A Survey of Recent Results on the Hardy Space of Dirichlet Series

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Notation

\( \mathbb{D} \) is the open unit disc.
\( \mathbb{T} \) is the unit circle.
\( \mathbb{C}_\rho \) is the right half plane with real part \( > \rho \).
\( \mathbb{C}_+ = \mathbb{C}_0 \).
We begin with the standard definition of the Hardy-Hilbert space on $\mathbb{D}$, a Hilbert space of holomorphic functions on $\mathbb{D}$ with a square summable power series.

$$H^2(\mathbb{D}) = \left\{ f(z) = \sum_{n=0}^{\infty} a_n z^n : \sum_{n=0}^{\infty} |a_n|^2 < \infty \right\}$$

where the inner product is given by

$$\langle f, g \rangle_{H^2(\mathbb{D})} = \left\langle \sum_{n=0}^{\infty} a_n z^n, \sum_{n=0}^{\infty} b_n z^n \right\rangle_{H^2(\mathbb{D})} = \sum_{n=0}^{\infty} a_n \overline{b_n}$$
This formulation of the Hardy-Hilbert space $H^2(\mathbb{D})$ is useful because it emphasizes the canonical equivalence of $H^2(\mathbb{D})$ and $\ell^2(\mathbb{N})$, namely

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \sim (a_0, a_1, a_2, \ldots)$$
Interestingly, the Hardy-Hilbert space norm is equivalent to a growth condition on the radial boundary values of its functions, so that if \( f(z) = \sum_{n=0}^{\infty} a_n z^n \),

\[
\|f\|_{H^2(\mathbb{D})}^2 = \sum_{n=0}^{\infty} |a_n|^2 = \sup_{0 \leq r < 1} \int_{0}^{2\pi} |f(re^{it})|^2 \frac{dt}{2\pi}
\]
Hardy Spaces

Hardy-Hilbert space functions living on $\mathbb{D}$ have nontangential boundary values almost everywhere on $\mathbb{T}$, allowing us to extend functions $f \in H^2(\mathbb{D})$ to functions $\tilde{f} \in H^2(\mathbb{T}) \subseteq L^2(\mathbb{T})$, where

$$\|f\|_{H^2(\mathbb{D})}^2 = \sup_{0 \leq r < 1} \int_0^{2\pi} |f(re^{it})|^2 \frac{dt}{2\pi} = \int_0^{2\pi} |\tilde{f}(e^{it})|^2 \frac{dt}{2\pi} = \|\tilde{f}\|_{H^2(\mathbb{T})}^2$$

and $H^2(\mathbb{T})$ is the subspace of $L^2(\mathbb{T})$ whose elements have only nonnegative Fourier coefficients.
For general $1 < p < \infty$, we can form the Hardy space $H^p$ similarly, with

$$
\|f\|_{H^p(D)}^p = \sup_{0 \leq r < 1} \int_0^{2\pi} |f(re^{it})|^p \frac{dt}{2\pi} = \int_0^{2\pi} |\tilde{f}(e^{it})|^p \frac{dt}{2\pi} = \|\tilde{f}\|_{H^p(T)}^p
$$

Here $H^p(D) \cong H^p(T) \subseteq L^p(T)$. We treat $H^p$ as both $H^p(D)$ and $H^p(T)$ interchangeably.
Hardy Spaces

The Hardy spaces $H^p$ can be thought of both as the holomorphic functions on $\mathbb{D}$ which satisfy a growth condition on the boundary, and the nontangential boundary functions which live inside of the $L^p$ space on that boundary.
There are three important properties possessed by the Hardy space $H^2$ as a Hilbert space which we would like to contrast:

- Reproducing kernels $k_\lambda$
- Zero sets $\{z_n\}$
- Multiplier algebra $\mathcal{M}(H^2)$
Reproducing Kernels for the space $H^2(\mathbb{D})$

Point evaluations are bounded linear functionals on $H^2(\mathbb{D})$, and can therefore be expressed as inner products with appropriate reproducing kernels.

If $\lambda \in \mathbb{D}$ and $f(z) = \sum_{n=0}^{\infty} a_n z^n \in H^2(\mathbb{D})$, then the reproducing kernel $k_{\lambda}(z) = \sum_{n=0}^{\infty} \lambda^n z^n$ is such that

$$f(\lambda) = \sum_{n=0}^{\infty} a_n \lambda^n = \sum_{n=0}^{\infty} a_n \overline{\lambda^n} = \langle f, k_{\lambda} \rangle_{H^2(\mathbb{D})}$$

Note that $\sum_{n=0}^{\infty} |\lambda^n|^2 < \infty$, so that $k_{\lambda} \in H^2(\mathbb{D})$. 
Zero Sets of $H^2(\mathbb{D})$

Given a sequence of points $\{z_n\} \subseteq \mathbb{D}$, there is a nontrivial function $f \in H^2(\mathbb{D})$ which vanishes at each $z_n$ if and only if $\{z_n\}$ satisfies the Blaschke condition

$$\sum_{n=1}^{\infty} (1 - |z_n|) < \infty$$
Multiplier Algebra of $H^2(\mathbb{D})$

The multipliers $\mathcal{M}(H^2(\mathbb{D}))$ are precisely $H^\infty(\mathbb{D})$, the bounded holomorphic functions on the disc.
Hardy Spaces in Half Planes

There is similarly a Hardy space for the half plane $\mathbb{C}_+$ using the growth condition on the imaginary line

$$H^p(\mathbb{C}_+) = \left\{ f \in \text{Hol}(\mathbb{C}_+) : \sup_{\sigma > 0} \int_{-\infty}^{\infty} |f(\sigma + it)|^p dt < \infty \right\}$$

One can also define spaces $H^p(\mathbb{C}_\rho)$ for arbitrary $\rho$. Like the Hardy space $H^2(\mathbb{D})$, $H^2(\mathbb{C}_+)$ has well understood reproducing kernels, zero sets, and a multiplier algebra.
Reproducing Kernels for the space $H^2(\mathbb{C}_+)$

Point evaluations are bounded linear functionals on $H^2(\mathbb{C}_+)$, and can therefore be expressed as inner products with appropriate reproducing kernels.

If $\lambda \in \mathbb{C}_+$ and $f \in H^2(\mathbb{C}_+)$, then the reproducing kernel $k_\lambda(z) = \frac{1}{z + \lambda}$ is such that

$$f(\lambda) = \langle f, k_\lambda \rangle_{H^2(\mathbb{D})}$$

Note that $\sup_{x>0} \int_{-\infty}^{\infty} \left| \frac{1}{x+it+\lambda} \right|^2 dt < \infty$, so that $k_\lambda \in H^2(\mathbb{D})$. 
Zero Sets of $H^2(\mathbb{C}_+)$

Given a sequence of points $\{z_n\} \subseteq \mathbb{C}_+$, there is a nontrivial function $f \in H^2(\mathbb{C}_+)$ which vanishes at each $z_n$ if and only if $\{z_n\}$ satisfies the following condition

$$\sum_{n=1}^{\infty} \frac{x_n}{1 + |z_n|^2} < \infty$$

where $z_n = x_n + iy_n$. If the sequence $\{z_n\}$ is bounded, then we recover a Blaschke-type condition

$$\sum_{n=1}^{\infty} x_n < \infty$$
Multiplier Algebra of $H^2(\mathbb{C}_+)$

The multipliers $\mathcal{M}(H^2(\mathbb{C}_+))$ are precisely $H^\infty(\mathbb{C}_+)$, the bounded holomorphic functions on the right half plane.
Dirichlet Series

A Dirichlet series is a series of the form $f(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$. We write $s = \sigma + it$, and let $\Omega_{\rho}$ denote the half plane with real part $> \rho$.

Unlike power series, the “radius” of convergence and absolute convergence may be different.
Dirichlet Series

For a particular \( f(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s} \), we write

\[
\sigma_c(f) = \inf \left\{ \mathcal{K}(s) : \sum_{n=1}^{\infty} \frac{a_n}{n^s} \text{ converges} \right\}
\]

\[
\sigma_b(f) = \inf \left\{ \rho : \sum_{n=1}^{\infty} \frac{a_n}{n^s} \text{ converges to a bounded function in } \Omega_{\rho} \right\}
\]

\[
\sigma_u(f) = \inf \left\{ \rho : \sum_{n=1}^{\infty} \frac{a_n}{n^s} \text{ converges uniformly in } \Omega_{\rho} \right\}
\]

\[
\sigma_a(f) = \inf \left\{ \mathcal{K}(s) : \sum_{n=1}^{\infty} \frac{a_n}{n^s} \text{ converges absolutely} \right\}
\]

\[
\sigma_c \leq \sigma_b = \sigma_u \leq \sigma_a \leq \sigma_c + 1
\]
The Riesz-Fischer theorem states that $\varphi(x) = \sqrt{2} \sin(\pi x)$ can be dilated to form a complete orthonormal basis

$$\left\{ \sqrt{2} \sin(\pi x), \sqrt{2} \sin(\pi 2x), \ldots \right\} = \left\{ \varphi(x), \varphi(2x), \ldots \right\}$$

for $L^2(0, 1)$. 
A natural extension of the theorem would be to ask which functions $\varphi$ can take the place of $\sin$ so that $\{\varphi(nx)\}_{n \geq 1}$ forms an orthonormal basis for $L^2(0, 1)$ under an equivalent norm. Such a sequence is called a Riesz basis.
Riesz-Basis

The characterization of Riesz-type sets which are complete in $L^2(0, 1)$ was characterized by Beurling in 1945, by transforming the expression

$$\varphi(x) = \sum_{n=1}^{\infty} a_n \sqrt{2} \sin(n\pi x)$$

into

$$S'\varphi(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$$

and analyzing properties of the analytic $S'\varphi$. In 1995, Hedenmalm, Lindqvist, and Seip solved the Reisz-basis problem completely by exploiting a Hilbert space of analytic functions of this form.
The system $\{\varphi(nx)\}_{n \geq 1}$ is a Riesz basis for $L^2(0,1)$ if and only if $S\varphi$ and $1/S\varphi$ are in the multiplier algebra $\mathcal{M}(\mathcal{H}^2)$. 
The proof used the Hardy space of Dirichlet series (or Hardy-Dirichlet space),

$$\mathcal{H}^2 = \left\{ f(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s} : \sum_{n=1}^{\infty} |a_n|^2 < \infty \right\}$$

along with the characterization of the multipliers $\mathcal{M}(\mathcal{H}^2)$ of the space. The paper also further established Bohr’s work on the connection between the Hardy-Dirichlet space $\mathcal{H}^2$ and the Hardy space of the infinite polycircle $H^2(\mathbb{T}^\infty)$. 
Hardy Space of Dirichlet Series

The work by Hedehmalm, Lindqvist, and Seip inspired an investigation of the space $\mathcal{H}^2$ and various related spaces over the next 15 years. Contributors in analysis include Aleman, Andersson, Bayart, McCarthy, Olsen, Saskman.

Topics included

- Multipliers
- Reproducing kernels
- Zero sets for $\mathcal{H}^2$ and related $\mathcal{H}^p$ spaces
- Boundary behavior (What happens on the line $\sigma = 1/2$? Can you look at behavior of the function on the line $\sigma = 0$?)
- Connections with the infinite polycircle $H^p(\mathbb{T}^\infty)$
- Carleson measures
The condition on the Hardy-Dirichlet space ensures that all functions \( f \in \mathcal{H}^2 \) have \( \sigma_a \leq \frac{1}{2} \), as by Cauchy-Schwarz

\[
\left( \sum_{n=1}^{\infty} \frac{|a_n|}{n^s} \right)^2 \leq \sum_{n=1}^{\infty} |a_n|^2 \sum_{n=1}^{\infty} \left| \frac{1}{n^{2s}} \right|
\]

This bound is sharp, since

\[
\sum_{n=1}^{\infty} \frac{1}{n^{s-1/2} \log(n + 1)} = \sum_{n=1}^{\infty} \frac{\sqrt{n}/\log(n + 1)}{n^s} \in \mathcal{H}^2
\]

so \( \mathcal{H}^2 \subset \text{Hol}(\mathbb{C}_{1/2}) \).
Hardy Space of Dirichlet Series

The Hardy-Dirichlet space $\mathcal{H}^2$ clearly mirrors the classical Hardy space $H^2$ of the disc

$$\mathcal{H}^2 = \left\{ f(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s} : \sum_{n=1}^{\infty} |a_n|^2 < \infty \right\}$$

$$H^2 = \left\{ f(z) = \sum_{n=0}^{\infty} a_n z^n : \sum_{n=0}^{\infty} |a_n|^2 < \infty \right\}$$
Reproducing Kernels in $\mathcal{H}^2$

The reproducing kernels on $\mathcal{H}^2$ are actually quite easy to define, since if $f(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s} \in \mathcal{H}^2$ and $\lambda \in \mathbb{C}_{1/2}$ then

$$f(\lambda) = \sum_{n=1}^{\infty} \frac{a_n}{n^\lambda} = \sum_{n=0}^{\infty} a_n e^{-\lambda \log(n)} = \sum_{n=0}^{\infty} a_n e^{-\bar{\lambda} \log(n)}$$

$$= \sum_{n=0}^{\infty} a_n \left( \frac{1}{n^\lambda} \right) = \left\langle f(s), \sum_{n=1}^{\infty} \frac{1}{n^{s+\bar{\lambda}}} \right\rangle_{\mathcal{H}^2}$$

so that $k_\lambda(s) = \zeta(s + \bar{\lambda})$. 
Zero Sets of $\mathcal{H}^2$

Like the space $H^2(\mathbb{C}_+)$, bounded sequences \{\(z_n\)\} have the same Blaschke-type condition that

$$\sum_{n=1}^{\infty} (x_n - 1/2) < \infty$$

On the other hand, Dirichlet series have strange vertical limit behavior which has made it difficult to fully classify unbounded sequences.
Almost Periodic Behavior of Dirichlet Series

If a function \( f \in \text{Hol}(\mathbb{C}_\rho) \), \( \epsilon > 0 \), then we say that \( t \) is an \( \epsilon \)-translation number for \( f \) if

\[
\sup_{s \in \mathbb{C}_\rho} |f(s + it) - f(s)| \leq \epsilon
\]

We say that \( f \) is uniformly almost periodic if for every \( \epsilon > 0 \) there is a length \( M \) such that every interval of length \( M \) contains at least one \( \epsilon \)-translation number for \( f \).

**Theorem**

*If \( f \in \text{Hol}(\mathbb{C}_\rho) \) is represented by a Dirichlet series which converges uniformly in \( \mathbb{C}_\rho \), then \( f \) is uniformly almost periodic.*
Zero Sets of $\mathcal{H}^2$

Since all functions in $\mathcal{H}^2$ are uniformly almost periodic in $\mathbb{C}_{1/2}$, they will either have no zeros or infinitely many zeros, and those zeros may be distributed quite wildly along vertical strips.
Multiplier Algebra of $\mathcal{H}^2$

The multiplier algebra of $\mathcal{H}^2$ consist precisely of those holomorphic functions in $\mathbb{C}_+$ which are bounded and representable by a Dirichlet series. If $\mathcal{D}$ is used to denote the collection of holomorphic functions representable by a convergence Dirichlet series on some half space, then we can write

$$\mathcal{M}(\mathcal{H}^2) = H^\infty(\mathbb{C}_+) \cap \mathcal{D} = \mathcal{M}(H^2(\mathbb{C}_+)) \cap \mathcal{D}$$
“Arms-Reach” Boundary Condition

Theorem (Carlson’s Lemma)

If \( f(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s} \) is convergent and bounded in \( \mathbb{C}_+ \), then

\[
\|f\|_{\mathcal{H}^2}^2 = \sum_{n=1}^{\infty} |a_n|^2 = \lim_{\sigma \to 0^+} \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |f(\sigma + it)|^2 dt
\]

Note that \( \sigma \) moves all the way back to \( \mathbb{C}_+ \) rather than just \( \mathbb{C}_{1/2} \), and requires that \( f \) be convergent and bounded in \( \mathbb{C}_+ \) to begin with. An interesting extension of \( \mathcal{H}^2 \) is as follows.
\( \mathcal{H}^p \) Spaces

For \( 1 \leq p < \infty \) we define the space \( \mathcal{H}^p \) as the closure of the Dirichlet polynomials under the norm

\[
\lim_{T \to \infty} \left( \frac{1}{2T} \int_{-\infty}^{\infty} \left| \sum_{n=1}^{\infty} \frac{a_n}{n^it} \right|^p dt \right)^{1/p}
\]

We will refer to this as the \( \mathcal{H}^p \) norm. Bayart (2002) showed that the Dirichlet series which represent such functions have \( \sigma_u \leq 1/2 \), so that they represent holomorphic functions on \( \mathbb{C}_{1/2} \).
Notes on $D^\infty$ and $T^\infty$

Let $T^k$ be the $k$-dimensional polycircle, and let $p_1, \ldots, p_k$ enumerate the first $k$ primes. Then the injection $(p_1^{it}, \ldots, p_k^{it})$ for $t \in \mathbb{R}$ has dense range in $T^k$.

This means that there is a dense subset of the infinite polydisc $D^\infty$ such that its elements can be expressed as $z = (p_1^{-s}, p_2^{-s}, \ldots)$, where $s = \sigma + it$ such that $t \in \mathbb{R}$ and $\sigma > 0$. 
Let each $n$ factor into $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}$. Setting $z = (p_1^{-s}, p_2^{-s}, \ldots)$ we have

$$f(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s} = \sum_{n=1}^{\infty} a_n z_1^{\alpha_1} \cdots z_r^{\alpha_r}$$

We consider the transformation $Df(z) = \sum_{n=1}^{\infty} a_n z_1^{\alpha_1} \cdots z_r^{\alpha_r}$ as being a function of $z \in \mathbb{D}^\infty$. 

Survey of Hardy-Dirichlet Series Spaces  
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**H^p** Spaces as \( H^p(\mathbb{T}^\infty) \)

Bohr showed that in fact for Dirichlet polynomials
\[
P(s) = \sum_{n=1}^{N} \frac{a_n}{n^s},
\]

\[
\|P\|^p_{H^p} = \lim_{T \to \infty} \frac{1}{2T} \left| \sum_{n=1}^{N} \frac{a_n}{n^{it}} \right|^p dt
\]

\[
= \lim_{T \to \infty} \frac{1}{2T} \left| \sum_{n=1}^{N} a_n p_1^{-it} \cdots p_k^{-it} \right|^p dt
\]

\[
= \int_{\mathbb{T}^\infty} \left| \sum_{n=1}^{N} a_n z_1^{\alpha_1} \cdots z_k^{p_k} \right|^p dm(z) = \|\mathcal{D}P\|_{H^p(\mathbb{T}^\infty)}
\]

Which follows from properties of the Kronecker flow and Birkhoff-Khintchin theorem.
**Proof that** $\mathcal{H}^p \subseteq \text{Hol}(\mathbb{C}_{1/2})$

Let $\omega \in \mathbb{C}_{1/2}$, and $z = (2^{-\omega}, 3^{-\omega}, \ldots) \in \mathbb{D}^\infty$. Note that since $\Re(\omega) > \frac{1}{2}$, $z \in \ell^2$ as well.

If $f \in \mathcal{H}^p$, then $\mathcal{D}f \in H^p(\mathbb{T}^\infty)$. On $H^p(\mathbb{T}^\infty)$ we have the inequality

$$|f(\omega)|^p = |\mathcal{D}f(z)|^p \leq \frac{\|\mathcal{D}f\|^p_{H^p(\mathbb{T}^\infty)}}{\prod_{j=1}^{\infty} (1 - |z_j|^2)} = \|f\|^p_{H^p} \prod_{j=1}^{\infty} \frac{1}{1 - |p_j^{-2}\omega|}$$
Proof that $\mathcal{H}^p \subseteq \text{Hol}(\mathbb{C}_{1/2})$

$$|f(\omega)|^p = |Df(z)|^p \leq \frac{\|Df\|_{\mathcal{H}^p(\mathbb{T}^\infty)}^p}{\prod_{j=1}^{\infty} (1 - |z_j|^2)} = \|f\|_{\mathcal{H}^p}^p \prod_{j=1}^{\infty} \frac{1}{1 - |p_j^{-2}\omega|}$$

However, Euler's identity concerning the Riemann zeta function says that the last term is precisely

$$\|f\|_{\mathcal{H}^p}^p \sum_{n=1}^{\infty} \frac{1}{n^{2\Re(\omega)}}$$

which is simply finite by $p$-series. Consequently,

$$|f(\omega)|^p \leq \|f\|_{\mathcal{H}^p}^p \zeta(2\Re(\omega))$$
Almost-Sure Properties of $\mathcal{H}^p$

An element $\chi \in \mathbb{T}^\infty$ can be thought of as a character in the sense that it acts on the prime elements in the canonical way. We define the function $f_\chi$ where $f \in \mathcal{H}^p$ is the function to be influence by the character as

$$f_\chi(s) = \sum_{n=1}^\infty \frac{a_n\chi(n)}{n^s}$$

**Theorem**

*For $f \in \mathcal{H}^p$ and for almost every $\chi \in \mathbb{T}^\infty$, $f_\chi$ is a Dirichlet series which converges in $\mathbb{C}_+$.***
Comparing $\mathcal{H}^p$ to $H^p(\mathbb{C}_{1/2})$

Since $\mathcal{H}^p$ are spaces of Dirichlet series which are well defined in $\mathbb{C}_{1/2}$, it is interesting to note comparisons between $\mathcal{H}^p$ and $H^p(\mathbb{C}_{1/2})$.

**Theorem (Hedenmalm, Lindqvist, Seip)**

If $f \in \mathcal{H}^2$ then $f(s)/s \in H^2(\mathbb{C}_{1/2})$.

In particular, this tells us that all functions in $\mathcal{H}^2$ have nontangential boundary values (as do functions in $\mathcal{H}^p$).
Comparing $\mathcal{H}^p$ to $H^p(\mathbb{C}_{1/2})$

Let $H^p_{\infty}(\mathbb{C}_{1/2})$ denote the uniform local $H^p$ space of the right half plane, defined as those elements such that

$$\|f\|_{H^p_{\infty}(\mathbb{C}_{1/2})}^p = \sup_{y \in \mathbb{R}} \sup_{\sigma > 1/2} \int_y^{y+1} |f(\sigma + it)|^p dt < \infty$$

**Theorem (Bayart)**

*If $p \geq 2$, then $\mathcal{H}^p \subset H^p_{\infty}(\mathbb{C}_{1/2})$ and the injection is continuous.*
Theorem (Bayart)

If $1 \leq p < \infty$ and $\mu$ is a positive measure on $\mathbb{C}_{1/2}$, and if $\mu$ is a Carleson measure for $\mathcal{H}^p$ then it is also a Carleson measure for $H^p(\mathbb{C}_{1/2})$. 
Open Problems

Does $\mathcal{H}^p$ embed in $H^p_{\infty}(C_{1/2})$ for $1 \leq p < 2$?

Is there a BMOA theory for the $\mathcal{H}^p$ spaces?

Can $\mathcal{H}^2$ be factored like $H^2(\mathbb{D})$?

What kind of classification for zero-sets can be achieved in the $\mathcal{H}^p$ setting?
Thank you!