*

Corollary. *Let X be a topological vector space, G open, nonempty, convex, and $0 \notin G$. Then there exists a closed hyperplane $\mathcal{M} \subseteq X$ such that $\mathcal{M} \cap G = \varnothing$.*

Theorem. *Let X be a topological vector space, and G an open, nonempty, convex subset. If Y is an affine subspace such that $Y \cap G = \varnothing$, then there exists an affine closed hyperplane $\mathcal{M}$ such that $Y \subseteq \mathcal{M}$.*

On $\mathbb{R}^n$ all norms are equivalent, and so there is essentially one kind of topology that we want to associate with it. However, different norms give rise to differently shaped unit balls, so it might be natural to ask what kind of shapes these can be. A unit ball must be open, convex, bounded, and balanced, and the converse is true: every open, convex, bounded, and balanced set gives rise to a unit ball through the norm defined by the Minkowski functional on the set.

What if we move away from normed spaces? The spaces $L^p[0,1]$ for $0 < p < 1$ can be equipped with a metric $d(f,g) = \int_0^1 |f-g|^p dx$ is a topological vector space, complete and translation invariant. However, if $G \subseteq L^p[0,1]$ is open and convex, then it is either empty or else $G = L^p[0,1]$. Consequently, the dual $(L^p[0,1])^*$ is actually empty.

As another example, consider $f \in L^p[0,1]$ for $1 < p < \infty$, and consider the question of finding $\inf \|f-q\|_p$ where q is a polynomial of degree at most n. This is the same as $\|[f]\|_{L^p[0,1]/P_n}$, which is precisely $\sup |\varphi([f])|$ where $\|\varphi\| \leq 1$ and $\varphi \in (L^p[0,1]/P_n)^* \cong P_n^\perp \subseteq L^q$. This is $\sup \left| \int_0^1 fh \right|$ where $h \in L^q[0,1]$, $\|h\|_q \leq 1$, and $\int_0^1 h(x)x^k dx = 0$ for all $k \leq n$.

Theorem. *Let X be a topological vector space. Suppose $G \subseteq X$ is open, convex, and nonempty, with $0 \notin G$. Then there exists a linear functional $f : X \to \mathbb{F}$ continuous and linear, such that $\ker f \cap G = \varnothing$.*

Corollary. *Let X be a topological vector space, $G \subseteq X$ open, convex, and nonempty. Then there exists a subspace $\mathcal{M} \subseteq X$ such that $\mathcal{M} \cap G = \varnothing$.*

Corollary. *Suppose X is a locally convex space, with $A \subseteq X$. Then the closed linear span of A is equal to $\bigcap_{f \in X^*, A \subseteq \ker f} \ker f = \bigcap_{A \subseteq \mathcal{M}} \mathcal{M}$ where the $\mathcal{M}$ are hperplanes.*

Corollary. *If (X, τ_1) and (X, τ_2) are two localy convex spaces, with $(X, \tau_1)^* = (X, \tau_2)^*$, then for all $A \subseteq X$ the closed linear span of A with respect to the two topologies are the same. Furthermore, if A is convex in X, then A is closed in one topology if and only if it is closed in the other.*

Definition. If $x \in X$ and $x^* \in X^*$, write $\langle x, x^* \rangle = x^*(x)$.

(i) $\sigma(X, X^*)$ is the weak topology on X, defined by $p_{x^*}(x) = |\langle x, x^* \rangle|$ and $\mathcal{P} = \{p_{x^*} : x^* \in X^*\}$.

(ii) $\sigma(X^*, X)$ is the weak-$*$ topology on X^* defiend similarly by $p_x(x^*) = |\langle x, x^* \rangle|$.

Locally Convex Spaces

Recall that if X is locally convex and $x \in X$, $x^* \in X^*$, then we write $\langle x, x^* \rangle = x^*(x)$. Although this looks like an inner product, we should note that it is linear in both components. We use $\sigma(X, X^*)$ to denote the weak topology on X, and $\sigma(X^*, X)$ to denote the weak-$*$ topology.

Theorem. *Let X be equipped with the weak topology, then its dual is X^* under the weak-$*$ topology.*

Theorem. *If X^* is equipped with the weak-$*$ topology, then its dual is X.*

Consider $L^\infty(\mathbb{D})$ using Lebesgue measure on $\mathbb{D}$. Let $A = \{\sum_{n=0}^N a_n z^n : N < \infty, a_n \in \mathbb{C}\}$ be the analytic polynomials. Then $\overline{A}$ (closed in the $L^\infty(\mathbb{D})$ norm) is equal to $C(\overline{\mathbb{D}}) \cap \mathrm{Hol}(\mathbb{D})$, and is a convex subspace of $L^\infty(\mathbb{D})$. We claim that the closure $\overline{A}$ in the $L^\infty(\mathbb{D})$ norm is the same as the closure $\overline{A}$ is the weak topology on $L^\infty(\mathbb{D})$. However, $(L^1(\mathbb{D})^* \cong L^\infty(\mathbb{D})$, and the closure $\overline{A}$ under the weak-$*$ topology on $L^\infty(\mathbb{D})$ is $H^\infty(\mathbb{D})$, which is strictly bigger than $C(\overline{\mathbb{D}}) \cap \mathrm{Hol}(\mathbb{D})$.

Note that if $\varphi \in H^\infty(\mathbb{D})$, then for any $r < 1$ we have that $\varphi_r(z) = \varphi(rz)$ is in $C(\overline{\mathbb{D}}) \cap \mathrm{Hol}(\mathbb{D})$. Furthermore, it is not hard to show that $\varphi_r \to \varphi$ in the weak-$*$ topology. Consequently, the closure $\overline{A}$ in the normal and weak topologies of $L^\infty(\mathbb{D})$ is dense in the closure $\overline{A}$ in the weak-$*$ topology.

Definition. We say that $A^0 = \{x^* \in X^* : \forall x \in A, |\langle x, x^* \rangle| \leq 1\}$ is the polar of A, and if ${}^0B = \{x \in X : \forall x^* \in B, |\langle x, x^* \rangle| \leq 1\}$ then ${}^0(A^0)$ is the bipolar of A. If $A = \mathcal{M} \subseteq X$ is a suspace, then $\mathcal{M}^0 = \mathcal{M}^\perp$. Note that if B_X and B_{X^*} are the unit balls on X and X^*, then $B_X^0 = B_{X^*}$.

Since $X \hookrightarrow X^{**}$, it follows that $B_X \hookrightarrow B_{X^{**}}$. Consequently, $({}^0B_X)^0 = B_{X^{**}}$. We will see that B_X is in fact dense in $B_{X^{**}}$.

If $A \subseteq X$ and $B \subseteq X^*$ then both A^0 and ${}^0B$ are convex and balanced, and we have that $A \subseteq^0 (A^0)$ and $B \subseteq ({}^0B)^0$. It actually follows that $A = ({}^0(A^0))^0$.

Theorem (Bipolar Theorem)**.** *Let X be a locally convex space. Then ${}^0(A^0)$ is the closed, convex, balanced hull of A. Similarly, $({}^0B)^0$ is the weak-$*$ closed, convex, balanced hull of B.*

Proof. ■

Corollary. *Let X be a normed space. Then B_X is weak-$*$ dense in $B_{X^{**}}$. Furthemore, X is weak-$*$ dense in X^{**}.*