A Survey of Recent Results on the Hardy Space of Dirichlet Series

Gregory Zitelli

University of Tennessee, Knoxville

September 2013

Notation

$\mathbb{D}$ is the open unit disc.

$\mathbb{T}$ is the unit circle.

$\mathbb{C}_\rho$ is the right half plane with real part $> \rho$.

$\mathbb{C}_+ = \mathbb{C}_0$.

Hardy Spaces

We begin with the standard definition of the Hardy-Hilbert space on $\mathbb{D}$, a Hilbert space of holomorphic functions on $\mathbb{D}$ with a square summable power series.

$$H^2(\mathbb{D}) = \left\{ f(z) = \sum_{n=0}^{\infty} a_n z^n : \quad \sum_{n=0}^{\infty} |a_n|^2 < \infty \right\}$$

where the inner product is given by

$$\langle f, g \rangle_{H^2(\mathbb{D})} = \left\langle \sum_{n=0}^{\infty} a_n z^n, \sum_{n=0}^{\infty} b_n z^n \right\rangle_{H^2(\mathbb{D})} = \sum_{n=0}^{\infty} a_n \overline{b_n}$$

Hardy Spaces

This formulation of the Hardy-Hilbert space $H^2(\mathbb{D})$ is useful because it emphasizes the canonical equivalence of $H^2(\mathbb{D})$ and $\ell^2(\mathbb{N})$, namely

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \sim (a_0, a_1, a_2, \ldots)$$

Hardy Spaces

Interestingly, the Hardy-Hilbert space norm is equivalent to a growth condition on the radial boundary values of its functions, so that if $f(z) = \sum_{n=0}^{\infty} a_n z^n$,

$$\|f\|_{H^2(\mathbb{D})}^2 = \sum_{n=0}^{\infty} |a_n|^2 = \sup_{0 \leq r < 1} \int_0^{2\pi} |f(re^{it})|^2 \frac{dt}{2\pi}$$

Hardy Spaces

Hardy-Hilbert space functions living on $\mathbb{D}$ have nontangential boundary values almost everywhere on $\mathbb{T}$, allowing us to extend functions $f \in H^2(\mathbb{D})$ to functions $\tilde{f} \in H^2(\mathbb{T}) \subseteq L^2(\mathbb{T})$, where

$$\|f\|^2_{H^2(\mathbb{D})} = \sup_{0 \le r < 1} \int_0^{2\pi} |f(re^{it})|^2 \frac{dt}{2\pi} = \int_0^{2\pi} |\tilde{f}(e^{it})|^2 \frac{dt}{2\pi} = \|\tilde{f}\|^2_{H^2(\mathbb{T})}$$

and $H^2(\mathbb{T})$ is the subspace of $L^2(\mathbb{T})$ whose elements have only nonnegative Fourier coefficients.

Hardy Spaces

For general $1 < p < \infty$, we can form the Hardy space H^p similarly, with

$$\|f\|_{H^p(\mathbb{D})}^p = \sup_{0 \le r < 1} \int_0^{2\pi} |f(re^{it})|^p \frac{dt}{2\pi} = \int_0^{2\pi} |\tilde{f}(e^{it})|^p \frac{dt}{2\pi} = \|\tilde{f}\|_{H^p(\mathbb{T})}^p$$

Here $H^p(\mathbb{D}) \cong H^p(\mathbb{T}) \subseteq L^p(\mathbb{T})$. We treat H^p as both $H^p(\mathbb{D})$ and $H^p(\mathbb{T})$ interchangeably.

Hardy Spaces

The Hardy spaces H^p can be thought of both as the holomorphic functions on $\mathbb{D}$ which satisfy a growth condition on the boundary, and the nontangential boundary functions which live inside of the L^p space on that boundary.

Hardy Spaces

There are three important properties posessed by the Hardy space H^2 as a Hilbert space which we would like to contrast:

- Reproducing kernels k_λ
- Zero sets $\{z_n\}$
- Multiplier algebra $\mathcal{M}(H^2)$

Reproducing Kernels for the space $H^2(\mathbb{D})$

Point evaluations are bounded linear functionals on $H^2(\mathbb{D})$, and can therefore be expressed as inner products with appropriate reproducing kernels.

If $\lambda \in \mathbb{D}$ and $f(z) = \sum_{n=0}^{\infty} a_n z^n \in H^2(\mathbb{D})$, then the reproducing kernel $k_\lambda(z) = \sum_{n=0}^{\infty} \overline{\lambda^n} z^n$ is such that

$$f(\lambda) = \sum_{n=0}^{\infty} a_n \lambda^n = \sum_{n=0}^{\infty} a_n \overline{(\overline{\lambda^n})} = \langle f, k_\lambda \rangle_{H^2(\mathbb{D})}$$

Note that $\sum_{n=0}^{\infty} \left|\overline{\lambda^n}\right|^2 < \infty$, so that $k_\lambda \in H^2(\mathbb{D})$.

Zero Sets of $H^2(\mathbb{D})$

Given a sequence of points $\{z_n\} \subseteq \mathbb{D}$, there is a nontrivial function $f \in H^2(\mathbb{D})$ which vanishes at each z_n if and only if $\{z_n\}$ satisfies the Blaschke condition

$$\sum_{n=1}^{\infty}(1 - |z_n|) < \infty$$

Multiplier Algebra of $H^2(\mathbb{D})$

The multipliers $\mathcal{M}(H^2(\mathbb{D}))$ are precisely $H^\infty(\mathbb{D})$, the bounded holomorphic functions on the disc.

Hardy Spaces in Half Planes

There is similarly a Hardy space for the half plane $\mathbb{C}_+$ using the growth condition on the imaginary line

$$H^p(\mathbb{C}_+) = \left\{ f \in \mathrm{Hol}(\mathbb{C}_+) : \quad \sup_{\sigma > 0} \int_{-\infty}^{\infty} |f(\sigma + it)|^p dt < \infty \right\}$$

One can also define spaces $H^p(\mathbb{C}_\rho)$ for arbitrary ρ. Like the Hardy space $H^2(\mathbb{D})$, $H^2(\mathbb{C}_+)$ has well understood reproducing kernels, zero sets, and a multiplier algebra.

Reproducing Kernels for the space $H^2(\mathbb{C}_+)$

Point evaluations are bounded linear functionals on $H^2(\mathbb{C}_+)$, and can therefore be expressed as inner products with appropriate reproducing kernels.

If $\lambda \in \mathbb{C}_+$ and $f \in H^2(\mathbb{C}_+)$, then the reproducing kernel $k_\lambda(z) = \frac{1}{z+\bar{\lambda}}$ is such that

$$f(\lambda) = \langle f, k_\lambda \rangle_{H^2(\mathbb{D})}$$

Note that $\sup_{x>0} \int_{-\infty}^{\infty} \left| \frac{1}{x+it+\bar{\lambda}} \right|^2 dt < \infty$, so that $k_\lambda \in H^2(\mathbb{D})$.

Zero Sets of $H^2(\mathbb{C}_+)$

Given a sequence of points $\{z_n\} \subseteq \mathbb{C}_+$, there is a nontrivial function $f \in H^2(\mathbb{C}_+)$ which vanishes at each z_n if and only if $\{z_n\}$ satisfies the following condition

$$\sum_{n=1}^{\infty} \frac{x_n}{1 + |z_n|^2} < \infty$$

where $z_n = x_n + iy_n$. If the sequence $\{z_n\}$ is bounded, then we recover a Blaschke-type condition

$$\sum_{n=1}^{\infty} x_n < \infty$$

Multiplier Algebra of $H^2(\mathbb{C}_+)$

The multipliers $\mathcal{M}(H^2(\mathbb{C}_+))$ are precisely $H^\infty(\mathbb{C}_+)$, the bounded holomorphic functions on the right half plane.

Dirichlet Series

A Dirichlet series is a series of the form $f(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$. We write $s = \sigma + it$, and let Ω_ρ denote the half plane with real part $> \rho$.

Unlike power series, the "radius" of convergence and absolute convergence may be different.

Dirichlet Series

For a particular $f(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$, we write

$$\sigma_c(f) = \inf \left\{ \Re(s) : \sum_{n=1}^{\infty} \frac{a_n}{n^s} \text{ converges} \right\}$$

$$\sigma_b(f) = \inf \left\{ \rho : \sum_{n=1}^{\infty} \frac{a_n}{n^s} \text{ converges to a bounded function in } \Omega_\rho \right\}$$

$$\sigma_u(f) = \inf \left\{ \rho : \sum_{n=1}^{\infty} \frac{a_n}{n^s} \text{ converges uniformly in } \Omega_\rho \right\}$$

$$\sigma_a(f) = \inf \left\{ \Re(s) : \sum_{n=1}^{\infty} \frac{a_n}{n^s} \text{ converges absolutely} \right\}$$

$$\sigma_c \leq \sigma_b = \sigma_u \leq \sigma_a \leq \sigma_c + 1$$

Riesz-Basis

The Riesz-Fischer theorem states that $\varphi(x) = \sqrt{2}\sin(\pi x)$ can be dilated to form a complete orthonormal basis

$$\left\{\sqrt{2}\sin(\pi x), \sqrt{2}\sin(\pi 2x), \ldots\right\} = \{\varphi(x), \varphi(2x), \ldots\}$$

for $L^2(0,1)$.

Riesz-Basis

A natural extension of the theorem would be to ask which functions φ can take the place of $\sin$ so that $\{\varphi(nx)\}_{n\geq 1}$ forms an orthonormal basis for $L^2(0,1)$ under an equivalent norm. Such a sequence is called a Riesz basis.

Riesz-Basis

The characterization of Riesz-type sets which are complete in $L^2(0,1)$ was characterized by Beurling in 1945, by transforming the expression

$$\varphi(x) = \sum_{n=1}^{\infty} a_n \sqrt{2} \sin(n\pi x)$$

into

$$S\varphi(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$$

and analyzing properties of the analytic $S\varphi$. In 1995, Hedenmalm, Lindqvist, and Seip solved the Reisz-basis problem completely by exploiting a Hilbert space of analytic functions of this form.

Riesz-Basis

Theorem (Hedenmalm, Lindqvist, Seip)

The system $\{\varphi(nx)\}_{n\geq 1}$ *is a Reisz basis for* $L^2(0,1)$ *if and only if* $S\varphi$ *and* $1/S\varphi$ *are in the multiplier algebra* $\mathcal{M}(\mathcal{H}^2)$.

Hardy Space of Dirichlet Series

The proof used the Hardy space of Dirichlet series (or Hardy-Dirichlet space),

$$\mathcal{H}^2 = \left\{ f(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s} : \quad \sum_{n=1}^{\infty} |a_n|^2 < \infty \right\}$$

along with the characterization of the multipliers $\mathcal{M}(\mathcal{H}^2)$ of the space. The paper also further established Bohr's work on the connection between the Hardy-Dirichlet space $\mathcal{H}^2$ and the Hardy space of the infinite polycircle $H^2(\mathbb{T}^\infty)$.

Hardy Space of Dirichlet Series

The work by Hedehmalm, Lindqvist, and Seip inspired an investigation of the space $\mathcal{H}^2$ and various related spaces over the next 15 years. Contributors in analysis include Aleman, Andersson, Bayart, McCarthy, Olsen, Saskman.

Topics included

- Multipliers
- Reproducing kernels
- Zero sets for $\mathcal{H}^2$ and related $\mathcal{H}^p$ spaces
- Boundary behavior (What happens on the line $\sigma = 1/2$? Can you look at behavior of the function on the line $\sigma = 0$?)
- Connections with the infinite polycircle $H^p(\mathbb{T}^\infty)$
- Carleson measures

Hardy Space of Dirichlet Series

The condition on the Hardy-Dirichlet space ensures that all functions $f \in \mathcal{H}^2$ have $\sigma_a \leq \frac{1}{2}$, as by Cauchy-Schwarz

$$\left(\sum_{n=1}^{\infty} \left|\frac{a_n}{n^s}\right|\right)^2 \leq \sum_{n=1}^{\infty} |a_n|^2 \sum_{n=1}^{\infty} \left|\frac{1}{n^{2s}}\right|$$

This bound is sharp, since

$$\sum_{n=1}^{\infty} \frac{1}{n^{s-1/2}\log(n+1)} = \sum_{n=1}^{\infty} \frac{\sqrt{n}/\log(n+1)}{n^s} \in \mathcal{H}^2$$

so $\mathcal{H}^2 \subset \mathrm{Hol}(\mathbb{C}_{1/2})$.

Hardy Space of Dirichlet Series

The Hardy-Dirichlet space $\mathcal{H}^2$ clearly mirrors the classical Hardy space H^2 of the disc

$$\mathcal{H}^2 = \left\{ f(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s} : \quad \sum_{n=1}^{\infty} |a_n|^2 < \infty \right\}$$

$$H^2 = \left\{ f(z) = \sum_{n=0}^{\infty} a_n z^n : \quad \sum_{n=0}^{\infty} |a_n|^2 < \infty \right\}$$

Reproducing Kernels in $\mathcal{H}^2$

The reproducing kernels on $\mathcal{H}^2$ are actually quite easy to define, since if $f(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s} \in \mathcal{H}^2$ and $\lambda \in \mathbb{C}_{1/2}$ then

$$f(\lambda) = \sum_{n=1}^{\infty} \frac{a_n}{n^{\lambda}} = \sum_{n=0}^{\infty} a_n e^{-\lambda \log(n)} = \sum_{n=0}^{\infty} a_n \overline{e^{-\overline{\lambda} \log(n)}}$$

$$= \sum_{n=0}^{\infty} a_n \overline{\left(\frac{1}{n^{\overline{\lambda}}} \right)} = \left\langle f(s), \sum_{n=1}^{\infty} \frac{1}{n^{s+\overline{\lambda}}} \right\rangle_{\mathcal{H}^2}$$

so that $k_\lambda(s) = \zeta(s + \overline{\lambda})$.

Zero Sets of $\mathcal{H}^2$

Like the space $H^2(\mathbb{C}_+)$, bounded sequences $\{z_n\}$ have the same Blaschke-type condition that

$$\sum_{n=1}^{\infty}(x_n - 1/2) < \infty$$

On the other hand, Dirichlet series have strange vertical limit behavior which has made it difficult to fully classify unbounded sequences.

Almost Periodic Behavior of Dirichlet Series

If a function $f \in \mathrm{Hol}(\mathbb{C}_\rho)$, $\epsilon > 0$, then we say that t is an ϵ-translation number for f if

$$\sup_{s \in \mathbb{C}_\rho} |f(s+it) - f(s)| \leq \epsilon$$

We say that f is uniformly almost periodic if for every $\epsilon > 0$ there is a length M such that every interval of length M contains at least one ϵ-translation number for f.

Theorem

If $f \in Hol(\mathbb{C}_\rho)$ is represented by a Dirichlet series which converges uniformly in $\mathbb{C}_\rho$, then f is uniformly almost periodic.

Zero Sets of $\mathcal{H}^2$

Since all functions in $\mathcal{H}^2$ are uniformly almost periodic in $\mathbb{C}_{1/2}$, they will either have no zeros or infinitely many zeros, and those zeros may be distributed quite wildly along vertical strips.

Multiplier Algebra of $\mathcal{H}^2$

The multiplier algebra of $\mathcal{H}^2$ consist precisely of those holomorphic functions in $\mathbb{C}_+$ which are bounded and representable by a Dirichlet series. If $\mathcal{D}$ is used to denote the collection of holomorphic functions representable by a convergence Dirichlet series on some half space, then we can write

$$\mathcal{M}(\mathcal{H}^2) = H^\infty(\mathbb{C}_+) \cap \mathcal{D} = \mathcal{M}(H^2(\mathbb{C}_+)) \cap \mathcal{D}$$

"Arms-Reach" Boundary Condition

Theorem (Carlson's Lemma)

If $f(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$ *is convergent and bounded in* $\mathbb{C}_+$*, then*

$$\|f\|_{\mathcal{H}^2}^2 = \sum_{n=1}^{\infty} |a_n|^2 = \lim_{\sigma \to 0^+} \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |f(\sigma + it)|^2 dt$$

Note that σ moves all the way back to $\mathbb{C}_+$ rather than just $\mathbb{C}_{1/2}$, and requires that f be convergent and bounded in $\mathbb{C}_+$ to begin with. An interesting extension of $\mathcal{H}^2$ is as follows.

$\mathcal{H}^p$ Spaces

For $1 \leq p < \infty$ we define the space $\mathcal{H}^p$ as the closure of the Dirichlet polynomials under the norm

$$\lim_{T\to\infty}\left(\frac{1}{2T}\int_{-\infty}^{\infty}\left|\sum_{n=1}^{\infty}\frac{a_n}{n^{it}}\right|^p dt\right)^{1/p}$$

We will refer to this as the $\mathcal{H}^p$ norm. Bayart (2002) showed that the Dirichlet series which represent such functions have $\sigma_u \leq 1/2$, so that they represent holomorphic functions on $\mathbb{C}_{1/2}$.

Notes on $\mathbb{D}^\infty$ and $\mathbb{T}^\infty$

Let $\mathbb{T}^k$ be the k-dimensional polycircle, and let $p_1, \ldots, p_k$ enumerate the first k primes. Then the injection $(p_1^{it}, \ldots, p_k^{it})$ for $t \in \mathbb{R}$ has dense range in $\mathbb{T}^k$.

This means that there is a dense subset of the infinite polydisc $\mathbb{D}^\infty$ such that its elements can be expressed as $z = (p_1^{-s}, p_2^{-s}, \ldots)$, where $s = \sigma + it$ such that $t \in \mathbb{R}$ and $\sigma > 0$.

$\mathcal{H}^p$ Spaces as $H^p(\mathbb{T}^\infty)$

Let each n factor into $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}$. Setting $z = (p_1^{-s}, p_2^{-s}, \dots)$ we have

$$f(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s} = \sum_{n=1}^{\infty} a_n z_1^{\alpha_1} \cdots z_r^{\alpha_r}$$

We consider the transformation $\mathcal{D}f(z) = \sum_{n=1}^{\infty} a_n z_1^{\alpha_1} \cdots z_r^{\alpha_r}$ as being a function of $z \in \mathbb{D}^\infty$.

$\mathcal{H}^p$ Spaces as $H^p(\mathbb{T}^\infty)$

Bohr showed that in fact for Dirichlet polynomials $P(s) = \sum_{n=1}^{N} \frac{a_n}{n^s}$,

$$\begin{aligned}
\|P\|_{\mathcal{H}^p}^p &= \lim_{T\to\infty} \frac{1}{2T} \int_{-\infty}^{\infty} \left| \sum_{n=1}^{N} \frac{a_n}{n^{it}} \right|^p dt \\
&= \lim_{T\to\infty} \frac{1}{2T} \int_{-\infty}^{\infty} \left| \sum_{n=1}^{N} a_n p_1^{-it} \cdots p_k^{-it} \right|^p dt \\
&= \int_{\mathbb{T}^\infty} \left| \sum_{n=1}^{N} a_n z_1^{\alpha_1} \cdots z_k^{p_k} \right|^p dm(z) = \|\mathcal{D}P\|_{H^p(\mathbb{T}^\infty)}
\end{aligned}$$

Which follows from properties of the Kronecker flow and Birkhoff-Khintchin theorem.

Proof that $\mathcal{H}^p \subseteq \mathrm{Hol}(\mathbb{C}_{1/2})$

Let $\omega \in \mathbb{C}_{1/2}$, and $z = (2^{-\omega}, 3^{-\omega}, \ldots) \in \mathbb{D}^\infty$. Note that since $\Re(\omega) > \frac{1}{2}$, $z \in \ell^2$ as well.

If $f \in \mathcal{H}^p$, then $\mathcal{D}f \in H^p(\mathbb{T}^\infty)$. On $H^p(\mathbb{T}^\infty)$ we have the inequality

$$|f(\omega)|^p = |\mathcal{D}f(z)|^p \leq \frac{\|\mathcal{D}f\|^p_{H^p(\mathbb{T}^\infty)}}{\prod_{j=1}^\infty (1 - |z_j|^2)} = \|f\|^p_{\mathcal{H}^p} \prod_{j=1}^\infty \frac{1}{1 - |p_j^{-2\omega}|}$$

Proof that $\mathcal{H}^p \subseteq \mathrm{Hol}(\mathbb{C}_{1/2})$

$$|f(\omega)|^p = |\mathcal{D}f(z)|^p \leq \frac{\|\mathcal{D}f\|^p_{H^p(\mathbb{T}^\infty)}}{\prod_{j=1}^\infty (1-|z_j|^2)} = \|f\|^p_{\mathcal{H}^p} \prod_{j=1}^\infty \frac{1}{1-|p_j^{-2\omega}|}$$

However, Euler's identity concerning the Riemann zeta function says that the last term is precisely

$$\|f\|^p_{\mathcal{H}^p} \sum_{n=1}^\infty \frac{1}{n^{2\Re(\omega)}}$$

which is simply finite by p-series. Consequently,

$$|f(\omega)|^p \leq \|f\|^p_{\mathcal{H}^p} \zeta(2\Re(\omega))$$

Almost-Sure Properties of $\mathcal{H}^p$

An element $\chi \in \mathbb{T}^\infty$ can be thought of as a character in the sense that it acts on the prime elements in the canonical way. We define the function f_χ where $f \in \mathcal{H}^p$ is the function to be influence by the character as

$$f_\chi(s) = \sum_{n=1}^{\infty} \frac{a_n \chi(n)}{n^s}$$

Theorem

For $f \in \mathcal{H}^p$ and for almost every $\chi \in \mathbb{T}^\infty$, f_χ is a Dirichlet series which converges in $\mathbb{C}_+$.

Comparing $\mathcal{H}^p$ to $H^p(\mathbb{C}_{1/2})$

Since $\mathcal{H}^p$ are spaces of Dirichlet series which are well defined in $\mathbb{C}_{1/2}$, it is interesting to note comparisons between $\mathcal{H}^p$ and $H^p(\mathbb{C}_{1/2})$.

Theorem (Hedenmalm, Lindqvist, Seip)

If $f \in \mathcal{H}^2$ then $f(s)/s \in H^2(\mathbb{C}_{1/2})$.

In particular, this tells us that all functions in $\mathcal{H}^2$ have nontangential boundary values (as do functions in $\mathcal{H}^p$).

Comparing $\mathcal{H}^p$ to $H^p(\mathbb{C}_{1/2})$

Let $H^p_\infty(\mathbb{C}_{1/2})$ denote the uniform local H^p space of the right half plane, defined as those elements such that

$$\|f\|^p_{H^p_\infty(\mathbb{C}_{1/2})} = \sup_{y\in\mathbb{R}} \sup_{\sigma>1/2} \int_y^{y+1} |f(\sigma+it)|^p dt < \infty$$

Theorem (Bayart)

If $p \geq 2$, then $\mathcal{H}^p \subset H^p_\infty(\mathbb{C}_{1/2})$ and the injection is continuous.

Carleson Measures

Theorem (Bayart)

If $1 \leq p < \infty$ and μ is a positive measure on $\mathbb{C}_{1/2}$, and if μ is a Carleson measure for $\mathcal{H}^p$ then it is also a Carleson measure for $H^p(\mathbb{C}_{1/2})$.

Open Problems

Does $\mathcal{H}^p$ embed in $H^p_\infty(\mathbb{C}_{1/2})$ for $1 \le p < 2$?

Is there a BMOA theory for the $\mathcal{H}^p$ spaces?

Can $\mathcal{H}^2$ be factored like $H^2(\mathbb{D})$?

What kind of classification for zero-sets can be achieved in the $\mathcal{H}^p$ setting?

Thank you!