MINIMAL SURFACES AS HOLOMORPHIC FUNCTIONS

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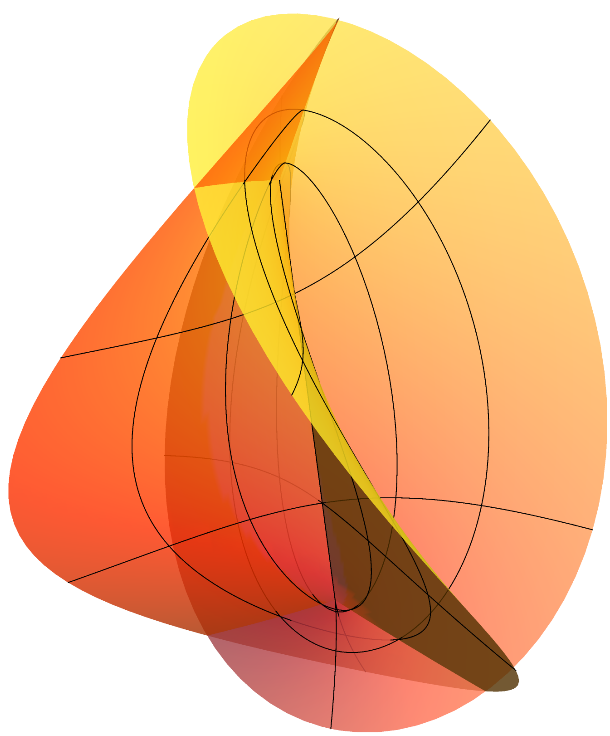
Abstract. In the context of differential geometry, minimal surfaces are defined as surfaces with vanishing mean curvature, and appear in problems related to finding surfaces of minimal area. An interesting consequence of their definition is that it is always possible to construct a coordinate patch for a minimal surface whose components are harmonic. Using complex analysis, we can connect these harmonic functions with the components of holomorphic complex functions. Furthermore, the representation formula of Weierstrass allows us to draw a one-to-one correspondence between holomorphic functions and the local coordinate patches of minimal surfaces in the absence of umbilic points.



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1. Introduction

Intuitively, a minimal surface is a surface with certain properties that generalize the notion of tightness and minimal area. Manifesting physically in the shape of soap films, a minimal surface's curvature is balanced, which agrees with physical concepts like surface tension. While not all minimal surfaces necessarily minimize area, they do often appear as the solutions to problems of that nature. Provided that we are given a sufficiently nice boundary and consider all surfaces stretched across it, if a surface with minimal area exists we will see that it must be a minimal surface. The converse, that a minimal surface exists for any nice curve, is known as Plateau's problem. Though it has been proven to be true, it is beyond the scope of this discussion.

Minimal surfaces are defined as surface with vanishing mean curvature, and so in order to analyze minimal surfaces further, we must briefly establish some elements of differential geometry.

1.1. **Principal Curvatures.**

Let M be a C^k regular surface in $\mathbb{R}^3$, with $k \geq 2$, and let $\vec{x} : U \to M$ be a local coordinate patch with $U \subseteq \mathbb{R}^2$ open, we can consider some point $\vec{x}(u^1, u^2) = \vec{p} \in M$ and the tangent space $T_{\vec{p}}M$. We defined the unit normal $\vec{n}$ and the first and second fundamental forms (g_{ij}) and (L_{ij}) with naming conventions as in [1], with subscripts denoting the partial derivative for the coordinate patch $\vec{x}$, while other subscripts like those for the fundamental forms may not.

$$\vec{n} = \frac{\vec{x}_1 \times \vec{x}_2}{|\vec{x}_1 \times \vec{x}_2|}$$

$$g_{ij} = \left\langle \vec{x}_i, \vec{x}_j \right\rangle \qquad i, j = 1, 2$$

$$L_{ij} = \left\langle \vec{n}, \vec{x}_{ij} \right\rangle \qquad i, j = 1, 2$$

Consider Figure 1, which shows a curve $\vec{\gamma}$ traversing part of a surface. The solid black lines show the tangent and normal vectors corresponding to the curve. The dotted line represents the surface normal $\vec{n}$ to the surface, as we just defined. If $\vec{\gamma}$ is a unit speed curve, then the tangent vector can be given by $\vec{T} = \vec{\gamma}'$, and we can define the curvature κ at the point to be the number $\kappa = |\vec{T}'|$.

Now in the context of surfaces, it is useful to consider how much a curve on a surface is curving in the natural direction of the surface. This is done by observing how much the vector $\kappa\vec{N}$ points in the direction of the normal $\vec{n}$. In other words, we wish to compute the following.

$$\kappa_n = \left\langle \kappa\vec{N}, \vec{n} \right\rangle$$

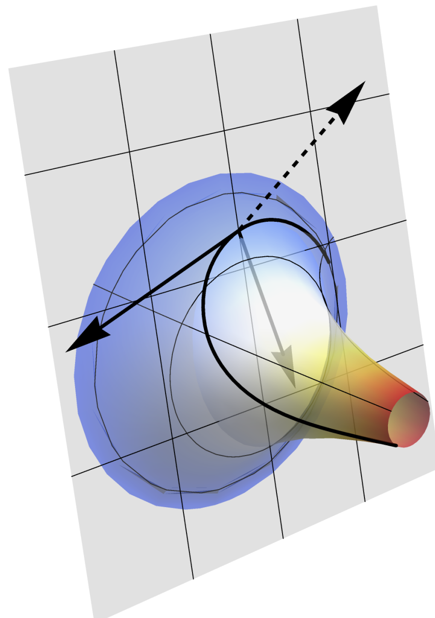

FIGURE 1. Part of a pseudosphere with a curve traced over it. The solid arrows show the tangent and normal vectors emanating from the point. The dotted line shows the normal vector to the surface. The normal curvature κ_n to the surface is the amount that the vector $\kappa\vec{N}$ points in the direction of the unit normal. In our picture, this quantity would be negative, since the normal vector to the curve and the unit normal to the surface point in opposite directions.

The Weingarten map $\mathsf{L} : T_{\vec{p}}M \to T_{\vec{p}}M$ is defined as the negative of the directional derivative of the unit normal $\vec{n}$ in the direction of vectors in $T_{\vec{p}}M$. The coefficients of the Weingarten map can be given by the following equation, where the matrix (g^{ij}) is the inverse of the first fundamental form (g_{ij}).

$$L^i{}_j = \sum_{k=1}^{2} L_{kj} g^{ki} \qquad i, j = 1, 2$$

The Weingarten map possesses a number of important properties that we shall give without proof. The map is self-adjoint, and in fact we have conveniently that for any $\vec{X}, \vec{Y} \in T_{\vec{p}}M$ the second fundamental form can be written as follows. Here, the superscripts denote the components of the vectors.

$$\mathrm{II}\left(\vec{X}, \vec{Y}\right) = \sum_{i,j=1}^{2} L_{ij} X^i Y^j = \left\langle \mathsf{L}\left(\vec{X}\right), \vec{Y} \right\rangle = \left\langle \vec{X}, \mathsf{L}\left(\vec{Y}\right) \right\rangle$$

It becomes natural to wonder which directions maximize and minimize the normal curvature, as expressed by the second fundamental form. By considering the set of unit vectors in $T_{\vec{p}}M$, which is itself a compact set, Lagrange multipliers can be used to determine that there exist eigenvectors and corresponding eigenvalues of L that do exactly that. We denote the eigenvalues, called *principal curvatures*, of M at point $\vec{p}$ by κ_1 (the minimum) and κ_2 (the maximum).

An important result of the Weingarten map begin self-adjoint is that the principal curvatures for any surface always occur in orthogonal directions. This is given by the following theorem, inspired by [2].

Theorem 1.1. *Given $\vec{p} \in M$, let $\vec{X} \in T_{\vec{p}}M$ be an eigenvector for L with eigenvalue λ_1. Let $\vec{Y} \in T_{\vec{p}}M$ be some nonzero vector orthogonal to $\vec{X}$. Then $\vec{Y}$ is also an eigenvector for L.*

Proof. The result follows from the fact that L is self-adjoint, and that $T_{\vec{p}}M$ has dimension 2. Using the second fundamental form, we have that

$$\mathrm{II}\left(\vec{X},\vec{Y}\right) = \left\langle \vec{X}, \mathsf{L}\left(\vec{Y}\right)\right\rangle = \left\langle \mathsf{L}\left(\vec{X}\right), \vec{Y}\right\rangle = \left\langle \lambda_1\vec{X}, \vec{Y}\right\rangle = 0$$

Thus, $\mathsf{L}(\vec{Y})$ is orthogonal to $\vec{X}$. Since $\vec{Y}$ is also orthogonal to $\vec{X}$, and L is non-degenerate, we have that $\mathsf{L}(\vec{Y})$ and $\vec{Y}$ are parallel. Therefore, there exists some λ_2 such that $\mathsf{L}(\vec{Y}) = \lambda_2\vec{Y}$. ■

Corollary 1.2. *The minimum and maximum values κ_1, κ_2 achieved by the second fundamental form over the set of unit vectors in $T_{\vec{p}}M$ occur in orthogonal directions.*

Definition 1.3. A point on a surface is said to be an *umbilical point* if $\kappa_1 = \kappa_2$.

1.2. **Mean Curvature.**

We now define the *mean curvature* H of the surface M as the average of κ_1 and κ_2. Because of their relationship with the Weingarten map, H can be expressed as follows.

$$H = \frac{\kappa_1 + \kappa_2}{2} = \frac{1}{2}\mathrm{Tr}(\mathsf{L})$$

Definition 1.4 (Minimal Surface)**.** A surface M as described above is a *minimal surface* if $H \equiv 0$ for all points on the surface.

Since we saw that the directions where κ_1 and κ_2 are achieved are always orthogonal, a minimal surface is one in which every point is curving equal and opposite amounts in two orthogonal directions. Therefore, each point on a minimal surface is a saddle point, although not all surfaces made up entirely of saddle points will necessarily be minimal.

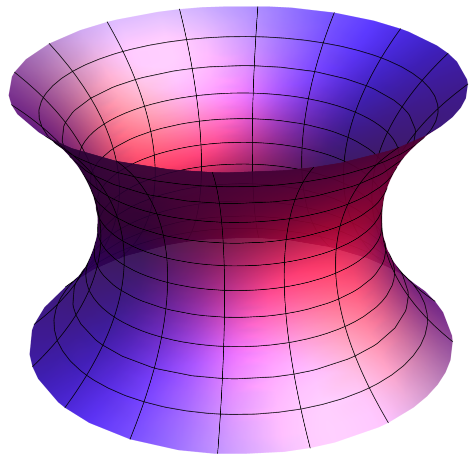

FIGURE 2. A portion of a catenoid, an example of a minimal surface. The lines shown run orthogonal to one another, and trace out curves with equal and opposite curvature.

2. Minimal Surfaces as Harmonic Maps

Because minimal surfaces are characterized by the vanishing of the mean curvature, we may wonder where the mean curvature may pop up in our calculations. Under certain, unusually pleasant coordinate patches, we will show that the Laplacian of the components of the coordinate patch involves the mean curvature, and when the mean curvature vanishes for minimal surfaces correlates it results in the vanishing of the Laplacian. Such a coordinate patch is known as an *isothermal* coordinate patch, and the involvement of the Laplacian allows us to show that the components of an isothermal coordinate patch for a minimal surface are harmonic. Using complex analysis, we can then connect the components of minimal surface coordinate patches to the real (or imaginary) parts of holomorphic functions.

2.1. **Isothermal Coordinates.**

Definition 2.1. A coordinate patch $\vec{x} : U \to M$ is said to be *isothermal* if the first fundamental form satisfies $g_{11} = g_{22}$, and $g_{12} = g_{21} = 0$ for all points on the surface M. That is to say, the first fundamental form can be expressed as follows:

$$\mathrm{I}\left(\vec{X}, \vec{Y}\right) = \sum_{i,j=1}^{2} g_{ij} X^i Y^j = c\left\langle \vec{X}, \vec{Y} \right\rangle$$

where $c(u,v) = g_{11} = g_{22}$ varies for different points $\vec{p} \in M$.

While it is true that there exist isothermal coordinates for all surfaces, the proof is far too lengthy for our purposes. Instead, it is fairly straightforward to show that such coordinates not only exist locally for any minimal surface, but also that they can be computed rather easily when the surface is expressed as a graph.

Theorem 2.2 (Osserman [2]). *For any minimal surface M, and point $\vec{p} \in M$, there exists some local coordinate patch $\vec{x} : U \to M$ such that $\vec{p}$ is contained in the image of $\vec{x}$, and that $\vec{x}$ is isothermal.*

Proof. As in [3], choose a Cartesian coordinate system such that M is locally given by the graph of some function $f(x, y) = z$. We can then construct the local coordinate patch $\vec{x}(x, y) = (x, y, f(x, y))$. The first fundamental form is computed as

$$\begin{bmatrix} 1+f_1^2 & f_1 f_2 \\ f_1 f_2 & 1+f_2^2 \end{bmatrix}$$

Now letting $W = \sqrt{\det(g_{ij})} = \sqrt{1+f_1^2+f_2^2}$, we can then construct the coefficients of the Weingarten map for this coordinate patch:

$$\left(L^i{}_j\right) = \frac{1}{W^3} \begin{bmatrix} f_{11}\left(1+f_2^2\right) - f_{12} f_1 f_2 & f_{12}\left(1+f_1^2\right) - f_{11} f_1 f_2 \\ f_{12}\left(1+f_2^2\right) - f_{22} f_1 f_2 & f_{22}\left(1+f_1^2\right) - f_{12} f_1 f_2 \end{bmatrix}$$

From the fact that M is a minimal surface, we then have that

$$f_{11}\left(1+f_2^2\right) - 2 f_{12} f_1 f_2 + f_{22}\left(1+f_1^2\right) = 0$$

Now observe that

$$\left(\frac{f_1 f_2}{W}\right)_1 - \left(\frac{1+f_1^2}{W}\right)_2 = \frac{f_2}{W}\Big(f_{11}\left(1+f_2^2\right) - 2 f_{12} f_1 f_2 + f_{22}\left(1+f_1^2\right)\Big) = 0$$
$$\left(\frac{f_1 f_2}{W}\right)_2 - \left(\frac{1+f_2^2}{W}\right)_1 = \frac{f_1}{W}\Big(f_{11}\left(1+f_2^2\right) - 2 f_{12} f_1 f_2 + f_{22}\left(1+f_1^2\right)\Big) = 0$$

Therefore, there exist functions $P(x, y)$ and $Q(x, y)$ such that

$$\begin{aligned} P_1 &= \left(1+f_1^2\right)/W & \qquad Q_1 &= \left(f_1 f_2\right)/W \\ P_2 &= \left(f_1 f_2\right)/W & \qquad Q_2 &= \left(1+f_2^2\right)/W \end{aligned}$$

Using the change of coordinates $u = x + P(x, y)$ and $v = y + Q(x, y)$, we have that the Jacobian of the transformation is (as in Osserman)

$$J = \frac{\partial\left(u, v\right)}{\partial\left(x, y\right)} = 2 + \frac{2+f_1^2+f_2^2}{W} > 0$$

Therefore, there exists a local inverse $\psi : V \to U$ such that the first fundamental form of the coordinate patch $\vec{x} \circ \psi : V \to M$ is described as $\hat{g}_{11} = \hat{g}_{22} = W/J$ and $\hat{g}_{12} = 0$. Therefore, the coordinate patch $\vec{x} \circ \psi$ is isothermal. ■

2.2. **Harmonic Components of Minimal Surfaces.**

Now consider the computation of the mean curvature for a surface expressed in isothermal coordinates. Through the use of the Weingarten map, we have the following.

$$H = \frac{1}{2}\mathrm{Tr}\left(\mathsf{L}\right) = \frac{L^1{}_1 + L^2{}_2}{2} = \frac{\sum_k L_{k1}g^{k1} + \sum_k L_{k2}g^{k2}}{2} = \frac{L_{11} + L_{22}}{2\sqrt{g}}$$

As promised, we will now see that the components of a coordinate patch written in isothermal coordinates of a minimal surface are all harmonic. The result occurs when we consider the "Laplacian" of the isothermal coordinate patch $\vec{x}$, given by $\vec{x}_{11} + \vec{x}_{22}$. Even if the coordinate patch is not minimal, the fact that is it isothermal eventually gives us that $\vec{x}_{11}+\vec{x}_{22} = 2\sqrt{g}H\vec{n}$, which only vanishes if the surface is minimal. As we mentioned, isothermal coordinates exist for any surface. However, since we have only proven their existence for minimal surfaces, we will restrict our consideration to them exclusively.

Theorem 2.3. *Let x^i be the components of an isothermal coordinate patch $\vec{x}$ for some minimal surface M. Then each x^i is harmonic. That is to say, $x^i_{11} + x^i_{22} = 0$.*

Proof. While it is not easy to show that each of the components is harmonic individually, we can show that $\vec{x}_{11} + \vec{x}_{22} = \vec{0}$, which will prove it for all three simultaneously. Using the fact that the coordinate patch $\vec{x}$ is isothermal, we have the following.

$$\langle \vec{x}_1, \vec{x}_1 \rangle = g_{11} = g_{22} = \langle \vec{x}_2, \vec{x}_2 \rangle \qquad \langle \vec{x}_1, \vec{x}_2 \rangle = g_{12} = 0$$

$$\langle \vec{x}_1, \vec{x}_2 \rangle_1 = \langle \vec{x}_{11}, \vec{x}_2 \rangle + \langle \vec{x}_1, \vec{x}_{12} \rangle = 0 \qquad \langle \vec{x}_2, \vec{x}_{11} \rangle = -\langle \vec{x}_1, \vec{x}_{12} \rangle$$

$$\langle \vec{x}_1, \vec{x}_2 \rangle_2 = \langle \vec{x}_{12}, \vec{x}_2 \rangle + \langle \vec{x}_1, \vec{x}_{22} \rangle = 0 \qquad \langle \vec{x}_2, \vec{x}_{12} \rangle = -\langle \vec{x}_1, \vec{x}_{22} \rangle$$

Furthermore, $\langle \vec{x}_1, \vec{x}_1 \rangle_1 = \langle \vec{x}_1, \vec{x}_{11} \rangle + \langle \vec{x}_{11}, \vec{x}_1 \rangle = 2\langle \vec{x}_1, \vec{x}_{11} \rangle$, and likewise $\langle \vec{x}_2, \vec{x}_2 \rangle_1 = 2\langle \vec{x}_2, \vec{x}_{12} \rangle$. By the equality of $\langle \vec{x}_1, \vec{x}_1 \rangle$ and $\langle \vec{x}_2, \vec{x}_2 \rangle$, we then have that

$$\langle \vec{x}_1, \vec{x}_{11} \rangle = \langle \vec{x}_2, \vec{x}_{12} \rangle = -\langle \vec{x}_1, \vec{x}_{22} \rangle$$

It follows that $\langle \vec{x}_1, \vec{x}_{11} + \vec{x}_{22} \rangle = 0$. Similarly, $\langle \vec{x}_2, \vec{x}_{11} + \vec{x}_{22} \rangle = 0$, and so $\vec{x}_{11} + \vec{x}_{22}$ is parallel to $\vec{n}$. Therefore,

$$\vec{x}_{11} + \vec{x}_{22} = \langle \vec{x}_{11} + \vec{x}_{22}, \vec{n} \rangle \vec{n} = (L_{11} + L_{22})\,\vec{n} = 2\sqrt{g}H\vec{n}$$

However, since M is minimal, we have that $\vec{x}_{11} + \vec{x}_{22} = \vec{0}$, and so each component of $\vec{x}$ is harmonic. ■

We can now describe a minimal surface locally as a triplet of harmonic functions of two variables. Not all triplets of harmonic functions will necessarily create a minimal surface, however. We've only shown that these components are harmonic when they are part of an isothermal coordinate patch, and so a triplet of harmonic functions must satisfy the appropriate conditions on its first fundamental form. As we will see, these restriction can be very elegantly described once these harmonic functions are described in terms of holomorphic functions.

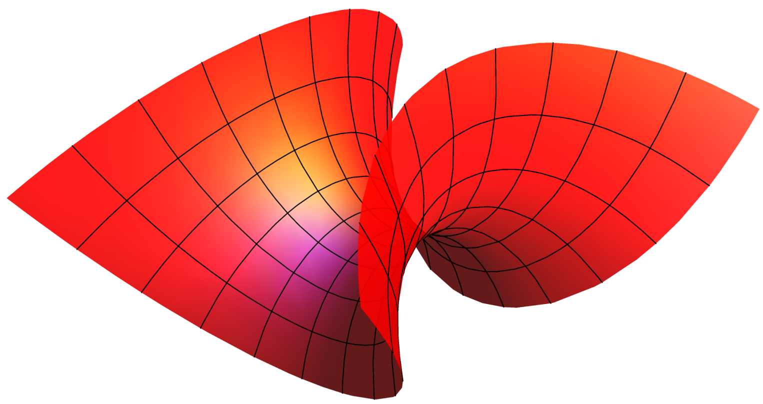

FIGURE 3. A portion of an Enneper surface, a minimal surface. The surface is described with harmonic components. This one is given as follows:

$$\vec{r}(u,v) = \left(x - x^3/3 + xy^2, y^3/3 - x^2y - y, x^2 - y^2\right)$$

3. The Weierstrass-Enneper Representation

In the context of the complex analysis, holomorphic functions of complex variables are intimately related to harmonic functions. Each holomorphic function $f(z) = u(x,y) + iv(x,y)$ of a single complex variable $z = x + iy$ can be represented by its real and imaginary components, which can in turn be described as real valued functions of two real variables x and y. As we will see, the fact that f is holomorphic places certain restrictions on these components, including that they be harmonic.

The converse of this result, that all harmonic functions can be used to construct holomorphic functions, is also true. This can be proven rather straightforwardly by performing the actual construction. While these facts can be found in [4] and [5], the construction we will use to create holomorphic functions mimics that which is typically found in the proof that they exist.

3.1. **Holomorphic Functions.**

We begin with some preliminary results from complex analysis. Our goal is to define what a holomorphic function is, and to construct holomorphic functions which will represent our local isothermal coordinate patch.

Definition 3.1. Let $f : D \to \mathbb{C} = \mathbb{R}^2$ be a function of a complex variable, with $D \subseteq \mathbb{C}$ open. Then f is said to be holomorphic on D if the derivative $f'(z_0) = \lim_{z \to z_0} \frac{f(z)-f(z_0)}{z-z_0}$ exists for all points $z_0 \in D$. If such a derivative exists for a function defined on all of D except for a set of isolated points, we say that function is meromorphic.

Theorem 3.2. *Suppose that $f : D \to \mathbb{C} = \mathbb{R}^2$ is a C^1 function, with D as before. Write $f(x+iy) = u(x,y) + iv(x,y)$, where $u, v : D \to \mathbb{R}$ are real valued functions of two variables. Then the necessary and sufficient condition for f to holomorphic on D is that the Cauchy-Riemann equations are met for all points $(x_0, y_0) \in D$:*

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad\qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Furthermore, the complex derivative f' can be expressed in terms of the partial derivatives with respect to the real variables:

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}$$

An example of these results can be seen if we consider $f : \mathbb{C} \to \mathbb{C}$ given by $f(z) = z^2$. We can express the function as $f(x+iy) = (x+iy)^2 = (x^2 - y^2) + i(2xy)$, and therefore our two real valued functions are $u(x,y) = x^2 - y^2$ and $v(x,y) = 2xy$. These functions do indeed satisfy the Cauchy-Riemann equations.

$$\frac{\partial u}{\partial x} = 2x = \frac{\partial v}{\partial y} \qquad\qquad \frac{\partial u}{\partial y} = -2y = -\frac{\partial v}{\partial x}$$

Additionally, the derivative $f'(z)$ may be written as $2x + 2iy = 2z$.

3.2. **Construction of the Representation Functions.**

Now we have the tools needed to construct holomorphic functions using the harmonic components or our coordinate patch. To avoid the confusion of using x and y as variables, we will write $u_1 + iu_2 = (u_1, u_2) \in D$ for arbitrary points in the domain.

Now given a minimal surface M, let $\vec{x} : D \to \mathbb{R}^3$ be an isothermal coordinate patch. We write $\vec{x} = (x^1, x^2, x^3)$. Then define the functions $\phi_j : D \to \mathbb{C}$ by

$$\phi_j = \mu_j + i\nu_j = \frac{\partial x^j}{\partial u_1} - i\frac{\partial x^j}{\partial u_2} \qquad j = 1, 2, 3$$

We write $\phi = (\phi_1, \phi_2, \phi_3)$ for the total triplet, $\phi^2 = \phi_1^2 + \phi_2^2 + \phi_3^2$ for its square, and $|\phi|^2 = |\phi_1|^2 + |\phi_2|^2 + |\phi_3|^2 = \phi_1\overline{\phi}_1 + \phi_2\overline{\phi}_2 + \phi_3\overline{\phi}_3$ for the square of the norm of ϕ.

The construction of these functions is chosen specifically so that they satisfy the Cauchy-Riemann equations. This can be computed immediately, relying on the fact that each component x^j is harmonic and mixed partials are equal.

$$\frac{\partial \mu_j}{\partial u_1} = \frac{\partial^2 x^j}{\partial u_1^2} = -\frac{\partial^2 x^j}{\partial u_2^2} = \frac{\partial \nu_j}{\partial u_2} \qquad \frac{\partial \mu_j}{\partial u_2} = \frac{\partial^2 x^j}{\partial u_2 \partial u_1} = \frac{\partial^2 x^j}{\partial u_1 \partial u_2} = -\frac{\partial \nu_j}{\partial u_2}$$

From the previous theorem, this shows that each function ϕ_j is holomorphic on D. One result of being holomorphic is that our functions have, at least locally, a well defined antiderivative given by the integral of the function. We define the local antiderivatives of our ϕ_j functions as follows.

$$\Phi_j(\omega) = \int_{\omega_0}^{\omega} \phi_j(z)dz + \alpha_j$$

for some $\omega_0 \in D$, and with α_j chosen such that $x^j(\omega_0) = \mathrm{Re}(\alpha_j)$. Then Φ_j is the unique complex antiderivative of ϕ_j such that $\Phi_j(\omega_0) = \alpha_j$. For a comprehensive theory of complex integration, see [5].

Theorem 3.3. *The coordinate functions x^j defined on the open interval $D \subseteq \mathbb{R}^2$ are identically equal to $\mathrm{Re}(\Phi_j)$. That is to say, for any $(u_1, u_2) \in D$,*

$$x^j(u_1, u_2) = \mathrm{Re}\left(\Phi_j(u_1 + iu_2)\right)$$

Proof. By properties of the complex integral, we have that $\Phi_j' = \phi_j$. From the Cauchy-Riemann equations, it follows that

$$\frac{\partial x^j}{\partial u_1} = \mathrm{Re}(\phi_j) = \mathrm{Re}\left(\frac{\partial \Phi_j}{\partial u_1}\right) = \frac{\partial\, \mathrm{Re}(\Phi_j)}{\partial u_1}$$

$$\frac{\partial x^j}{\partial u_2} = \mathrm{Re}(i\phi_j) = \mathrm{Re}\left(\frac{\partial \Phi_j}{\partial u_2}\right) = \frac{\partial\, \mathrm{Re}(\Phi_j)}{\partial u_2}$$

Furthermore, $x^j(\omega_0) = \mathrm{Re}(\alpha_j) = \mathrm{Re}(\Phi_j(\omega_0))$ by definition. Since the two functions are both the real-valued anti-derivatives of the function ϕ_j, they must be equal by its uniqueness. ■

Our results thus far can be summarized in the following theorem, which we will expand as our discussion continues. In particular, we will begin to examine the restrictions placed on the functions ϕ_j that we have constructed.

Theorem 3.4 (Preliminary Representation Theorem)**.** *Given a minimal surface M, and some point $\vec{p} \in M$, there exists a triplet of complex analytic functions $\phi = (\phi_1, \phi_2, \phi_3)$ such that the triplet*

$$\mathrm{Re}\left(\int_{\omega_0}^{\omega} \phi(z)dz + \alpha\right)$$

is an isothermal coordinate patch for M over some open neighborhood $D \subseteq \mathbb{R}^2$ for some $\omega_0 \in D$ and $\alpha \in \mathbb{C}^3$.

The converse of this statement will now become our main concern, and therefore expand our ability to represent minimal surfaces as holomorphic functions. Given a triplet of complex holomorphic functions, will the real component of such an integral produce isothermal coordinates of some minimal surface? If the triplet is chosen randomly, certainly not. What, then, should be expected of such a triplet?

3.3. **Refinement of the Representation Functions.**

The easiest way to explore this question is to determine what properties are present when we construct such functions from a minimal surface with an isothermal coordinate patch. Consider the previously constructed ϕ_j functions, and the isothermal coordinate patch $\vec{x}$. Since the surface is non-degenerate, the two partials of $\vec{x}$ cannot be $\vec{0}$, and so $|\phi|^2 \neq 0$. From the isothermal conditions, we also have the following equation. Recall that x^i_j refers to derivative with respect to u_j of the i^{th} component of the coordinate patch $\vec{x}$.

$$\begin{aligned}
\phi_1^2 + \phi_2^2 + \phi_3^2 &= \left(x^1_1 - ix^1_2\right)^2 + \left(x^2_1 - ix^2_2\right)^2 + \left(x^3_1 - ix^3_2\right)^2 \\
&= \left(x^1_1\right)^2 + \left(x^2_1\right)^2 + \left(x^3_1\right)^2 \\
&\quad - \left(x^1_2\right)^2 - \left(x^2_2\right)^2 - \left(x^3_2\right)^2 \\
&\qquad - 2i\left(x^1_1 x^1_2\right) - 2i\left(x^2_1 x^2_2\right) - 2i\left(x^3_1 x^3_2\right) \\
&= \langle \vec{x}_1, \vec{x}_1 \rangle - \langle \vec{x}_2, \vec{x}_2 \rangle - 2i \langle \vec{x}_1, \vec{x}_2 \rangle = 0
\end{aligned}$$

The immediate consequence of this observation is that we can now reduce such a triplet of functions by solving for one of them in terms of the other two. The most obvious way that this can be done is by transforming the equation into $\phi_3^2 = -\phi_1^2 - \phi_2^2$. If we solve for ϕ_3, we would have a representation of the minimal surface as the triplet $\left(\phi_1, \phi_2, i\sqrt{\phi_1^2 + \phi_2^2}\right)$, which only depends on two holomorphic functions. The disadvantage of this representation is that it relies on the complex square root, which is multivalued. This would make things considerably more difficult, and so we concern ourselves with finding another way to eliminate the need for all three functions. We can accomplish this by observing that the above equation can be factored, such that we have the following.

$$(\phi_1 - i\phi_2)(\phi_1 + i\phi_2) + \phi_3^2 = 0$$

$$\frac{\phi_1 + i\phi_2}{\phi_3} = -\frac{\phi_3}{\phi_1 - i\phi_2}$$

From this we could construct other holomorphic functions, our goal being to represent all three of these functions by only two, simpler functions. As an example, consider the functions $P = \phi_1 - i\phi_2$ and $Q = \phi_3$. Then we can reconstruct our functions using the equations above.

$$(\phi_1, \phi_2, \phi_3) = \left(-\frac{1}{2}\frac{Q^2 - P^2}{P}, \frac{i}{2}\frac{Q^2 + P^2}{P}, Q \right)$$

While this makeshift representation accomplishes our goal of reducing our problems to two holomorphic functions, it is in no way intuitive. In order to find a more appropriate formulation, we consider the complex formulation of the Gauss map.

3.4. **The Gauss Map and Stereographic Projection.**

Given a coordinate patch $\vec{x} : U \to M$ to a surface (not necessarily minimal), we can construct the Gauss map $G : U \to S^2$ from the domain of the coordinate patch onto the unit sphere, by assigning to each points $\vec{p} \in U$ the unit normal $\vec{n}$ corresponding to the point $\vec{x}(\vec{p})$. In calculating the unit normal, we must take the cross product of the two partial coordinate patches $\vec{x}_1, \vec{x}_2$. However, by our construction we have that $x_1^j = \text{Re}(\phi_j) = \mu_j$ and $x_2^j = \text{Im}(\phi_j) = -\nu_j$.

$$\begin{aligned}
\vec{x}_1 \times \vec{x}_2 &= \left(x_1^2 x_2^3 - x_1^3 x_2^2,\ x_1^3 x_2^1 - x_1^1 x_2^3,\ x_1^1 x_2^2 - x_1^2 x_2^1 \right) \\
&= \Big(\mu_2(-\nu_3) - \mu_3(-\nu_2),\ \mu_3(-\nu_1) - \mu_1(-\nu_3),\ \mu_1(-\nu_2) - \mu_2(-\nu_1) \Big) \\
&= \Big(\mu_3\nu_2 - \mu_2\nu_3,\ \mu_1\nu_3 - \mu_3\nu_1,\ \mu_2\nu_1 - \mu_1\nu_2 \Big) \\
&= \left(\text{Im}(\phi_2\overline{\phi}_3),\ \text{Im}(\phi_3\overline{\phi}_1),\ \text{Im}(\phi_1\overline{\phi}_2) \right)
\end{aligned}$$

By normalizing this expression, we have the following unit normal:

$$\vec{n} = \frac{2\left(\text{Im}(\phi_2\overline{\phi}_3),\ \text{Im}(\phi_3\overline{\phi}_1),\ \text{Im}(\phi_1\overline{\phi}_2) \right)}{|\phi|^2}$$

Because the Gauss map sends elements in the domain of the coordinate patch onto the unit sphere, we have the opportunity to take advantage of stereographic projection between the sphere and the complex plane. See [3] for more details on stereographic projection. The projection of a point on the sphere S^2 with coordinates $(\hat{x}, \hat{y}, \hat{z})$ is given by the complex point $\Gamma = (\hat{x} + i\hat{y})/(1 - \hat{z})$. By considering the coordinates of $\vec{n}$ as our point on S^2, we can compute the corresponding point on the complex plane. Following from the fact that $\mathrm{Im}(z) = (z - \overline{z})/2i$ for a complex number z, we have the following:

$$\Gamma = \frac{2\,\mathrm{Im}(\phi_2\overline{\phi}_3)/|\phi|^2 + 2i\,\mathrm{Im}(\phi_3\overline{\phi}_1)/|\phi|^2}{1 - 2\,\mathrm{Im}(\phi_1\overline{\phi}_2)/|\phi|^2} = \frac{2\,\mathrm{Im}(\phi_2\overline{\phi}_3) + 2i\,\mathrm{Im}(\phi_3\overline{\phi}_1)}{|\phi|^2 - 2\,\mathrm{Im}(\phi_1\overline{\phi}_2)}$$

$$= \frac{\left(\phi_2\overline{\phi}_3 - \overline{\phi}_2\phi_3 + i\phi_3\overline{\phi}_1 - i\overline{\phi}_3\phi_1\right)/i}{|\phi|^2 - 2\,\mathrm{Im}(\phi_1\overline{\phi}_2)} = \frac{\phi_3(\overline{\phi}_1 + i\overline{\phi}_2) - \overline{\phi}_3(\phi_1 + i\phi_2)}{|\phi|^2 - 2\,\mathrm{Im}(\phi_1\overline{\phi}_2)}$$

As in [3], we can use the relationship between the representation functions to simplify the numerator.

$$\Gamma = \frac{\phi_3(\overline{\phi}_1 + i\overline{\phi}_2) - \overline{\phi}_3(\phi_1 + i\phi_2)}{|\phi|^2 - 2\,\mathrm{Im}(\phi_1\overline{\phi}_2)}$$

$$= \frac{\phi_3(\overline{\phi}_1 + i\overline{\phi}_2) - \overline{\phi^2}_3\left(-\phi_3/(\phi_1 - i\phi_2)\right)}{|\phi|^2 - 2\,\mathrm{Im}(\phi_1\overline{\phi}_2)}$$

$$= \left(\frac{\phi_3}{\phi_1 - i\phi_2}\right)\frac{\left(\overline{\phi}_1 + i\overline{\phi}_2\right)(\phi_1 - i\phi_2) + \phi_3\overline{\phi}_3}{|\phi|^2 - 2\,\mathrm{Im}(\phi_1\overline{\phi}_2)}$$

$$= \left(\frac{\phi_3}{\phi_1 - i\phi_2}\right)\frac{\phi_1\overline{\phi}_1 + \phi_2\overline{\phi}_2 + \phi_3\overline{\phi}_3 + i\phi_1\overline{\phi}_2 - i\overline{\phi}_1\phi_2}{\phi_1\overline{\phi}_1 + \phi_2\overline{\phi}_2 + \phi_3\overline{\phi}_3 + i\phi_1\overline{\phi}_2 - i\overline{\phi}_1\phi_2}$$

Therefore, the expression reduces simply to $\phi_3/(\phi_1 - i\phi_2)$. Consequently, given a minimal surface and representation functions (ϕ_1, ϕ_2, ϕ_3), the function $\phi_3/(\phi_1 - i\phi_2) : D \to \mathbb{C}$ is a holomorphic map that takes points in the original parameter domain D, looks at their image on the unit sphere S^2 through the Gauss map, and uses stereographic projection to project the points back down to the complex plane.

Theorem 3.5 (Weierstrass-Enneper Representation)**.** *Given a minimal surface M, and some point $\vec{p} \in M$, we can construct a holomorphic function F and meromorphic function G, where FG^2 is holomorphic, such that the triple of functions*

$$\mathrm{Re}\left(\int_{\omega_0}^{\omega} \frac{1}{2}F\left(1 - G^2\right)\ dz + \alpha_1\right) \quad \mathrm{Re}\left(\int_{\omega_0}^{\omega} \frac{i}{2}F\left(1 + G^2\right)\ dz + \alpha_2\right) \quad \mathrm{Re}\left(\int_{\omega_0}^{\omega} FG\ dz + \alpha_3\right)$$

is an isothermal coordinate patch for M over some open neighborhood $D \subseteq \mathbb{R}^2$ for some $\omega_0 \in D$ and $\alpha \in \mathbb{C}^3$. Such a construction is obtained by defining

$$F = \phi_1 - i\phi_2 \qquad\qquad G = \frac{\phi_3}{\phi_1 - i\phi_2}$$

Furthermore, any pair of functions (F, G) satisfying these conditions can be used to construct an isothermal coordinate patch for a minimal surface from the integrals above.

The function G is taken to represent the Gauss map. The other function, F, appears to be somewhat arbitrarily chosen, however it allows us to reconstruct our original representation by the relation $(\phi_1, \phi_2, \phi_3) = \big(\frac{1}{2}F(1-G^2), \frac{i}{2}F(1+G^2), FG\big)$. Most texts demand that F is not identically zero in order to ensure that $|\phi|^2 \neq 0$, however since F appears in the denominator of G, the restriction that G is meromorphic guarantees this for us.

4. Single Function Representations

The advantage of reducing minimal surfaces down to two complex functions, one holomorphic and one meromorphic, is that we can smoothly manipulate such surfaces by manipulating their representations. We proceed as in [6] to show that we can further reduce the representation to a single holomorphic function, as long as there are no umbilic points close by. This, in fact, shows that there exists a local one-to-one correspondence between minimal surfaces and holomorphic functions. This is known as the representation formula of Weierstrass.

4.1. The Inverse Gauss Map.

Consider a minimal surface M with coordinate patch $\vec{x}$ and appropriate functions (F, G), where G represents the Gauss map. Since G is given to be meromorphic, the condition that $G'(\omega) \neq 0$ is enough to ensure that there exists a local inverse Ψ, also meromorphic, in some neighborhood of ω.

Theorem 4.1. *Given the function G representing the Gauss map, $G'(\omega) = 0$ if and only if $\vec{x}(\omega)$ is an umbilic point of the minimal surface M.*

We provide this theorem without proof, as it would involve the introduction of Gaussian curvature, and numerous calculations. We differ the interested reader to [6]. Now consider the new coordinate patch $\vec{x} \circ \Psi$. The work to confirm that such a reparametrization is still isothermal is trivial. Computing the representation functions $\hat{\phi}$ for this coordinate patch yields the following:

$$\hat{\phi}(\omega) = \phi\big(\Psi(\omega)\big)\Psi'(\omega)$$

As in [6], the Weierstrass-Enneper representation for such a reparametrization yields $\hat{F}(\omega) = F\big(\Psi(\omega)\big)\Psi'(\omega)$ and $\hat{G}(\omega) = \omega$. Then we let $\Xi(\omega) = \frac{1}{2}\hat{F}(\omega) = \frac{1}{2}F\big(\Psi(\omega)\big)\Psi'(\omega)$, with a factor of 1/2 added for convenience. This leads us to the following theorem.

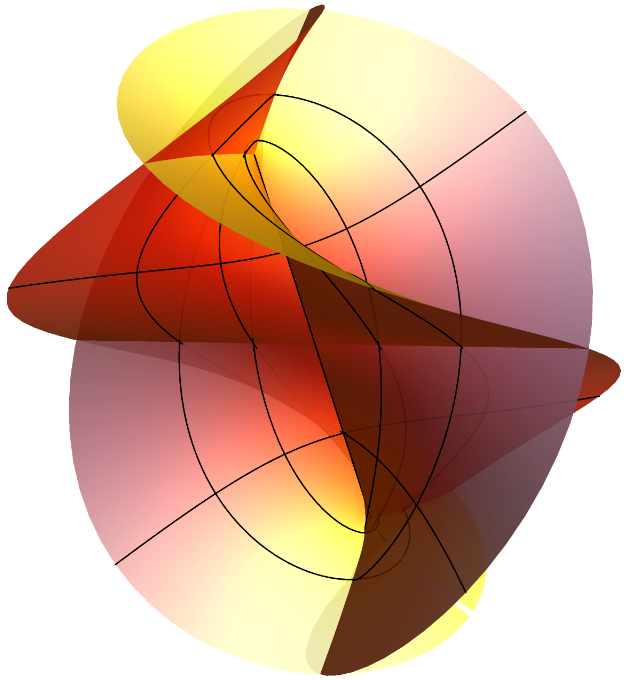

FIGURE 4. A minimal surface given by the function $\Xi(\omega) = 1/\omega - 1/\omega^3$, the adjoint of Catalan's surface. Though the meromorphic function has a pole at the origin, the coordinate patch is defined over part of an annulus that avoids it.

Theorem 4.2 (Representation Formula of Weierstrass)**.** *Given a minimal surface M, and some point $\vec{p} \in M$, there exists a holomorphic function Ξ such that*

$$\text{Re}\left(\int_{\omega_0}^{\omega} \Xi(z)\left(1 - z^2\right)\ dz + \alpha_1\right)$$

$$\text{Re}\left(\int_{\omega_0}^{\omega} i\ \Xi(z)\left(1 + z^2\right)\ dz + \alpha_2\right)$$

$$\text{Re}\left(\int_{\omega_0}^{\omega} 2z\ \Xi(z)\ dz + \alpha_3\right)$$

is an isothermal coordinate patch for M over some open neighborhood $D \subseteq \mathbb{R}^2$ for some $\omega_0 \in D$ and $\alpha \in \mathbb{C}^3$. Furthermore, any holomorphic function Ξ can be used to construct an isothermal coordinate patch for a minimal surface from the integrals above.

Beginning with two arbitrary functions (F, G), we have essentially chosen G permanently to be the identity function. Such a choice does not make it terribly easy for us to construct the Ξ function from a given minimal surface, however it does allow us to create minimal surfaces given only one holomorphic function.

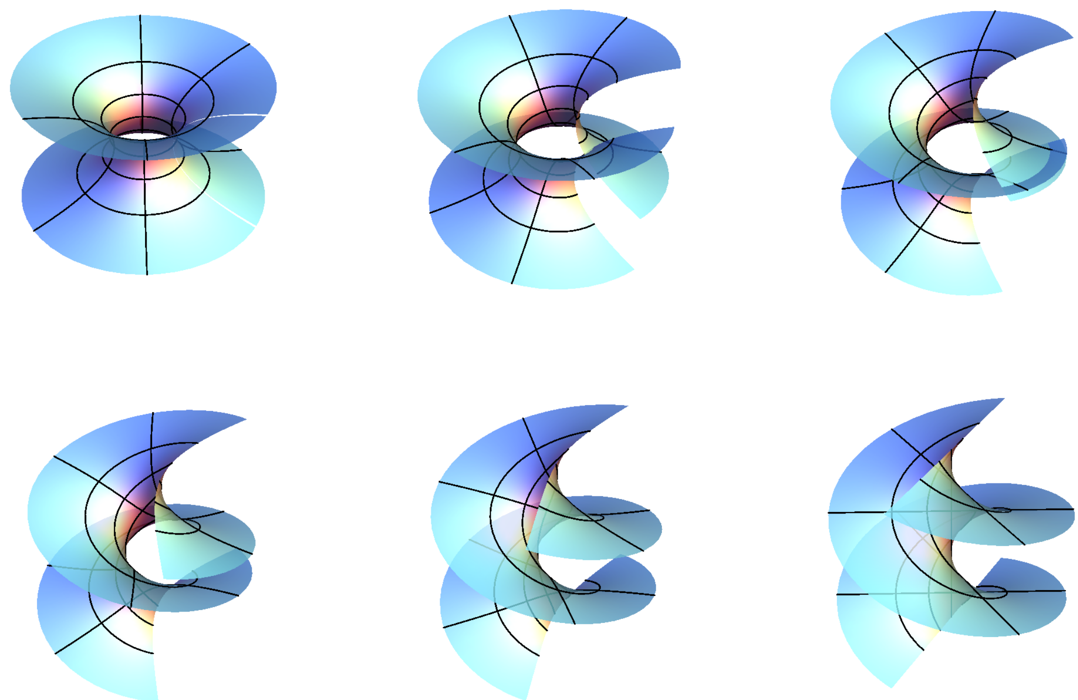

FIGURE 5. The deformation of a catenoid into a helicoid using the function $\Xi_\theta(\omega) = e^{i\theta} \cdot 1/\omega^2$. Images show $\theta = 0, \frac{\pi}{10}, \frac{\pi}{5}, \frac{3\pi}{10}, \frac{2\pi}{5}, \frac{\pi}{2}$. Each intermediate surface is minimal, due to the construction.

4.2. **Adjoint Surfaces.**

Recall that the original representation for a minimal surface essentially reduced to showing that its isothermal coordinate patch was the real component of a holomorphic function. More specifically, we constructed a triplet of holomorphic functions $\Phi = (\Phi_1, \Phi_2, \Phi_3)$, and showed that $\vec{x}_j = \text{Re}(\phi_j)$. We may now ask, what does the imaginary part of Φ represent? Clearly, it too could be used to construct a minimal surface, using the fact that $\text{Im}(\Phi) = \text{Re}(-i\Phi)$. Such a surface is called the adjoint of the original surface. Our new representation of minimal surfaces as a single holomorphic function Ξ makes it incredibly easy to compute adjoint surfaces, as they can be obtained simply by multiplying Ξ by a factor of $\pm i$.

The traditional example of this concept is the helicoid and the catenoid, the latter of which we have already seen. The catenoid can be given by the function $\Xi(\omega) = 1/\omega^2$, while the helicoid can similarly be described by $\hat{\Xi}(\omega) = i/\omega^2$. Initially looking at the two surfaces, it is not clear how they are connected to one another. Our representation becomes even more advantage, however, when one considers that the multiplication of holomorphic functions is itself holomorphic. We can define the holomorphic function $\Xi_\theta = e^{i\theta} \cdot \Xi$ for arbitrary $0 \leq \theta < 2\pi$. Such a function is the rotation of the function Ξ

through the angle θ in the complex plane, and in fact we have that $\Xi_0 = \Xi$ and $\Xi_{\pi/2} = \hat{\Xi}$. Additionally, Ξ_θ also represents a one parameter family of minimal surfaces, smoothly transforming the catenoid into the helicoid. Such a family is called the associated family of minimal surfaces with the catenoid (or helicoid, as they are part of the same families).

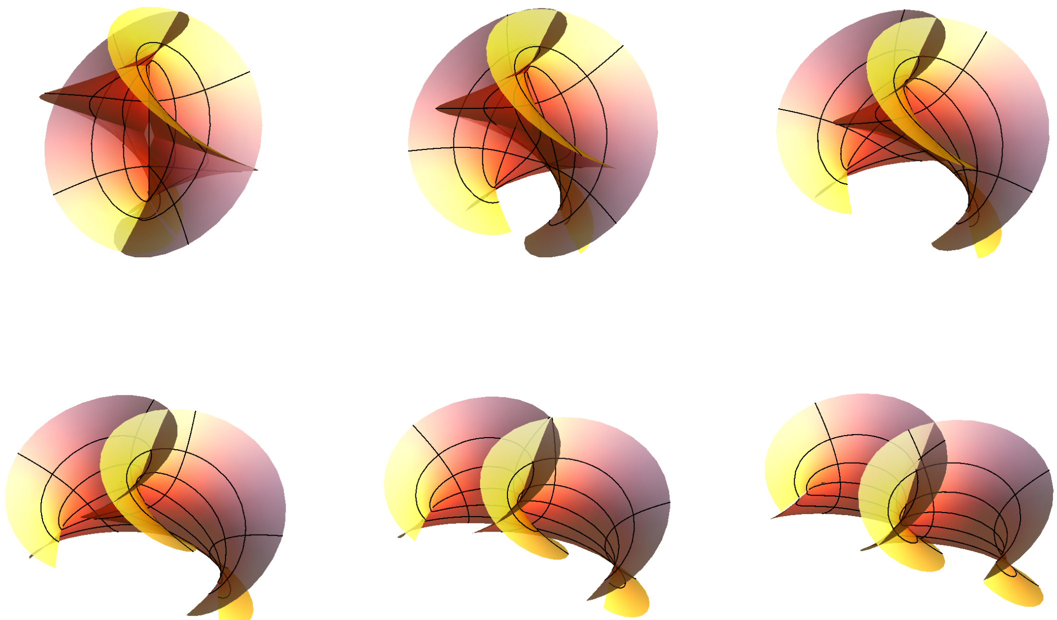

Figure 6. The deformation of Catalan's surface, given by the function $\Xi_\theta(\omega) = e^{i\theta} \cdot \left(\frac{1}{\omega} - \frac{1}{\omega^3}\right)$. The final surface on the lower right side is periodic, although our coordinate patch does not extend to further periods.

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