THE ALMOST SURE THEORY OF FINITE METRIC SPACES

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ABSTRACT. We introduce a metric space $A_S$ and show that its (continuous) theory is the almost-sure theory of finite metric spaces of diameter at most 1.

1. Introduction

Recall that the Urysohn sphere $U$ is the unique Polish metric space of diameter 1 satisfying two properties: universality: all Polish metric spaces of diameter at most 1 embed into $U$; and ultrahomogeneity: any isometry between finite subspaces of $U$ extends to a self-isometry of $U$. From the model-theoretic perspective, $U$ is the Fraïssé limit of the class of all finite metric spaces of diameter at most 1 and is the model completion of the pure theory of metric spaces.

A lingering question about $U$ is whether or not it is pseudofinite, that is, elementarily equivalent to an ultraproduct of finite metric spaces, or, equivalently, whether or not, given a sentence $\sigma$ for which $\sigma^U = 0$ and $\epsilon > 0$, there is a finite metric space $X$ of diameter at most 1 such that $\sigma^X < \epsilon$. In an earlier preprint, we claimed that not only is $U$ pseudofinite, but indeed a stronger result is true, namely $\text{Th}(U)$ is the almost-sure theory of finite metric spaces, which means, given any sentence $\sigma$ and any $\epsilon > 0$, almost all sufficiently large metric spaces $X$ of diameter at most 1 satisfy $|\sigma^X - \sigma^U| < \epsilon$.

However, a serious flaw in our argument was discovered by Alex Kruckman and thus the pseudofiniteness of the Urysohn sphere is still in question. It is the purpose of this note to rescue the latter fact, namely that there is an almost-sure theory of finite metric spaces of diameter at most 1. The motivation for the definition of this theory comes from the fact that almost all sufficiently large metric spaces of diameter at most 1 have all nontrivial distances at least $\frac{1}{2} - O(n^C)$ (see [3] and [2]). This led us to consider a space defined just like $U$ except with all nontrivial distances being at least $\frac{1}{2}$. Since any assignment of distances between distinct points taking values at least $\frac{1}{2}$ automatically satisfies the triangle inequality, this allowed us to salvage a version of our argument in this context.

In Section 2, we precisely define this modified version of $U$, which we denote by $A_S$, and list its relevant model-theoretic properties. In Section 3, we show that the theory of $A_S$ is indeed the almost-sure theory of finite metric

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spaces of diameter at most 1. In an appendix, we review an amalgamation result that we need in our arguments.

2. The almost-sure metric space

Suppose that \( \mathcal{C} \) is the class of finite metric spaces in which the distance between any two distinct points lies in the interval \([1/2, 1]\). Note that all such metric spaces are discrete as any ball of radius \(1/4\) consists just of its center. We let \( L \) denote the “empty” metric language, that is, the language that only consists of the metric symbol.

**Theorem 1.** \( \mathcal{C} \) is a metric Fraïssé class with separably categorical Fraïssé limit \( \mathcal{A}_S \). Th(\( \mathcal{A}_S \)) has quantifier elimination and is the model completion of the \( L \)-theory

\[
\{ \sup_{x,y} \min(\frac{1}{2} - d(x, y), d(x, y)) = 0 \}.
\]

The notation \( \mathcal{A}_S \) for the Fraïssé limit standards for “almost-sure.” This notation will become clear when we show that Th(\( \mathcal{A}_S \)) is the almost-sure theory of finite metric spaces with values in \([0, 1]\).

The proof of the preceding theorem proceeds just as in the case of the class of all metric spaces with distances in \([0, 1]\) (whose corresponding limit is the Urysohn sphere). We refer the reader to [5]. However, we will spell out an axiomatization of Th(\( \mathcal{A}_S \)) as this will be important for what is to follow.

Given a finite metric space \( X = \{x_1, \ldots, x_n\} \), we let Conf\(_X\)(\( \vec{v}_1, \ldots, \vec{v}_n \)) denote the formula

\[
\max_{1 \leq i < j \leq n} |d(x_i, x_j) - d(\vec{v}_i, \vec{v}_j)|.
\]

We use the notation \( X \sqsubseteq Y \) when \( X \) is a finite metric space and \( Y \) is a one-point extension of \( X \), in which case the extra point is denoted by \( y \).

Given \( X \sqsubseteq Y \) with \( X, Y \in \mathcal{C} \), we let \( \Psi_{X \sqsubseteq Y}^\epsilon \) denote the sentence

\[
\sup_{\vec{v}} \min_{\vec{w}} (\epsilon - \text{Conf}_X(\vec{v}), \inf \text{Conf}_Y(\vec{v}, \vec{w}) - \epsilon).
\]

**Theorem 2.** The set of conditions \( \{ \Psi_{X \sqsubseteq Y}^\epsilon : \epsilon > 0, X \sqsubseteq Y, X, Y \in \mathcal{C} \} \) axiomatizes Th(\( \mathcal{A}_S \)).

**Proof.** That these axioms hold in Th(\( \mathcal{A}_S \)) follows from the definition of \( \mathcal{A}_S \) and a simple amalgamation result that we discuss in the appendix. It remains to note that any separable model of these axioms must be isometric to \( \mathcal{A}_S \) by a simple back-and-forth argument. \( \Box \)

We end this section with one result for the model-theorists:

**Theorem 3.** Th(\( \mathcal{A}_S \)) is not stable but is supersimple of \( U \)-rank 1. Moreover, forking independence is characterized by

\[
A \perp_C B \iff A \cap B \subseteq C,
\]

where \( A, B, C \) are small subsets of some monster model \( \mathcal{A}_S^* \) of Th(\( \mathcal{A}_S \)).
Proof. It is straightforward to verify that the independence relation in the above display satisfies all of the axioms of forking independence in simple theories. We verify only the Independence Theorem over Models. Suppose that \( M \) is a model, \( M \subseteq A, B, A \downarrow_{M} B \) and \( p(x) \in S(A) \) and \( q(x) \in S(B) \) are types with a common restriction to \( M \). Since we are free to amalgamate \( A \) and \( B \) in \( A\S^* \) over \( M \) (again, see the appendix), the types \( p \) and \( q \) can be amalgamated.

Since \( A\S \) is discrete, to see that \( \text{Th}(A\S) \) is supersimple, it suffices to show that any type does not fork over a finite subset of its domain. If \( p(x) \in S_n(A) \) and \( a = (a_1, \ldots, a_n) \models p \), then setting \( B := \{a_1, \ldots, a_n\} \cap A \), we see that \( p \) does not fork over \( B \).

To see that the \( U \)-rank of the theory is 1, suppose that \( p \in S_1(A) \) is a type with \( U(p) \geq 1 \). Take a forking extension \( q \in S_1(B) \) of \( p \) and let \( a \models p \). Then \( a \in B \setminus A \) and thus the condition \( d(x, a) = 0 \) belongs to \( q \). It follows that \( q \) is algebraic, whence \( U(p) = 1 \).

To see that \( \text{Th}(A\S) \) is not stable, let \( p(x) \) be any 1-type over a model \( M \), let \( a \in A\S^* \) realize \( p \) and take \( b \in A\S^* \setminus Ma \). Then we can assign \( d(x, b) \) to be any number in \( \left(\frac{1}{2}, 1\right) \) and obtain an extension of \( p \) to \( Mb \) in this manner. Thus, there are continuum many different nonforking extensions of \( p \) to \( Mb \).

The reader should contrast the previous result with the case of the Urysohn sphere, which is not simple (see [5]).

3. THE ALMOST-SURE THEORY OF FINITE METRIC SPACES

We set \( \lambda_n \) to be Lebesgue measure on \([0, 1]\) \( \overline{\cap}_{i=1}^{n} C_{n} := \left(\frac{1}{2}, 1\right) \), \( \mu_n := \lambda_n \restriction C_{n} \), and \( \mu_n' := \frac{\mu_n}{\mu_n(C_{n})} = 2^{n} \mu_n \), a probability measure on \( C_{n} \).

We identify \( \tilde{d} = (d_{ij}) \in C_{n} \) with the metric space on \( \{1, \ldots, n\} \) with \( d(i, j) := d_{ij} \). In this manner, if \( X \subseteq \mathcal{C} \), we write \( \text{Conf}_{X}(\tilde{d}) \), with the interpretation that the appearance of \( \text{d}(v_1, v_j) \) gets replaced with \( d_{ij} \). We perform a similar identification with \( \Psi_{X \subseteq Y}^{c}(\tilde{d}) \).

Theorem 4. For any \( X_1 \subseteq Y_1, \ldots, X_m \subseteq Y_m \) from \( \mathcal{C} \) and any \( \epsilon > 0 \), we have

\[
\lim_{n \to \infty} \mu_n' \left( \left\{ \tilde{d} \in C_{n} : \max_{i=1, \ldots, n} \Psi_{X \subseteq Y_1}^{c}(\tilde{d}) = 0 \right\} \right) = 1.
\]

Proof. Fix \( i \in \{1, \ldots, m\} \) and set \( X := X_i \) and \( Y := Y_i \). We decompose

\[
C_{n} = C_{k} \times \left[\frac{1}{2}, 1\right]^{k} \cdots \times \left[\frac{1}{2}, 1\right]^{k} \times C_{n-k}
\]

and likewise decompose \( \tilde{d} = (\tilde{d}', \tilde{d}', \ldots, \tilde{d}'_{n-k}, \tilde{d}'_{n}) \). The intention is that \( \tilde{d}' \) represents \( d_{ij} \) for \( 1 \leq i < j \leq k \), \( \tilde{d}'_{t} \) represents \( d_{ii} \) for \( i = 1, \ldots, k \) and \( t = k + 1, \ldots, n \), and \( \tilde{d}'_{n} \) represents \( d_{ij} \) for \( k + 1 \leq i < j \leq n \). Set

\[
A := \{ \tilde{d} \in C_{n} : \text{Conf}_{X}(\tilde{d}') \leq \epsilon \text{ and } \text{Conf}_{Y}(\tilde{d}', \tilde{d}'_{t}) \geq \epsilon \text{ for all } t = 1, \ldots, n-k \}.
\]
Let $B$ be the projection of $A$ onto $C_k$ and $B'$ be the projection of $A$ onto $C_k \times [\frac{1}{2}, 1]^k \cdots \times [\frac{1}{2}, 1]^k$. Since we are free to assign the distances to $x_{k+1}, \ldots, x_n$ in any way we want (as the triangle inequality is always satisfied), we have $\mu_n(A) = (\mu_k \times \eta^{n-k})(B') \cdot \mu_{n-k}(C_{n-k})$; here, $\eta$ is Lebesgue measure on $[\frac{1}{2}, 1]^k$. For $\tilde{d}' \in B$, set $f(\tilde{d}') := \eta([\tilde{d}' \in [\frac{1}{2}, 1]^k : \operatorname{Conf}_Y(\tilde{d}', \tilde{d}') \leq \varepsilon])$. Note that $f(\tilde{d}') < (\frac{1}{2})^k$ for each $\tilde{d}' \in B$. Since $f$ is upper semi-continuous, $\sup_{\tilde{d}' \in B} f(\tilde{d}') = \max_{\tilde{d}' \in B} f(\tilde{d}')$, whence there is $\delta > 0$ such that $f(\tilde{d}') \leq (\frac{1}{2})^k - \delta$ for all $\tilde{d}' \in B$. Now notice that, since we can independently assign the values of $d_{1t}$, we have $(\mu_k \times \eta^{n-k})(B') \leq (\frac{1}{2})^k - \delta)^{n-k} \mu_k(B)$. After normalizing, we have

$$\mu'_n(A) \leq 2^n \mu_k(B)((\frac{1}{2})^k - \delta)^{n-k} \mu_{n-k}(C_{n-k}).$$

Since

$$\binom{n}{2} = \binom{k}{2} + k(n-k) + \binom{n-k}{2},$$

we rewrite the right-hand side of the above inequality as

$$\mu'_n(B)(1 - 2^k \delta)^{n-k}.$$ 

This entire expression is then of the form $C_d p^m$ for some constants $C_i > 0$ and $p_i < 1$, which are independent of $n$ (but do depend on $i$).

Now $\mu'_n(A)$ is an upper bound of the measure of those $\tilde{d} \in C_n$ which make $\Psi^e_{X,Y}$ false as witnessed by $x_1, \ldots, x_k$, that is, the “first” $k$ points of the space. However, the same reasoning applies to any $k$ points from our space, whence

$$\mu_n([\tilde{d} \in C_n : \Psi^e_{X,Y} > 0]) \leq n(n-1) \cdots n(n-k+1) C_d p^m.$$ 

Set $C := \max_{1 \leq i \leq m} C_i$ and $p := \max_{1 \leq i \leq m} p_i$. We then have

$$\mu'_n([\tilde{d} \in C_n : \max_{i=1,\ldots,m} \Psi^e_{X_i,Y_i}(\tilde{d}) > 0] \leq m \cdot n(n-1) \cdots n(n-k+1) C_d p^m.$$ 

Since the quantity on the right goes to $0$ as $n$ tends to $\infty$, we have the desired result.

Let $M_n \subseteq [0, 1]^{[n]}$ denote the set of all metric spaces on $\{1, \ldots, n\}$ with values in $[0, 1]$. Note that $C_n \subseteq M_n$. We let $\nu_n$ be Lebesgue measure normalized to $M_n$, that is, $\nu_n(A) = \frac{\lambda_n(A)}{\lambda_n(M_n)}$.

**Lemma 5.** $\lim_{n \to \infty} \nu_n(C_n) = 1$.

**Proof.** Let $C^n_\tau$ be the set of $1/\tau$-grid points in $C_n$ and similarly for $M^n_\tau$, that is, the $n$-element $[0, 1]$-metric spaces with distances that are multiples of $1/\tau$. By [3], Theorem 1.2, for even $\tau \geq 4$, there is $\beta > 0$ such that for all $n$,

$$|C^n_\tau| \leq |M^n_\tau| \leq \frac{1}{1 - 2^{-\beta n}|C^n_\tau|}.$$
From this we conclude that for large, even \( r \), \( \frac{|C_n^r|}{|M_n^r|} \) tends to 1 as \( n \) tends to infinity. Since \( \frac{|C_n^r|}{|M_n^r|} \) approximates \( \nu(C_n) \) as \( r \to \infty \) (by grid approximation to Lebesgue measure), we have the desired result. \( \square \)

Given \( \tilde{d} \in M_n \) and an \( L \)-sentence \( \sigma \), we write \( \sigma^{\tilde{d}} \) for the value of \( \sigma \) in the metric space on \( \{1, \ldots, n\} \) corresponding to \( \tilde{d} \).

**Theorem 6 (0-1 law).** For any sentence \( \sigma \) and any \( \delta > 0 \), we have
\[
\nu_n(\{|\tilde{d} \in M_n : |\sigma^{\tilde{d}} - \sigma^{A^S}| < \delta\}) = 1.
\]

**Proof.** Set \( r := \sigma^{A^S} \). Take \( X_1 \subset Y_1, \ldots, X_m \subset Y_m \), finitely many 1-point extensions of finite \([1/2, 1]\)-metric spaces, and \( \epsilon > 0 \) such that
\[
\max_{1 \leq i \leq m} \psi_{[X_i,Y_i]}(\tilde{d}) = 0 \implies |\sigma^{\tilde{d}} - r| < \delta.
\]

Let \( A_n := \{\tilde{d} \in M_n : \max_{1 \leq i \leq m} \psi_{[X_i,Y_i]}(\tilde{d}) = 0\} \). We then have
\[
\nu_n(A_n) = \frac{\lambda_n(A_n)}{\lambda_n(M_n)} = \frac{\lambda_n(A_n \cap C_n)}{\lambda_n(C_n)} \cdot \frac{\lambda_n(C_n)}{\lambda_n(M_n)} + \frac{\lambda_n(A_n \setminus C_n)}{\lambda_n(M_n)}.
\]

By Theorem 4 and Lemma 5, the right-hand side of the previous display approaches 1 as \( n \) approaches \( \infty \), whence the desired result follows. \( \square \)

**Appendix A. An amalgamation reminder**

We remind the reader of the following construction: suppose that \( X \sqsubset Y \) is a 1-point extension, with \( y \) as the additional point from \( Y \), and suppose that \( X \sqsubset Z \) is another finite extension of \( X \). Then \( Y \) and \( Z \) can be amalgamated over \( X \) by setting, for every \( z \in Z \),
\[
d(y,z) := \min_x (d(x,y) + d(x,z)).
\]

Moreover, if \( X = \{x_1, \ldots, x_n\} \) and \( Z = X \cup \{z_1, \ldots, z_n\} \) then with the above amalgamation, we have
\[
|d(y,x_i) - d(y,z_i)| \leq d(x_i, z_i).
\]

This calculation shows that if we have two \( n \)-element metric spaces \( X = \{x_1, \ldots, x_n\} \) and \( X' = \{x'_1, \ldots, x'_n\} \) that have been jointly embedded in some metric space \( Z \) such that \( d(X,X') \leq \epsilon \) (that is, \( \max_i d(x_i,x'_i) \leq \epsilon \) for all \( i = 1, \ldots, n \)), then for \( Y \) as above, it is possible to jointly embed \( Y \) and \( Z \) in such a way that \( d(Y,X' \cup \{y\}) \leq \epsilon \).

**References**


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