COMPUTABLE K-THEORY FOR C*-ALGEBRAS II: AF ALGEBRAS

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Abstract. We continue the study of the effective content of K-theory for C*-algebras, with a focus on AF algebras. We show that from a c.e. presentation of an AF algebra it is possible to compute a representation of the algebra as an inductive limit of finite-dimensional algebras. Using this, and an analogous result for dimension groups, we show that the computable K_0 functor provides a computable equivalence of categories between c.e. presentations of AF algebras and c.e. presentations of unital (scaled) dimension groups, giving an effective version of Elliott's classification theorem. We use our results to determine the complexity of the index set and isomorphism problems for various classes of AF algebras.

Contents

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1. Introduction

In our previous work [9] with R. Miller, we initiated the effective study of K-theory for C*-algebras. More precisely, we established a computable functor which associated, to each computably enumerable (c.e.) presentation $\mathbf{A}^\#$ of a C*-algebra $\mathbf{A}$, a c.e. presentation $K_0(\mathbf{A}^\#)$ of its K_0 group $K_0(\mathbf{A})$. In addition, this functor maps computable $*$-homomorphisms to computable group homomorphisms. These statements are made precise in Section 3.2.2 below. A similar such computable functor was also defined for K_1 as well. As a proof of concept, the special case of the K_0 functor restricted to *uniformly hyperfinite (UHF)* algebras was completely analyzed.

In this paper, we broaden our scope and consider our computable K_0 functor restricted to the class of *approximately finite-dimensional (AF)* algebras, which are the C*-algebras isomorphic to those obtained as inductive limits of finite-dimensional C*-algebras. Every AF algebra $\mathbf{A}$ has $K_1(\mathbf{A}) = 0$, so the K-theory for AF algebras is entirely focused on K_0. Since AF algebras are stably finite, their K_0 group has a natural ordering which renders it a (partially) ordered abelian group. In addition, for a c.e. presentation $\mathbf{A}^\#$, the positive cone of this group, which we denote $K_0(\mathbf{A})^+$, is a c.e. set of the presentation $K_0(\mathbf{A}^\#)$; accordingly, we refer to $(K_0(\mathbf{A}^\#), K_0(\mathbf{A})^+)$ as the *ordered presented group of* $\mathbf{A}^\#$. The ordered abelian groups that arise as ordered K_0 groups of AF algebras are known as *dimension groups*. By the Effros-Handelman-Shen Theorem, these are precisely the ordered abelian groups isomorphic to an inductive limit of *simplicial groups*, that is, groups of the form $\mathbb{Z}^n$ with their canonical ordering. By distinguishing the image $K_0(\mathbf{1}_\mathbf{A})$ of the unit of $\mathbf{A}$ in $K_0(\mathbf{A})$, one views $K_0(\mathbf{A})$ as a *unital (or scaled) group*, that is, as an ordered abelian group with a distinguished order unit. Elliott's classification theorem [10] states that AF algebras are characterized, up to isomorphism, by their unital ordered K_0 groups. In fact, the K_0 functor yields an equivalence of categories between the category of AF algebras, with approximate unitary equivalence classes of $*$-algebra homomorphisms as morphisms, and the category of unital dimension groups, with order-unit preserving positive group homomorphisms as morphisms.

While K_0 groups provide an algebraic representation of AF algebras, Bratteli diagrams provide a combinatorial method of representing AF algebras, namely as infinite multi-graphs arranged in a sequence of levels. Labeled Bratteli diagrams assign numbers to the vertexes, and every AF algebra is completely described by a labeled Bratteli diagram.

Our first overarching goal is to classify the c.e. presentations of AF algebras. To this end, we introduce the concept of an *AF certificate* (defined in Section 2.3.5

below). These essentially describe an AF algebra as a specific inductive limit of finite dimensional algebras. Our first main result is the following.

Main Theorem 1. *If a unital AF algebra has a c.e. presentation, then it also has a computable AF certificate.*

In classical mathematics, when an AF algebra is considered, a representation of it as an inductive limit of finite-dimensional algebras is assumed. However, a c.e. presentation of an AF algebra, which merely provides information about the norm on a certain dense set, does not in and of itself readily provide such a description. Rather, from the information provided by a c.e. presentation of an AF algebra $\mathbf{A}$, one must algorithmically discern the finite-dimensional $*$-subalgebras of $\mathbf{A}$ (which are not the same as the finitely generated subalgebras). Locating these $*$-subalgebras turns out to be no mean feat, and as an intermediate step we go to great lengths to first prove an effective version of Glimm's Lemma (Theorem 6.1 below). We note that our proof of Main Theorem 1 is uniform in that it provides an algorithm for producing AF certificates from c.e. presentations of AF algebras.

Our second main result, which will also contribute to our first overarching goal, is an effective version of the Effros-Handelman-Shen Theorem referred to above.

Main Theorem 2. *Suppose $\mathcal{G}^\#$ is a c.e. presentation of a dimension group.*

(1) *There is a c.e. presentation $\mathbf{A}^\#$ of an AF algebra so that the presented ordered K_0 group of $\mathbf{A}^\#$ is computably isomorphic to $\mathcal{G}^\#$.*
(2) *If u is an order unit of $\mathcal{G}$, then there is a c.e. presentation $\mathbf{A}^\#$ of a unital AF algebra so that the presented unital ordered K_0 group of $\mathbf{A}^\#$ is computably isomorphic to $(\mathcal{G}, u)^\#$.*

Again, the proof is uniform. With these two results in hand, we then achieve our goal of classifying the c.e. presentations of unital AF algebras.

Main Theorem 3 (Classification of computably presentable unital AF algebras)**.** *Suppose $\mathbf{A}$ is a unital AF algebra. Then, the following are equivalent.*

(1) $\mathbf{A}$ *has a c.e. presentation.*
(2) *There is a computable inductive sequence of finite dimensional algebras whose inductive limit is $*$-isomorphic to $\mathbf{A}$ and so that each bonding map is unital.*
(3) $\mathbf{A}$ *is computably presentatable.*
(4) *The unital ordered K_0 group of $\mathbf{A}$ is computably presentatable.*
(5) $\mathbf{A}$ *has a computably presentable labeled Bratteli diagram.*

Once again, the proof is uniform. Our next overarching goal is to strengthen this classification result by showing that computably isomorphic objects map to computably isomorphic objects. This is the content of the following.

Main Theorem 4. *Suppose $\mathbf{A}^\#$ and $\mathbf{B}^\#$ are c.e. presentations of unital AF algebras. Then, the following are equivalent.*

(1) $\mathbf{A}^\#$ *is computably $*$-isomorphic to $\mathbf{B}^\#$.*
(2) *The presented unital ordered K_0 groups of $\mathbf{A}^\#$ and $\mathbf{B}^\#$ are computably isomorphic.*
(3) *Every Bratteli diagram of a computable AF certificate of $\mathbf{A}^\#$ is computably equivalent to every Brattelii diagram of a computable AF certificate of $\mathbf{B}^\#$.*

Not surprisingly, the proof is uniform. Our results so far suggest that it should be possible to demonstrate an effective version of Elliott's equivalence of categories, and in fact it is:

Main Theorem 5. *The following categories are computably equivalent:*

(1) *The category of c.e. presentations of unital AF algebras.*
(2) *The category of c.e. presentations of unital dimension groups.*

We formally define the terminology in this theorem in Section 11 below; for now let us say that computable equivalence means the required functors and natural isomorphisms can be computed. The proof is very much based on the uniformity of the proofs of the prior theorems.

Our main theorems have several interesting consequences. For one, every c.e. presentation of a unital AF algebra is computable (Corollary 7.3). Also, in contrast to UHF algebras, there is an AF algebra that is not computably categorical (Corollary 10.3). We can use this observation to prove similar statements for dimension groups and Bratteli diagrams. Finally, we use our results to determine the complexity of the index set and isomorphism problems for AF and UHF algebras.

This paper brings together material from several areas of mathematics: computability theory, category theory, group theory, and the theory of C^*-algebras. In order to make the presentation fairly self-contained and navigable, we have chosen the following organization. We first cover background material from these areas; this consists of information that is already in the literature but which we attempt to organize and summarize so that the reader need not constantly consult an array of other sources. We divide this coverage into two components: the background from classical mathematics (Section 2) and from effective mathematics (Section 3). The latter presents a picture of the computability theory of the topics covered in the former with a parallel organization. We then attend to the development of preliminary matters from classical and effective mathematics (Sections 4 and 5). The Effective Glimm Lemma is proven in Section 6, and our main theorems are proven in Sections 7 through 11. Section 12 contains our results on index sets

2. Background from classical mathematics

In this section, we summarize relevant background material from four subjects which will be intertwined throughout the rest of this paper: inductive limits in categories, ordered abelian groups, C^*-algebras, and Bratteli diagrams. We begin with inductive limits as this topic supports much of our work with regards to the second and third of these topics.

2.1. Inductive limits in categories.

Inductive limits are ubiquitous in mathematics by virtue of their ability to construct large objects from chains of smaller ones. Here, we discuss how category theory provides a framework for unifying these constructions so that one need not repeat definitions and arguments that are essentially identical. Accordingly, we now assume $\mathcal{C}$ denotes a category with a zero object.

Suppose that for each $n \in \mathbb{N}$, A_n is an object of $\mathcal{C}$ and ϕ_n is a morphism from A_n to A_{n+1}. The sequence $(A_n, \phi_n)_{n\in\mathbb{N}}$ is called an *inductive sequence of* $\mathcal{C}$. Through composition, these morphisms yield morphisms $\phi_{n,n'}$ from A_n to $A_{n'}$ whenever $n' > n$. Additionally, we define $\phi_{n,n}$ to be the identity morphism of A_n, and when $\ell < n$, let $\phi_{n,\ell}$ denote the zero morphism.

Now suppose A is an object of $\mathcal{C}$, and assume that for each $n \in \mathbb{N}$, ν_n is a morphism from A_n to A so that $\nu_n = \nu_{n+1} \circ \phi_n$. We then refer to $(A, (\nu_n)_{n\in\mathbb{N}})$ as an *inductive upper limit* of $(A_n, \phi_n)_{n\in\mathbb{N}}$. If $(B, (\mu_n)_{n\in\mathbb{N}})$ is also an inductive upper limit of $(A_n, \phi_n)_{n\in\mathbb{N}}$, then a *reduction* of $(A, (\nu_n)_{n\in\mathbb{N}})$ to $(B, (\mu_n)_{n\in\mathbb{N}})$ is a morphism λ from A to B so that $\lambda \circ \nu_n = \mu_n$ for all $n \in \mathbb{N}$. We say that $(A, (\nu_n)_{n\in\mathbb{N}})$ is an *inductive limit* of $(A_n, \phi_n)_{n\in\mathbb{N}}$ if for every inductive upper limit $(B, (\mu_n)_{n\in\mathbb{N}})$ of $(A_n, \phi_n)_{n\in\mathbb{N}}$, there is a unique reduction from $(A, (\nu_n)_{n\in\mathbb{N}})$ to $(B, (\mu_n)_{n\in\mathbb{N}})$. The following is a fairly direct consequence of the definitions.

Proposition 2.1. *Suppose $(A, (\nu_n)_{n\in\mathbb{N}})$ and $(B, (\mu_n)_{n\in\mathbb{N}})$ are inductive limits of $(A_n, \phi_n)_{n\in\mathbb{N}}$. Then the reduction from $(A, (\nu_n)_{n\in\mathbb{N}})$ to $(B, (\mu_n)_{n\in\mathbb{N}})$ is an isomorphism.*

As a result of the previous proposition, one may then speak of *the* inductive limit of an inductive sequence (when it exists).

2.2. **Ordered abelian groups.** We now move on to discussing the second of our four topics, ordered abelian groups. Our goal here is to summarize the pertinent concepts and results from the theory of ordered abelian groups. We refer the reader to standard sources such as [18] for more comprehensive treatments.

2.2.1. *Basic definitions and principles.* Fix an abelian group G which we write additively and whose identity element we denote $\mathbf{0}_G$. Traditionally, the ordered abelian groups studied by computability theorists are linearly ordered. However, the investigation of AF algebras necessitates the consideration of partial orders that are consistent with the group operation. The most direct way to establish such an order on G is via a *positive cone* which is defined to be a subset P of G that satisfies the following:

(1) P is closed under addition.
(2) $P \cap (-P) = \{\mathbf{0}_G\}$.
(3) $P + (-P) = G$.

Suppose P is a positive cone of G. This positive cone defines a partial order $\leq$ on G so that $x \leq y$ if and only if $y - x \in P$. Accordingly, we call the pair (G, P) an *ordered abelian group*.

We now set $\mathcal{G} = (G, P)$; we may sometimes write $\mathcal{G}^+$ instead of P. Suppose $\mathcal{G}' = (G', P')$ is an ordered abelian group. We say that a map ϕ is a *homomorphism* from $\mathcal{G}$ to $\mathcal{G}'$ if ϕ is a homomorphism from G to G' (as groups) so that $\phi[P] \subseteq P'$; we may also say that ϕ is a *positive* morphism. The ordered abelian groups form a category in which the morphisms are these homomorphisms. We refer to $\mathcal{G}$ as a *subgroup* of $\mathcal{G}'$ if G is a subgroup of G' and $P \subseteq P'$. If $\mathcal{G}$ is a subgroup of $\mathcal{G}'$, and if every element of $\mathcal{G}'$ between g and g' belongs to $\mathcal{G}$ whenever g and g' belong to $\mathcal{G}$, then we say that $\mathcal{G}$ is *convex*.

A positive element u of $\mathcal{G}$ is called an *order unit (or a scale) of* $\mathcal{G}$ if for every $g \in \mathcal{G}$ there exists a positive integer n so that $-nu \leq g \leq nu$. If u is an order unit of $\mathcal{G}$, then $(\mathcal{G}, u)$ is referred to as a *unital ordered abelian group*. When $(\mathcal{G}, u)$ and $(\mathcal{H}, v)$ are unital ordered abelian groups, we say that a map ϕ is a *homomorphism* from $(\mathcal{G}, u)$ to $(\mathcal{H}, v)$ if it is a homomorphism from $\mathcal{G}$ to $\mathcal{H}$ for which $\phi(u) = u'$. The unital ordered abelian groups form a category in which the morphisms are these homomorphisms.

The following principle will be very useful in our proof of Main Theorem 2. We include a proof for the sake of completeness.

Lemma 2.2. *Suppose u is a positive element of an ordered abelian group $\mathcal{G}$. Then u is an order unit of the convex subgroup of $\mathcal{G}$ that is generated by u.*

Proof. Suppose $\mathcal{G} = (G, P)$. Let $\mathcal{H}$ denote the convex subgroup of $\mathcal{G}$ that is generated by u. Set

$$H' = \bigcup_{n \in \mathbb{N}} [-nu, nu].$$

Let $\mathcal{H}' = (H', H' \cap P)$. It follows that $\mathcal{H}'$ is a subgroup of $\mathcal{G}$.

We claim that $\mathcal{H}'$ is convex. To see this, let $g, g' \in H'$. Assume $h \in G$ and $g \leq h \leq g'$. There exist $n, n' \in \mathbb{N}$ so that $-nu \leq g \leq nu$ and so that $-n'u \leq g' \leq n'u$. Thus, $h \leq n'u \leq (n+n')u$ and $h \geq -nu \geq -(n+n')u$. Thus, $\mathcal{H}'$ is convex. Hence, $\mathcal{H}$ is a subgroup of $\mathcal{H}'$ and the lemma is proven. □

2.2.2. *Simplicial groups.* Fix a positive integer n. There is a natural positive cone on the group $\mathbb{Z}^n$, namely the set of all n-tuples of nonnegative integers. We denote this positive cone by $\mathbb{Z}^n_{\geq 0}$, and we identify $\mathbb{Z}^n$ with $(\mathbb{Z}^n, \mathbb{Z}^n_{\geq 0})$. These ordered groups are referred to as *simplicial* and they are the building blocks for many of the ordered groups we wish to consider.

We also establish a bit of notation. We let $e^n_1, \ldots, e^n_n$ denote the standard basis elements of $\mathbb{Z}^n$ and when $h \in \mathbb{Z}^n$ and $g = e^n_j$, we set $\pi_g(h) = h(j)$.

2.2.3. *Inductive limits.* It is fairly well-known that every inductive sequence of abelian groups has an inductive limit; see the proof of Lemma 4.1 below. Similarly, every inductive sequence in the category of ordered abelian groups has an inductive limit; see, for example, [20, Proposition 6.2.6]. A similar argument establishes that every inductive sequence in the category of unital ordered abelian groups has an inductive limit.

2.2.4. *Dimension groups.* We have now established sufficient material for a formal discussion of dimension groups, which are treated extensively in [8] and in [18]. Fix an ordered abelian group $\mathcal{G}$. A *certificate of dimensionality* of $\mathcal{G}$ is a sequence $(n_s, \phi_s, \nu_s)_{s \in \mathbb{N}}$, so that $(\mathcal{G}, (\nu_s)_{s \in \mathbb{N}})$ is an inductive limit of $(\mathbb{Z}^{n_s}, \phi_s)_{s \in \mathbb{N}}$ in the category of ordered abelian groups. A *dimension group* is an ordered abelian group that has a certificate of dimensionality.

The Shen property, which we define momentarily, is a key property of dimension groups. The definition is based on the concept of a factoring of a homomoprhism, which we define as follows. Suppose $\theta : \mathbb{Z}^n \to G$, $\phi : \mathbb{Z}^n \to \mathbb{Z}^p$, and $\theta' : \mathbb{Z}^p \to G$ are group homomorphisms and let $\alpha \in \ker(\theta)$. We say that (ϕ, θ') *factors θ at α* if $\alpha \in \ker(\phi)$ and $\theta = \theta' \circ \phi$.

Suppose $\mathcal{G}$ is an ordered abelian group. We then say that $\mathcal{G}$ has the *Shen Property* if for every homomorphism $\theta : \mathbb{Z}^n \to \mathcal{G}$ and every $\alpha \in \ker(\theta)$, there is a pair of ordered group homomorphisms $\phi : \mathbb{Z}^n \to \mathbb{Z}^p$ and $\theta' : \mathbb{Z}^p \to G$ that factors θ at α.

We will need the following fact (see [8, Lemma 3.4.4]):

Proposition 2.3. *Every dimension group has the Shen property.*

2.3. C***-algebras.** We now proceed to the third of our four background topics: C*-algebras. We assume the reader is familiar with the basic definitions and results of the theory of C*-algebras which can be found in [6]. We begin by reviewing some additional general background information before proceeding to the more specific topics of K-theory, matrix algebras, finite-dimensional algebras, and AF algebras. We conclude this section with a collection of perturbation principles which will be very useful in our proof of the Effective Glimm Lemma alluded to in the introduction.

2.3.1. *General background.* If U is a subset of a metric space X, and if $\epsilon > 0$, we let $B(U;\epsilon) = \{p \in X \ : \ d(p, U) < \epsilon\}$.

Let $\mathbf{A}$ be a C*-algebra. If $\mathbf{A}$ is unital, we let $\mathbf{1_A}$ denote the unit of $\mathbf{A}$. Let $\mathrm{Proj}(\mathbf{A})$ denote the set of all projections of $\mathbf{A}$. We note that, by the C*-identity, every element of $\mathrm{Proj}(\mathbf{A})$ has norm 0 or 1. We say that $v \in \mathbf{A}$ is a *partial isometry* if $v^*v \in \mathrm{Proj}(\mathbf{A})$. Partial isometries will play a role in the perturbation principles we discusss below and hence in the proof of the Effective Glimm Lemma.

One of the consequences of the C*-identity is that every $*$-homomorphism of C*-algebras is 1-Lipschitz. This principle simplifies many arguments concerning the computability of these maps.

When $u \in \mathbf{A}$ is unitary, we let $\mathrm{Ad}_u : \mathbf{A} \to \mathbf{A}$ denote the map given by $\mathrm{Ad}_u(a) = uau^*$ for all $a \in \mathbf{A}$. These maps will prove advantageous when classifying $*$-homomorphisms of finite-dimensional algebras.

An $n \times n$ array $(a_{r,s})_{r,s}$ of self-adjoint elements of $\mathbf{A}$ is an $n \times n$ *system of matrix units of* $\mathbf{A}$ if $a_{r,s}a_{r',s'} = \delta_{s,r'}a_{r,s'}$ for all $r, s, r', s' \in \{1, \ldots, n\}$. If $\mathbf{A}$ is unital, then we call such a system *unital* if $\sum_j a_{j,j} = \mathbf{1_A}$. The construction of these systems plays a key role in the computability theory of finite-dimensional algebras and AF algebras.

Suppose $\mathbf{B}$ is a C*-algebra. The direct sum $\mathbf{A} \bigoplus \mathbf{B}$ is also a C*-algebra; in particular, the norm is defined by $\|(a, b)\| = \max\{\|a\|, \|b\|\}$. This algebra is also denoted by $\mathbf{A} \times \mathbf{B}$.

Assume ϕ and ψ are $*$-homomorphisms from $\mathbf{A}$ to $\mathbf{B}$. We say that ϕ and ψ are *approximately unitarily equivalent* if there is a sequence $(u_n)_{n\in\mathbb{N}}$ of unitaries of B so that $(\mathrm{Ad}_{u_n} \circ \phi)_{n\in\mathbb{N}}$ converges pointwise to ψ. We will rely on this equivalence relation when we discuss AF algebras as a category in the proof of Main Theorem 5.

2.3.2. *K-theory.* We now provide a quick synopsis of the fundamental principles of K-theory which are expounded much more completely in [20]. The reader may also consult our previous paper [9] for an explanation of just the definitions and results we rely on here. To start, K-theory defines a functor K_0 from the category of C*-algebras to the category of abelian groups. If $\mathbf{A}$ is a separable C*-algebra, then $K_0(\mathbf{A})$ is countable. If $\mathbf{A}$ is stably finite[1], then K-theory also associates to $\mathbf{A}$ a canonical positive cone of $K_0(\mathbf{A})$, which is denoted $K_0(\mathbf{A})^+$. If $\mathbf{A}$, $\mathbf{B}$ are C*-algebras and ϕ is a $*$-homomorphism from $\mathbf{A}$ to $\mathbf{B}$, then $K_0(\phi)$ is a homomorphism from $K_0(\mathbf{A})$ to $K_0(\mathbf{B})$; if, in addition, $\mathbf{A}$ and $\mathbf{B}$ are stably finite, then $K_0(\varphi)$ is an ordered group homomorphism from $(K_0(\mathbf{A}), K_0(\mathbf{A})^+)$ to $(K_0(\mathbf{B}), K_0(\mathbf{B})^+)$. If $\mathbf{A}$ is stably finite and unital, then $(K_0(\mathbf{A}), K_0(\mathbf{A})^+)$ has an order unit; in fact, K-theory

[1]Due to its complexity, we demur from precisely defining 'stably finite'; the curious reader may find a definition in standard sources such as [20]. Suffice it to say, AF algebras are stably finite.

associates $\mathbf{A}$ with a particular order unit of $(K_0(\mathbf{A}), K_0(\mathbf{A})^+)$, which we denote by $K_0(\mathbf{1_A})$ (since it is in fact obtained from the unit of $\mathbf{A}$). Furthermore, if $\mathbf{A}$ and $\mathbf{B}$ are stably finite and unital, and if ϕ is a unital $*$-homormorphism from $\mathbf{A}$ to $\mathbf{B}$, then $K_0(\phi)(K_0(\mathbf{1_A})) = K_0(\mathbf{1_B})$. Hence, *we have a functor from the category of stably finite unital* C*-*algebras (in which the morphisms are assumed to be unital) to the category of unital ordered abelian groups*; we denote this functor by K_0^{sc} (sc stands for *scaled* as the triple consisting of an ordered abelian group together with an order unit is often called a *scaled group*).

2.3.3. *Matrix algebras.* We now discuss matrix algebras, which are the building blocks of AF algebras. Fix a positive integer n. Let $M_n(\mathbb{C})$ denote the C*-algebra of $n \times n$ complex matrices, which is a C*-algebra under the operator norm and in which the involution is the adjoint operation.

Suppose that ℓ is also a positive integer. It is well-known that there is a nonzero $*$-homomorphism from $M_n(\mathbb{C})$ to $M_\ell(\mathbb{C})$ if and only if $\ell \geq n$ (see, for example, [6]). Suppose $\ell \geq n$. For k a positive integer for which $kn \leq \ell$ define, for each $A \in M_n(\mathbb{C})$, $\mathcal{E}_{n,\ell,k}(A)$ to be the $\ell \times \ell$ matrix obtained by repeating A k times along the diagonal. It is well-known that if $\psi : M_n(\mathbb{C}) \to M_\ell(\mathbb{C})$ is a nonzero $*$-homomorphism, then there is a unique positive integer k and a unitary $U \in M_\ell(\mathbb{C})$ so that $\psi(A) = \mathrm{Ad}_U(\mathcal{E}_{n,\ell,k}(A))$ (again, see [6]). The number k is called the *multiplicity* of ψ.

We now discuss the K-theory of these algebras. To begin, there is a standard isomorphism ζ_n from $K_0^{\mathrm{sc}}(M_n(\mathbb{C}))$ to $(\mathbb{Z}, n)$ (that is, the ordered group $\mathbb{Z}^n$ together with the order unit n). If ϕ is a homomorphism from $K_0^{\mathrm{sc}}(M_n(\mathbb{C}))$ to $K_0^{\mathrm{sc}}(M_\ell(\mathbb{C}))$, then there is a canonical homomorphism $\eta_{n,\ell}(\phi)$ from $(\mathbb{Z}, n)$ to $(\mathbb{Z}, \ell)$ so that $\eta_{n,\ell}(\phi) \circ \zeta_n = \zeta_\ell \circ \phi$.

2.3.4. *Finite-dimensional* C*-*algebras.* We state the following well-known result as a theorem for future reference; a proof can be found in standard resources such as [6].

Theorem 2.4 (Classification of finite-dimensional algebras)**.** *A* C*-*algebra* $\mathbf{A}$ *is finite-dimensional if and only if there is a finite multiset* F *of positive integers so that* $\mathbf{A}$ *is* $*$-*isomorphic to* $\bigoplus_{n \in F} M_n(\mathbb{C})$.

Suppose F is a finite multiset of positive integers and set $\mathbf{B} = \bigoplus_{n \in F} M_n(\mathbb{C})$. When π is the projection map associated with a summand of $\mathbf{B}$, we associate π with an injection map $\iota_\pi : \mathrm{ran}(\pi) \to \mathbf{B}$ so that $\pi \circ \iota_\pi = \mathrm{Id}_{\mathrm{ran}(\pi)}$. In addition, we ensure that $\pi \circ \iota_{\pi'} = \mathbf{0}$ if $\pi \neq \pi'$.

We now discuss the K-theory of finite-dimensional algebras. Let $k =$ the cardinality of F. Then, by the properties of the K_0 functor (see, e.g., [20, Section 3.2]), it is possible to construct from the isomorphisms and order units described in Section 2.3.3 an order unit v_F for $\mathbb{Z}^k$ and a canonical isomorphism ζ_F from $K_0^{\mathrm{sc}}(\mathbf{B})$ to $(\mathbb{Z}^k, v_F)$. Moreover, this construction provides a canonical bijective mapping ξ_F from the projection maps of the summands of $\mathbf{B}$ to the generators of $\mathbb{Z}^k$.

Suppose F' is a finite multiset of positive integers and set $\mathbf{C} = \bigoplus_{n \in F'} M_n(\mathbb{C})$. Let $k' =$ the cardinality of F'. Suppose ϕ is a $*$-homomorphism from $\mathbf{B}$ to $\mathbf{C}$. The properties of the K_0 functor allow one to construct, from the maps $\eta_{m,n}$ described in Section 2.3.3, a canonical homomorphism $\eta_{F,F'}(\phi)$ from the ordered group $\mathbb{Z}^k$ to $\mathbb{Z}^{k'}$ so that $\zeta_{F'} \circ \phi = \eta_{F,F'}(\phi) \circ \zeta_F$. In addition, the construction of $\eta_{F,F'}$

ensures that whenever π is the projection map for a summand of $\mathbf{B}$ and π' is a projection map for a summand of $\mathbf{C}$, $\pi_{\xi_{F'}(\pi')}(\eta_{F,F'}(\phi)(\xi_F(\pi)))$ = the multiplicity of $\pi' \circ \phi \circ \iota_\pi$. It follows form the discussion in Section 2.3.3 that $\eta_{F,F'}(\phi)(v_F) \leq v_{F'}$ and so $K_0(\phi)(K_0(\mathbf{1_B})) \leq K_0(\mathbf{1_C})$. Conversely, if $\gamma : (K_0(\mathbf{B}), K_0(\mathbf{B})^+) \to (K_0(\mathbf{C}), K_0(\mathbf{C})^+)$ is a homomorphism and if $\gamma(K_0(\mathbf{1_B})) \leq K_0(\mathbf{1_C})$, then there is a $*$-homomorphism δ from $\mathbf{A}$ to $\mathbf{B}$ so that $K_0(\delta) = \gamma$.

2.3.5. *AF algebras.* We finally arrive at a formal treatment of AF algebras.

Definition 2.5. Suppose $\mathbf{A}$ is a C*-algebra. An *AF certificate* of $\mathbf{A}$ is a sequence $(F_j, \psi_j)_{j\in\mathbb{N}}$ that satisfies the following:

1. Each F_j is a finite multiset of positive integers.
2. For each $j \in \mathbb{N}$, ψ_j is a unital $*$-embedding of $\bigoplus_{n\in F_j} M_n(\mathbb{C})$ into $\mathbf{A}$.
3. For all j, $\operatorname{ran}(\psi_j) \subseteq \operatorname{ran}(\psi_{j+1})$.
4. $\mathbf{A} = \overline{\bigcup_{j\in\mathbb{N}} \operatorname{ran}(\psi_j)}$.

A C*-algebra is an *AF algebra* if it has an AF certificate. We remark that this definition is not, but nevertheless is equivalent to, the standard definition of an AF algebra as an inductive limit of finite-dimensional algebras. Our choice of this definition is motivated by forthcoming computabilty considerations.

We can now state Elliott's Theorem formally, a proof of which can be found in [20].

Theorem 2.6 (Elliott's Theorem)**.** *Suppose $\mathbf{A}$ and $\mathbf{B}$ are AF algebras. If $K_0^{\mathrm{sc}}(\mathbf{A})$ and $K_0^{\mathrm{sc}}(\mathbf{B})$ are isomorphic, then $\mathbf{A}$ and $\mathbf{B}$ are $*$-isomorphic. Moreover, if α is an isomorphism from $K_0^{\mathrm{sc}}(\mathbf{A})$ to $K_0^{\mathrm{sc}}(\mathbf{B})$, then there is a unital $*$-isomorphism $\phi : \mathbf{A} \to \mathbf{B}$ so that $K_0(\phi) = \alpha$.*

2.3.6. *Some perturbation principles.* The results in this section will support our proof of the Effective Glimm Lemma. They generally have the flavor of saying that if one or more elements of a C*-algebra $\mathbf{A}$ are "sufficently close" to some closed $*$-subalgebra $\mathbf{B}$, then there is a small unitary u so that when the map Ad_u is applied to these elements, one obtains elements of $\mathbf{B}$ that have certain desired relations to each other. Our aim is to state these principles in such a way as to make precise how close is "sufficiently close" and how to obtain the required unitaries. This level of precision might not have much utility for purely classical considerations, but for future computability developments it is essential. Our source for these results is [21].

We begin with the following, which is essentially [21, Lemma 8.4.1].

Lemma 2.7. *Let $\mathbf{A}$ be a unital C*-algebra. Assume $b \in \mathbf{A}$ is self-adjoint and $p \in B(b; 1/2) \cap \operatorname{Proj}(\mathbf{A})$. Then, there exists $q \in \operatorname{Proj}(\mathbf{A}) \cap \mathrm{C}^*(\{b\})$ so that $\|q - b\| \leq 2\|p - b\|$. Moreover, p and q are equivalent.*

Define $\delta_0 : (0, 1) \times (\mathbb{N} \setminus \{0\}) \to (0, \infty)$ by recursion as follows:

$$\begin{aligned} \delta_0(\epsilon, 1) &:= \min\{\epsilon/2, 1/2\} \\ \delta_0(\epsilon, n+1) &:= \min\{4^{-1}\epsilon(n+1)^{-1}, \delta_0(12^{-1}\epsilon(n+1)^{-2}, n), 1\} \end{aligned}$$

The following is very similar to [21, Lemma 8.4.2], and our proof is based on the proof therein.

Lemma 2.8. *Let* $\mathbf{A}$ *be a unital* C^**-algebra, and assume* $\mathbf{B}$ *is a closed* $*$*-subalgebra of* $\mathbf{A}$. *Suppose* $p_1, \ldots, p_n$ *are mutually orthogonal projections of* $\mathbf{A}$ *so that* $\max_j d(p_j, \mathbf{B}) < \delta_0(\epsilon, n)$. *Then,* $\mathbf{B}$ *contains mutually orthogonal projections* $q_1, \ldots q_n$ *so that* $\max_j \|p_j - q_j\| < \epsilon$. *Furthermore, if* $\sum_j p_j = \mathbf{1_A}$, *then* $\sum_j q_j = \mathbf{1_A}$.

Proof. We proceed by induction. Suppose $n = 1$. Choose $b \in \mathbf{B}$ so that $\|p_1 - b\| < \delta_0(\epsilon, 1)$. Since p_1 is a projection, $\left\|p_1 - \frac{1}{2}(b^* + b)\right\| \leq \|p_1 - b\|$. So, we can simply apply Lemma 2.7 and if $\mathbf{1_A} \in \mathbf{B}$, we just set $p_1 = \mathbf{1_A}$.

Suppose $n > 1$, and set $\delta = \delta_0(\epsilon, n)$. For each $j \in \{1, \ldots, n\}$, choose $b_j \in \mathbf{B}$ so that $\|p_j - b_j\| < \delta$. By definition, $\delta \geq \delta_0(\epsilon/(12n^2), n-1)$. Thus, by way of induction, there exist mutually orthogonal projections $q_1, \ldots, q_{n-1} \in \mathbf{B}$ so that $\max_{j<n} \|p_j - q_j\| < \epsilon/(12n^2)$. Set $p = \sum_{j=1}^{n-1} p_j$, and set $q = \sum_{j=1}^{n-1} q_j$. We first estimate $\|p_n - (\mathbf{1_A} - q)b_n(\mathbf{1_A} - q)\|$ as follows. We first note that

$$\|p_n - (\mathbf{1_A} - q)b_n(\mathbf{1_A} - q)\| \quad = \quad \|(\mathbf{1_A} - p)p_n(\mathbf{1_A} - p) - (\mathbf{1_A} - q)b_n(\mathbf{1_A} - q)\|$$

Set

$$\begin{aligned} E_1 &= \|(\mathbf{1_A} - p)p_n(\mathbf{1_A} - p) - (\mathbf{1_A} - q)p_n(\mathbf{1_A} - p)\| \\ E_2 &= \|(\mathbf{1_A} - q)p_n(\mathbf{1_A} - p) - (\mathbf{1_A} - q)b_n(\mathbf{1_A} - p)\| \\ E_3 &= \|(\mathbf{1_A} - q)b_n(\mathbf{1_A} - p) - (\mathbf{1_A} - q)b_n(\mathbf{1_A} - q)\| \end{aligned}$$

By the Triangle Inequality,

$$\|(\mathbf{1_A} - p)p_n(\mathbf{1_A} - p) - (\mathbf{1_A} - q)b_n(\mathbf{1_A} - q)\| \leq E_1 + E_2 + E_3.$$

However, by factoring out $p_n(\mathbf{1_A} - p)$, we see that $E_1 \leq \|p - q\| < \epsilon/(12n)$. Similarly, by factoring out $\mathbf{1_A} - q$ and $\mathbf{1_A} - p$, we see that $E_2 \leq \|p_n - b_n\| < \delta$. Finally, by factoring out $\|(\mathbf{1_A} - q)b_n\|$, we obtain that

$$E_3 \leq \|b_n\| \|p - q\| \leq (\|p_n - b_n\| + \|p_n\|) \|p - q\| < 2 \|p - q\|.$$

Hence, we now have that

$$\begin{aligned} \|p_n - (\mathbf{1_A} - q)b_n(\mathbf{1_A} - q)\| &\leq 3 \|p - q\| + \delta \\ &< (\epsilon/(4n)) + \delta \\ &< \epsilon/(2n). \end{aligned}$$

In addition, since p_n, p, q are projections, we now have that

$$\left\| p_n - (\mathbf{1_A} - q)\frac{b_n + b_n^*}{2}(\mathbf{1_A} - q) \right\| < \epsilon/(2n).$$

Therefore, by Lemma 2.7, there is a projection $q_n \in (\mathbf{1_A} - q)\mathbf{B}(\mathbf{1_A} - q) \cap B(p_n; \epsilon/n)$. It follows that $q_n q_j = q_j q_n = \mathbf{0}$ when $j < n$.

Suppose $\sum_j p_j = \mathbf{1_A}$. It then follows that $\left\|\mathbf{1_A} - \sum_j q_j\right\| < \epsilon < 1$. Since $q_1, \ldots, q_n$ are mutually orthogonal, $\sum_j q_j$ is a projection. Therefore $\mathbf{1_A} - \sum_j q_j$ is a projection. Hence, $\mathbf{1_A} = \sum_j q_j$. ☐

Define $\delta_1 : (0, 1) \times (\mathbb{N} \setminus \{0\}) \to (0, 1)$ by recursion as follows:

$$\begin{aligned} \delta_1(\epsilon, 1) &:= \frac{1}{2} \\ \delta_1(\epsilon, n+1) &:= \min\{3^{-1}, \epsilon(48n)^{-1}, \delta_1(\epsilon(48n)^{-1}, n)\}. \end{aligned}$$

The following is similar to [21, Lemma 8.4.4], the only difference being our "δ function" is defined somewhat differently for the sake of future developments.

Lemma 2.9. *Let $\mathbf{A}$ be a unital* C*-*algebra, and suppose $p_1, \ldots, p_n$ are projections of $\mathbf{A}$ so that $\max_{i\neq j} \|p_i p_j\| < \delta_1(\epsilon, n)$. Then, there exist mutually orthogonal projections $q_1, \ldots, q_n$ of $\mathbf{A}$ so that $\max_j \|p_j - q_j\| < \epsilon$.*

Proof. We proceed by induction. The case $n = 1$ is vacuous. So, suppose n is a positive integer, and let $p_1, \ldots, p_{n+1}$ be projections of $\mathbf{A}$ so that $\max_{i\neq j} \|p_i p_j\| < \delta_1(\epsilon, n+1)$. By way of induction, there exist mutually orthogonal projections $q_1, \ldots, q_n$ of $\mathbf{A}$ so that $\max_{j\leq n} \|q_j - p_j\| < \epsilon/(48n)$. Set $q = \sum_j q_j$. Thus, $q \in \mathrm{Proj}(\mathbf{A})$. We estimate $\|p_{n+1} - (\mathbf{1_A} - q)p_{n+1}(\mathbf{1_A} - q)\|$ as follows.

$$\begin{aligned}
\|p_{n+1} - (\mathbf{1_A} - q)p_{n+1}(\mathbf{1_A} - q)\| &= \|qp_{n+1} - qp_{n+1}q + p_{n+1}q\| \\
&\leq 3\|qp_{n+1}\| \\
&< 3\left(\sum_{j=1}^{n} \|p_j p_{n+1}\| + \epsilon/(48n)\right) \\
&< 3(\epsilon/48 + \epsilon/48) \\
&< \epsilon/8.
\end{aligned}$$

By Lemma 2.7, there is a projection $q_{n+1} \in (\mathbf{1_A} - q)\mathbf{A}(\mathbf{1_A} - q)$ so that $\|q_{n+1} - p_{n+1}\| < \epsilon$. It follows from direct computation that $q_{n+1}q_j = q_j q_{n+1} = 0$ when $j \leq n$. ☐

For all $\epsilon \in (0, 1)$ and every positive integer n, set

$$\delta_2(\epsilon, n) = \min\{5^{-1}, \epsilon(8 - 5\epsilon), \delta_1(\epsilon, n)\}.$$

The following is essentially [21, Lemma 8.4.7], and there is very little difference in the proof which we therefore choose to omit.

Lemma 2.10. *Suppose $\mathbf{A}$ is a unital* C*-*algebra and suppose $\mathbf{B}$ is a closed $*$-subalgebra of $\mathbf{A}$. Assume $(e_{i,j})_{i,j}$ is an $n \times n$ system of matrix units of $\mathbf{A}$ so that $\max_{i,j} d(e_{i,j}, \mathbf{B}) < \delta_2(\epsilon, n)$. Then, there is an $n \times n$ system of matrix units $(f_{i,j})_{i,j}$ of $\mathbf{B}$ so that $\max_{i,j} \|e_{i,j} - f_{i,j}\| < \epsilon$. In addition, if $\mathbf{1_A} \in \mathbf{B}$, and if $\sum_i e_{i,i} = \mathbf{1_A}$, then $\sum_i f_{i,i} = \mathbf{1_A}$.*

2.4. **Bratteli diagrams.** We now arrive at our final background topic: Bratteli diagrams. Bratteli diagrams were introduced in [2], and they provide a combinatorial way to visualize the structure of AF algebras. They also provide a visualization of the structure of dimension groups [8]. Our goal here is to provide a definition of this concept and to explain the association of Bratteli diagrams with AF algebras by means of AF certificates. We also discuss the relation between these diagrams and dimension groups.

2.4.1. *Fundamental definitions.*

Definition 2.11. A *Bratteli diagram* consists of a set V of *vertices*, a *level function* $L : V \to \mathbb{N}$, and an *edge function* $E : V \times V \to \mathbb{N}$ that satisfy the following:

(1) For all $u, v \in V$, if $E(u, v) > 0$, then $L(v) = L(u) + 1$.
(2) For each $n \in \mathbb{N}$, $L^{-1}[\{n\}]$ is finite.

When $\mathcal{D}$ is a Bratteli diagram, let $V_{\mathcal{D}}$ denote its vertex set, $E_{\mathcal{D}}$ its edge function, and $L_{\mathcal{D}}$ its level function. We refer to $L_{\mathcal{D}}^{-1}[\{n\}]$ as the *n-th level* of $\mathcal{D}$. We conceive of a Bratteli diagram as representing a multigraph where the edge function counts the number of edges between two vertexes; there is no need to formally represent the edges themselves. Bratteli diagrams provide most of the information required to represent an AF algebra. To give a complete representation, one considers labelings, which we define as follows. In the case of AF algebras, these will assist in representing the dimensions of the summands, and in the case of unital dimension groups they will describe the order units.

Definition 2.12.

(1) A *labeling* of a Bratteli diagram $\mathcal{D}$ is a function $\Lambda : V_{\mathcal{D}} \to \mathbb{N}$.
(2) A *labeled Bratteli diagram* consists of a Bratteli diagram $\mathcal{D}$ and a labelling of $\mathcal{D}$.

We regard a labeled Bratteli diagram as a kind of Bratteli diagram. When $\mathcal{D}$ is a labeled Bratteli diagram, let $\Lambda_{\mathcal{D}}$ denote its labeling.

There is an obvious notion of isomorphism for Bratteli diagrams (labeled and unlabeled) which is often too restrictive. The definition of a more useful equivalence relation requires the following.

Definition 2.13. Let $\mathcal{D}$ be a Bratteli diagram. For all $u, v \in V_{\mathcal{D}}$, let

$$P_{\mathcal{D}}(u,v) = \begin{cases} 0 & L_{\mathcal{D}}(u) \geq L_{\mathcal{D}}(v) \\ E(u,v) & L_{\mathcal{D}}(v) = L_{\mathcal{D}}(u) + 1 \\ \sum_{w \in L_{\mathcal{D}}^{-1}[\{L(u)+1\}]} E(u,w) P_{\mathcal{D}}(w,v) & L_{\mathcal{D}}(v) > L_{\mathcal{D}}(u) + 1 \end{cases}$$

Informally speaking, $P_{\mathcal{D}}(u,v)$ counts the number of paths from u to v in $\mathcal{D}$. By means of this path-counting function we introduce the concept of a telescoping.

Definition 2.14. Suppose $\mathcal{D}$ and $\mathcal{D}'$ are Bratteli diagrams. We say that $\mathcal{D}$ is a *telescoping* of $\mathcal{D}'$ if there is an increasing sequence $(n_k)_{k \in \mathbb{N}}$ of integers so that $n_0 = 0$, $L_{\mathcal{D}}^{-1}[\{k\}] = L_{\mathcal{D}'}^{-1}[\{n_k\}]$, and $E_{\mathcal{D}}(u,v) = P_{\mathcal{D}'}(u,v)$. If $\mathcal{D}$ and $\mathcal{D}'$ are labelled, then we also require that $\Lambda_{\mathcal{D}}(v) = \Lambda_{\mathcal{D}'}(v)$.

We say that two Bratteli diagrams $\mathcal{D}$ and $\mathcal{D}'$ are *equivalent* if there is a sequence $\mathcal{D}_0 = \mathcal{D}$, ..., $\mathcal{D}_n = \mathcal{D}'$ so that for each $j < n$, $\mathcal{D}_j$ and $\mathcal{D}_{j+1}$ are isomorphic or one is a telescoping of the other.

2.4.2. *Bratteli diagrams of AF algebras.* We now give a formal definition of the relationship between Bratteli diagrams and AF algebras by means of AF certificates. We start with the Bratteli diagram of an inductive sequence of finite-dimensional algebras.

Definition 2.15. Suppose $(\bigoplus_{n \in F_k} M_n(\mathbb{C}), \psi_k)_{k \in \mathbb{N}}$ is an inductive sequence of finite-dimensional C*-algebras. We define the *standard labeled Bratteli diagram of* $(\bigoplus_{n \in F_k} M_n(\mathbb{C}), \psi_k)_{k \in \mathbb{N}}$ as follows:

(1) The vertices are the pairs of the form (k, π), where π is the projection map of a summand of $\bigoplus_{n \in F_k} M_n(\mathbb{C})$.
(2) The edge function is defined by defining $E((k,\pi),(k+1,\pi'))$ to be the multiplicity of $\pi' \circ \psi_k \circ \iota_\pi$.
(3) The level function is defined by setting $L(k,\pi) = k$.
(4) The labeling function is defined by defining $\Lambda(k,\pi)$ to be $\sqrt{\dim(\mathrm{dom}(\pi))}$.

We say that a labeled Bratteli diagram $\mathcal{D}$ is a *Bratteli diagram of* $(\bigoplus_{n\in F_k} M_n(\mathbb{C}), \psi_k)_{k\in\mathbb{N}}$ if it is isomorphic to the standard Bratteli diagram of this inductive sequence.

Now let $\mathcal{D}$ be a Bratteli diagram. We define $\mathcal{D}$ to be a Bratteli diagram of an AF certificate $(F_s, \phi_s)_{s\in\mathbb{N}}$ if it is a Bratteli diagram of the inductive sequence $(\bigoplus_{n\in F_k} M_n(\mathbb{C}), \phi_{k+1}^{-1}\circ\phi_k)_{k\in\mathbb{N}}$. Then we define $\mathcal{D}$ to be a *Bratteli diagram of* $\mathbf{A}$ if it is a Bratteli diagram of an AF certificate of $\mathbf{A}$.

2.4.3. *Bratteli diagrams of inductive sequences of simplicial groups.* In the proof of Main Theorem 4, it will be useful to have a notion of a labeled Bratteli diagram of an inductive sequence of unital simplicial groups. Given such a sequence $((\mathbb{Z}^{n_s}, u_s), \phi_s)_{s\in\mathbb{N}}$, we define its *standard Bratteli diagram* as follows:

(1) The vertex set is $\{(s, e_j^{n_s}) \ : \ s\in\mathbb{N} \text{ and } j\in\{1,\ldots,n_s\}\}$.
(2) We define the level function by setting $L(s,g) = s$.
(3) We define the edge function by setting $E((s,g),(s+1,g')) = \pi_{g'}(\phi_s(g))$.
(4) Finally, we define the labeling function by setting $\Lambda(s, e_j^{n_s}) = u_s(j)$.

We then say that a Bratteli diagram $\mathcal{D}$ is a Bratteli diagram of $((\mathbb{Z}^{n_s}, u_s), \phi_s)_{s\in\mathbb{N}}$ if it is isomorphic to the standard Bratteli diagram of this inductive sequence.

This completes our picture of the classical material to be considered in this paper. We now discuss its computability theory.

3. Background from effective (computable) mathematics

Our goal in this section is to summarize relevant prior work from the computability theory of groups and C*algebras. We assume knowledge of the fundamentals of computability theory as expounded in [5].

3.1. **Algorithmic group theory.** The objective of this section is to review the concepts of group presentation and associated computable maps, c.e. sets, and computable sets. We follow the framework that we used in [9].

Fix a countably infinite set $X = \{x_0, x_1, \ldots\}$ of indeterminates and let F_ω denote the free group generated by X. We implicitly view F_ω as equipped with a computable bijection with $\mathbb{N}$, so that it makes sense to speak of, for example, c.e. and computable subsets of F_ω.

Fix a group G, and assume ν is an epimorphism from F_ω onto G. The pair (G,ν) is called a *presentation* of G. Set $G^\# = (G,\nu)$. If $w\in F_\omega$ and $\nu(w) = a$, then we call w an $G^\#$ *label* of a. The *kernel* of $G^\#$ is the kernel of ν.

Some groups admit "standard presentations". For example, we always view the group $\mathbb{Z}^n$ as equipped with its standard presentation induced by the labeling $\nu : F_\omega \to \mathbb{Z}^n$ given by sending the first n elements of X to the standard generators of $\mathbb{Z}^n$ and mapping the remaining elements of X to the identity in $\mathbb{Z}^n$. In the sequel, we identify groups which admit standard presentations with said standard presentation.

Suppose $G^\#$ is a presentation of a group. We say that $G^\#$ is *c.e.* (resp. *computable*) if its kernel is a c.e. (resp. computable) subset of F_ω. We remark that the class of computably presentable groups is closed under isomorphism as is the class of groups with c.e. presentations.

Suppose $S\subseteq G$. We say that S is a *computable (resp. c.e.)* set of $G^\#$ if the set of all $G^\#$ labels of elements of S is computable (resp. c.e.). If S is a c.e. set of

$G^\#$, then we say $e \in \mathbb{N}$ is a $G^\#$ *index* of S if e is a code of a Turing machine that enumerates all $G^\#$ labels of elements of S.

Suppose $H^\#$ is a group presentation. A map $\phi : G \to H$ is a *computable map from $G^\#$ to $H^\#$* if there is an algorithm that, given a $G^\#$ label of $g \in G$, computes an $H^\#$ label of $\phi(g)$. A $(G^\#, H^\#)$ *index* of ϕ is an index of such an algorithm.

The following simple observation will streamline our discussion of the computability theory of dimension groups.

Proposition 3.1. *If $G^\#$ is a group presentation, then every homomorphism from the abelian group $\mathbb{Z}^n$ to G is a computable map from $\mathbb{Z}^n$ to $G^\#$.*

Proof sketch. Such a map is completely described by $G^\#$ labels of its values on the standard basis for $\mathbb{Z}^n$. □

3.2. C*-algebras.

3.2.1. *Basic background.* We recall the setting for studying C*-algebras from the perspective of computability theory. Much of this basic material is taken from our earlier work [9]. The reader seeking more details on this background material may also consult [12] and [14].

Definition 3.2. Let $\mathbf{A}$ be a C*-algebra. A *presentation* of $\mathbf{A}$ is a sequence $(a_0, a_1, \ldots)$ of points in $\mathbf{A}$ such that the *-subalgebra of $\mathbf{A}$ generated by $\{a_i : i \in \mathbb{N}\}$ is dense in $\mathbf{A}$. The points $a_0, a_1, \ldots$ are called the *special points* of the presentation, while the points of the form $p(a_0, \ldots, a_n)$, where $p(x_0, \ldots, x_n)$ is a $*$-polynomial with coefficients from $\mathbb{Q}(i)$ (as n varies), are called the *generated points* of the presentation. We typically denote a presentation of $\mathbf{A}$ by $\mathbf{A}^\#$ or $\mathbf{A}^\dagger$.

In earlier work, such as [4] and [9], the generated points of a presentation are referred to as *rational points*; we now prefer to use the term generated points, following the convention established in [14].

By standard techniques (see, for example, [4, Section 2]), we can obtain an effective (but typically non-injective) list of the generated points of a presentation of a C*-algebra. When we say, for example, that an algorithm takes a generated point q of $\mathbf{A}^\#$ as input, we really mean that we have fixed an effective list $(q_i)_{i\in\mathbb{N}}$ of generated points of $\mathbf{A}^\#$ and the algorithm takes input an index i for which $q = q_i$.

By the Church-Turing thesis, every algorithm can be represented as a Turing machine. An *index* (or *code*) of an algorithm is a natural number that codes a representation of the algorithm as a Turing machine.

There is a natural notion of a (finite) product of presentations. When we have a presentation $\mathbf{A}^\#$ of $\mathbf{A}$, if we need a presentation on $\mathbf{A}^n$ we use this product presentation, though the specific details of its construction will not be crucial for us. See [9, Subsection 1.1] for details.

Definition 3.3. Fix a presentation $\mathbf{A}^\#$.

1. $\mathbf{A}^\#$ is *computable* if there is an algorithm that, given a generated point q and a $k \in \mathbb{N}$, outputs a rational number r such that $|\|q\| - r| < 2^{-k}$; an index of such an algorithm is an *index* of $\mathbf{A}^\#$.
2. $\mathbf{A}^\#$ is *(right) c.e.* if there is an algorithm that, given a generated point q, enumerates a decreasing sequence $(r_n)_{n\in\mathbb{N}}$ of rational numbers such that $\lim_{n\to\infty} r_n = \|q\|$; an index of such an algorithm is called a *(right) c.e. index* of $\mathbf{A}^\#$.

There is a corresponding notion of left c.e. presentation, but we will not make use of it here.

With each presentation of $\mathbf{A}$, there is an associated class of computable points. These are defined as follows.

Definition 3.4. An element $a \in \mathbf{A}$ is a *computable point of* $\mathbf{A}^\#$ if there is an algorithm such that, given $k \in \mathbb{N}$, returns a generated point b of $\mathbf{A}^\#$ such that $\|a - b\| < 2^{-k}$; an index of such an algorithm is a $\mathbf{A}^\#$*-index for* a.

In other words, a computable point of $\mathbf{A}^\#$ is one that can be effectively approximated by generated points of $\mathbf{A}^\#$ with arbitrarily good precision.

We remark that the class of computably presentable C*-algebras is closed under $*$-isomorphism, that is, if $\mathbf{A}$ is a computably presentable C*-algebra and $\mathbf{B}$ is $*$-isomorphic to $\mathbf{A}$, then $\mathbf{B}$ is computably presentable. The same principle holds for the class of C*-algebras that have c.e. presentations.

A *rational open ball* of a presentation $\mathbf{A}^\#$ is an open ball in $\mathbf{A}$ that is centred at a generated point of $\mathbf{A}^\#$ and has rational radius. Each rational open ball of $\mathbf{A}^\#$ has a *code* consisting of the radius and a code for the centre of the ball. If $C \subseteq \mathbf{A}$ is closed, we say that C is a *c.e.-closed subset of* $\mathbf{A}^\#$ if the set of all (codes for) open rational balls of $\mathbf{A}^\#$ that intersect C is c.e. If $U \subseteq \mathbf{A}$ is open, we say that U is a *c.e.-open set of* $\mathbf{A}^\#$ if there is a c.e. set $\mathcal{S}$ of rational open balls so that $U = \bigcup \mathcal{S}$.

By an index of a c.e.-open set we mean an index for the c.e. set of open balls whose union comprises the set; the index of a c.e.-closed set is defined analogously.

The following is [9, Lemma 1.10]:

Lemma 3.5. *Suppose that $\mathbf{A}^\#$ is a c.e. presentation. Further suppose that U is a c.e.-open subset of $\mathbf{A}^\#$ and C is a c.e.-closed subset of $\mathbf{A}^\#$ such that $U \cap C \neq \emptyset$. Then $U \cap C$ contains a computable point. Moreover, an index of this computable point can be computed from indexes of U and C.*

The following is folklore and is a simple consequence of the pertinent definitions.

Proposition 3.6. *Suppose $\mathbf{A}^\#$ and $\mathbf{B}^\#$ are c.e. presentations of* C**-algebras, and let U be a c.e. open set of $\mathbf{B}^\#$. If f is a computable map from $\mathbf{A}^\#$ to $\mathbf{B}^\#$, then $f^{-1}[U]$ is a c.e. open set of $\mathbf{A}^\#$. Moreover, it is possible to compute an $\mathbf{A}^\#$ index of $f^{-1}[U]$ from a $\mathbf{B}^\#$ index of U and an $(\mathbf{A}^\#, \mathbf{B}^\#)$ index of f.*

Suppose $\mathbf{A}^\#$ is a c.e. presentation of a stably finite unital C*-algebra. In [9], it is shown that $\mathbf{1_A}$ is a computable point of $\mathbf{A}^\#$. We say that $e \in \mathbb{N}$ is a *unital index of* $\mathbf{A}^\#$ if e is a code of a pair (e_0, e_1) that consists of an index of $\mathbf{A}^\#$ and an $\mathbf{A}^\#$ index of $\mathbf{1_A}$. Our motivation for introducing such indexes is that even for the class of stably finite unital C*-algebras, an $\mathbf{A}^\#$-index of $\mathbf{1_A}$ may not be computable from an index of $\mathbf{A}^\#$; see [19].

We now discuss prior developments regarding the computability of K-theory.

3.2.2. *K-theory.* In [9], it is shown that there is a *computable* functor from the category of c.e. presentations of unital C*-algebras (where the morphisms are computable unital $*$-morphisms) to the category of c.e. presentations of abelian groups. This functor is denoted K_0. The functor is computable in the sense that it is possible to compute an index of $K_0(\mathbf{A}^\#)$ from an index of $\mathbf{A}^\#$. and from an $(\mathbf{A}^\#, \mathbf{B}^\#)$ index of a computable $*$-homomorphism ϕ, it is possible to compute a $(K_0(\mathbf{A}^\#), K_0(\mathbf{B}^\#))$-index of $K_0(\phi)$.

The application of computable K-theory to UHF algebras relied heavily on the concept of computable weak stability, which will also support our proof of the Effective Glimm Lemma. Hence, it is our next topic for review.

3.2.3. *Computable weak stability.* The concept of computable weak stability was introduced by Fox, Goldbring, and Hart [13] based on the corresponding notion of weak stability in continuous model theory, which was introduced in [11]. In [9], we presented computable weak stability in a way that illuminated its connection with c.e. closedness. For the sake of referencing some of its details, we recall this definition now.

Definition 3.7. Suppose that $\vec{x} = (x_1, \ldots, x_N)$ is a tuple of variables, $C_1, \ldots, C_N \in \mathbb{N}$, and $p_1(\vec{x}), \ldots, p_M(\vec{x})$ are rational $*$-polynomials. Consider the set of relations

$$\mathcal{R} = \{p_i(\vec{x}) = 0 \ : \ i = 1, \ldots, M\} \cup \{\|x_j\| \leq C_j \ : \ j = 1, \ldots, N\}$$

.

(1) For a C*-algebra $\mathbf{B}$ and $w_1, \ldots, w_N \in \mathbf{B}$, let $\mathcal{R}^{\mathbf{B}}$ denote the quantity

$$\max(\{\|p_j(\vec{w})\| \ : \ j \in \{1, \ldots, M\}\} \cup \{\|w_j\| - C_j \ : j \in \{1, \ldots, N\}\}$$

and write $\mathbf{B} \models \mathcal{R}(\vec{w})$ if $p_j(\vec{w}) = 0$ for all $j \in \{1, \ldots, M\}$ and $\|w_j\| \leq C_j$ for all $j \in \{1, \ldots, N\}$.

(2) A function $g : \mathbb{N} \to \mathbb{N}$ is a *modulus of weak stability* for $\mathcal{R}$ provided that, for every $k \in \mathbb{N}$, every C*-algebra $\mathbf{B}$, and every $w_1, \ldots, w_N \in \mathbf{B}$, if $\mathcal{R}^{\mathbf{B}}(\vec{w}) < 2^{-g(k)}$, then there exists $z_1, \ldots, z_N \in \mathbf{B}$ so that $\max_j \|w_j - z_j\| < 2^{-k}$ and $\mathbf{B} \models \mathcal{R}(\vec{z})$.

(3) We say that $\mathcal{R}$ is *weakly stable* if it has a modulus of weak stability and *computably weakly stable* if it has a computable modulus of weak stability.

In [9, Theorem 1.12] we showed that computably weakly stable relations define c.e. closed sets in c.e. presentations of C*-algebras.

We have now completed our description of relevant prior developments in classical and computable mathematics. We now attend to some purely prelminary matters in these realms which will support our main efforts later.

4. Preliminaries from classical mathematics

4.1. **Ordered abelian groups.**

4.1.1. *Inductive limits.* Our goal here is to arrive at a more concrete understanding of inductive limits of ordered abelian groups which will be valuable when developing their computability theory. We begin with abelian groups.

Lemma 4.1. *Suppose $(G_s, \phi_s)_{s \in \mathbb{N}}$ is an inductive sequence of abelian groups and let $(G, (\nu_s)_{s \in \mathbb{N}})$ be an inductive limit of $(G_s, \phi_s)_{s \in \mathbb{N}}$. Then:*

(1) $G = \bigcup_{s \in \mathbb{N}} \operatorname{ran}(\nu_s)$.
(2) *For each $s \in \mathbb{N}$,* $\ker(\nu_s) = \bigcup_{k \in \mathbb{N}} \ker(\phi_{s,s+k})$.

Proof sketch. We first review a standard construction of an inductive limit of $(G_s, \phi_s)_{s \in \mathbb{N}}$. For each $s \in \mathbb{N}$, let $\rho_s : G_s \to \prod_n G_n$ be the homomorphism defined by setting $\rho_s(a) = (\phi_{s,n}(a))_{n \in \mathbb{N}}$. When $f, g \in \prod_n G_n$, write $f \sim g$ if there exists $k \in \mathbb{N}$ so that $f(k+n) = g(k+n)$ for all $n \in \mathbb{N}$. Thus, $\sim$ is a congruence relation on $\prod_n G_n$. Let $\pi : \prod_n G_n \to \prod_n G_n / \sim$ denote the canonical epimorphism, and set $\mu_s = \pi \circ \rho_s$ for all $s \in \mathbb{N}$.

Define H to be $\bigcup_{s\in\mathbb{N}} \operatorname{ran}(\mu_s)$. By construction, $\operatorname{ran}(\mu_s) \subseteq \operatorname{ran}(\mu_{s+1})$. Hence, H is an abelian group. It is well-known, and easy to verify, that $(H, (\mu_n)_{n\in\mathbb{N}})$ is an inductive limit of $(G_s, \phi_s)_{s\in\mathbb{N}}$. Furthermore, for each $s \in \mathbb{N}$, $\ker(\mu_s) = \bigcup_{k\in\mathbb{N}} \ker(\phi_{s,s+k})$.

Let λ be the reduction of $(G, (\nu_s)_{s\in\mathbb{N}})$ to $(H, (\mu_s)_{s\in\mathbb{N}})$. By Proposition 2.1, λ is an isomorphism. Since $(H, (\mu_s)_{s\in\mathbb{N}})$ satisfies (1) and (2), it follows that $(G, (\nu_s)_{s\in\mathbb{N}})$ does as well. □

We now proceed to describe inductive limits of ordered abelian groups.

Lemma 4.2. *Suppose $(\mathcal{G}_s, \phi_s)_{s\in\mathbb{N}}$ is an inductive sequence of ordered abelian groups. Assume $(\mathcal{G}, (\nu_s)_{s\in\mathbb{N}})$ is an inductive upper limit of $(\mathcal{G}_s, \phi_s)_{s\in\mathbb{N}}$. Then $(\mathcal{G}, (\nu_s)_{s\in\mathbb{N}})$ is an inductive limit of $(\mathcal{G}_s, \phi_s)_{s\in\mathbb{N}}$ if and only if it satisfies the following:*

(1) $\mathcal{G}^+ = \bigcup_{s\in\mathbb{N}} \nu_s[\mathcal{G}_s^+]$.
(2) *For each* $s \in \mathbb{N}$, $\ker(\nu_s) = \bigcup_{k\in\mathbb{N}} \ker(\phi_{s,s+k})$.

Proof. Assume $\mathcal{G} = (G, P)$ and $\mathcal{G}_s = (G_s, P_s)$ for all $s \in \mathbb{N}$. Let $(H, (\mu_s)_{s\in\mathbb{N}})$ be an inductive limit of $(G_s, \phi_s)_{s\in\mathbb{N}}$, and let λ be the reduction of $(H, (\mu_s)_{s\in\mathbb{N}})$ to $(G, (\nu_s))_{s\in\mathbb{N}}$. Let $P' = \bigcup_{s\in\mathbb{N}} \mu_s[P_s]$ and set $\mathcal{H} = (H, P')$. By [20, Proposition 6.2.6], $(\mathcal{H}, (\mu_s)_{s\in\mathbb{N}})$ is an inductive limit of $(\mathcal{G}_s, \phi_s)_{s\in\mathbb{N}}$. It also follows that λ is positive and so λ is the reduction of $(\mathcal{H}, (\mu_s)_{s\in\mathbb{N}})$ to $(\mathcal{G}, (\nu_s)_{s\in\mathbb{N}})$. Furthermore, by construction, $\mathcal{H}^+ = \bigcup_s \mu_s[\mathcal{G}_s^+]$, and by Lemma 4.1, $\ker(\mu_s) = \bigcup_{k\in\mathbb{N}} \ker(\phi_{s,s+k})$ for all $s \in \mathbb{N}$.

Suppose $(\mathcal{G}, (\nu_s)_{s\in\mathbb{N}})$ is an inductive limit of $(\mathcal{G}_s, \phi_s)_{s\in\mathbb{N}}$. Then λ is an isomorphism. Since $(\mathcal{H}, (\mu_s)_{s\in\mathbb{N}})$ satisfies (1) and (2), it follows that $(\mathcal{G}, (\nu_s)_{s\in\mathbb{N}})$ does as well.

Finally, suppose $(\mathcal{G}, (\nu_s)_{s\in\mathbb{N}})$ satisfies (1) and (2). It follows from (1) that λ is surjective. Let $h \in \ker(\lambda)$. Then, there exists s and $g \in \mathcal{G}_s$ so that $\mu_s(g) = h$. Hence, $g \in \ker(\nu_s)$, and so there exists $k \in \mathbb{N}$ so that $g \in \ker(\phi_{s,s+k})$. Thus, $h = \mu_{s+k}(\phi_{s,s+k}(g)) = \mathbf{0}_G$, and so λ is an isomorphism. Hence, $(\mathcal{G}, (\nu_s)_{s\in\mathbb{N}})$ is an inductive limit of $(\mathcal{G}_s, \phi_s)_{s\in\mathbb{N}}$. □

4.1.2. *Unital dimension groups.* Unital dimension groups are central to the K-theory of unital AF algebras, and accordingly we elaborate on our definition of certificate of dimensionality as follows.

Definition 4.3. Suppose $(\mathcal{G}, u)$ is a unital dimension group. We say that $(n_s, u_s\nu_s, \phi_s)_{s\in\mathbb{N}}$ is a *unital certificate of dimensionality* of $(\mathcal{G}, u)$ if u_s is an order unit of $\mathbb{Z}^{n_s}$ and if $((\mathcal{G}, u), (\nu_s)_{s\in\mathbb{N}})$ is an inductive limit of $((\mathbb{Z}^{n_s}, u_s), \phi_s)_{s\in\mathbb{N}}$ in the category of unital ordered abelian groups.

These certificates will form the backbone of our proof of Main Theorem 2.

4.2. C*-algebras.

4.2.1. *Inductive limits.* As in the previous section, we give a concrete description of inductive limits of C*-algebras.

Lemma 4.4. *Suppose $(\mathbf{A}_s, \phi_s)_{s\in\mathbb{N}}$ is an inductive sequence of* C**-algebras, and let $(\mathbf{A}, (\nu_s)_{s\in\mathbb{N}})$ be an inductive upper limit of $(\mathbf{A}_s, \phi_s)_{s\in\mathbb{N}}$. Then, $(\mathbf{A}, (\nu_s)_{s\in\mathbb{N}})$ is an inductive limit of $(\mathbf{A}_s, \phi_s)_{s\in\mathbb{N}}$ if and only if it satisfies the following:*

(1) $\mathbf{A} = \overline{\bigcup_{s\in\mathbb{N}} \operatorname{ran}(\nu_s)}$.
(2) *For every* $s \in \mathbf{A}_s$, $\|\nu_s(a)\| = \lim_k \|\phi_{s,s+k}(a)\|$.
(3) *For every* $s \in \mathbb{N}$, $\ker(\nu_s) = \{a \in A_s \ : \lim_k \|\phi_{s,s+k}(a)\| = 0\}$.

Proof. The forward direction is [20, Proposition 6.2.4].

Suppose (1), (2), and (3) are satisfied. Let $(\mathbf{B}, (\mu_s)_{s\in\mathbb{N}})$ be an inductive upper limit of $(\mathbf{A}_s, \phi_s)_{s\in\mathbb{N}}$.

We first demonstrate that $\ker(\nu_s) \subseteq \ker(\mu_s)$. Let $a \in \ker(\nu_s)$. Then, by (2), $\lim_k \|\phi_{s,s+k}(a)\| = 0$. Since every $*$-homomorphism is 1-Lipschitz, we have $\lim_k \|\mu_{s+k}(\phi_{s,s+k}(a))\| = 0$. However, since $(\mathbf{B}, (\mu_s)_{s\in\mathbb{N}})$ is an inductive upper limit of $(\mathbf{A}_s, \phi_s)_{s\in\mathbb{N}}$, it follows that $\mu_{s+k}(\phi_{s,s+k}(a)) = \mu_s(a)$ and so $a \in \ker(\mu_s)$.

It now follows that if $\nu_s(a_0) = \nu_{s+k}(a_1)$, then $\mu_s(a_0) = \mu_{s+k}(a_1)$.

We now construct a $*$-homomorphism λ' from $\bigcup_{s\in\mathbb{N}} \operatorname{ran}(\nu_s)$ to $\mathbf{B}$. To begin, let $a \in \operatorname{ran}(\nu_s)$, and choose any $a' \in \mathbf{A}_s$ so that $\nu_s(a') = a$. Define $\lambda'(a)$ to be $\mu_s(a')$. It follows from what has just been shown that λ' is well defined. Since each ν_s and each μ_s is a $*$-homomorphism, it also follows that λ' is a $*$-homomorphism.

We now claim that λ' is 1-Lipschtiz. Let $a \in \operatorname{ran}(\nu_s)$, and suppose $\nu_s(a') = a$. By (2), $\|\nu_s(a')\| = \lim_k \|\phi_{s,s+k}(a')\|$. Since each μ_{s+k} is 1-Lipschitz, it follows that $\|\nu_s(a')\| \geq \lim_k \|\mu_{s+k}(\phi_{s,s+k}(a'))\|$. But, $\mu_{s+k}(\phi_{s,s+k}(a')) = \mu_s(a') = \lambda'(a)$.

It now follows that λ' has a unique continuous extension λ to $\mathbf{A}$. Furthermore, λ is a $*$-homomorphism and by construction $\lambda \circ \nu_s = \mu_s$ for all $s \in \mathbb{N}$.

Finally, it directly follows from the definitions that any two reductions of $(\mathbf{A}, (\nu_s)_{s\in\mathbb{N}})$ to $(\mathbf{B}, (\mu_s)_{s\in\mathbb{N}})$ agree on $\bigcup_{s\in\mathbb{N}} \operatorname{ran}(\nu_s)$ and hence must be identical. □

An immediate consequence of Lemma 4.4 is the following connection between AF certificates and inductive limits.

Corollary 4.5. *If $(F_j, \psi_j)_{j\in\mathbb{N}}$ is an AF certificate of $\mathbf{A}$, then $(\mathbf{A}, (\psi_j)_{j\in\mathbb{N}})$ is an inductive limit of $(\bigoplus_{n\in F_j} M_n(\mathbb{C}), (\psi_{j+1}^{-1} \circ \psi_j)_{j\in\mathbb{N}})$.*

4.2.2. *Matricial systems.* Our only item here is to present some terminology which will support our proof of the Effective Glimm Lemma.

Definition 4.6. Suppose $\mathbf{A}$ is a C*-algebra and let $n_1, \ldots, n_m$ be positive integers. An array $(e^s_{i,j})_{s,i,j}$ is a *type-*$(n_1, \ldots, n_m)$ *matricial system of* $\mathbf{A}$ if it satisfies the following:

(1) For each $s \in \{1, \ldots, m\}$, $(e^s_{i,j})_{i,j}$ is an $n_s \times n_s$ system of matrix units of $\mathbf{A}$.
(2) When $s, s' \in \{1, \ldots, m\}$ are distinct, then $e^s_{i,j} e^{s'}_{i',j'} = 0$ for all $i, j \in \{1, \ldots, n_s\}$ and all $i', j' \in \{1, \ldots, n_{s'}\}$.

If, in addition, $\mathbf{A}$ is unital and $\sum_{s,i} e^s_{i,i} = \mathbf{1}_\mathbf{A}$, then we say that the system $(e^s_{i,j})_{s,i,j}$ is *unital.*

If $\mathbf{F}$ is a C*-algebra that is generated by a matricial system $(e^s_{i,j})_{s,i,j}$, then we call $(e^s_{i,j})_{s,i,j}$ a *matricial generating system for* $\mathbf{F}$. Thus a C*-algebra is finite-dimensional if and only if it has a matricial generating system.

5. Preliminaries from effective mathematics

Herein, we present our frameworks for the computability of inductive limits in a computably indexed category and for the computability of ordered abelian groups.

5.1. **Inductive limits in computably indexed categories.** We now present a computable picture of the material discussed in 2.1. Thus, we again assume $\mathcal{C}$ is a category with a zero object. However, we assume we have fixed an *indexing* of $\mathcal{C}$. This consists of an assignment of natural numbers to the objects of $\mathcal{C}$. The number assigned to an object is referred to as one of its indexes. Each object has at least one index. A number may not index more the one object. Each morphism is indexed by at least one triple of natural numbers. If (e_0, e_1, e) indexes a morphism γ, then e_0 must index its domain and e_1 must index its co-domain.

We also assume this indexing is *computable* in the sense that there is an algorithm that given an index (e_0, e_1, e) of $\phi : A \to B$ and an index (e_1, e_2, e') of $\psi : B \to C$ computes an index of $\psi \circ \phi$ of the form (e_0, e_2, e''). Furthermore, we require that from an index e of an object A it is possible to compute an index of its identity morphism.

Now that the computable indexing is fixed, our remaining definitions are straightforward modifications of those in Section 2.1. Fix an inductive sequence $(A_n, \phi_n)_{n\in\mathbb{N}}$ of $\mathcal{C}$. We say that $(A_n, \phi_n)_{n\in\mathbb{N}}$ is *computable* if there is an algorithm that given $n \in \mathbb{N}$ computes indexes of A_n and ϕ_n. Now suppose $(A, (\nu_n)_{n\in\mathbb{N}})$ is an inductive upper limit of $(A_n, \phi_n)_{n\in\mathbb{N}}$, and assume it is possible to compute an index of ν_n from n. We say that an upper inductive limit $(A, (\nu_n)_{n\in\mathbb{N}})$ of $(A_n, \phi_n)_{n\in\mathbb{N}}$ is *computable* with index $e \in \mathbb{N}$ if e is the code of a pair (e_0, e_1) so that e_0 is an index of A and e_1 is an index of an algorithm that computes an index of ν_n from n. We say that $(A, (\nu_n)_{n\in\mathbb{N}})$ is *a computable inductive limit* if it is a computable inductive upper limit and if it is possible to compute from an index of a computable inductive upper limit $(B, (\mu_n)_{n\in\mathbb{N}})$ of $(A_n, \phi_n)_{n\in\mathbb{N}}$ an index of the reduction from $(A, (\nu_n)_{n\in\mathbb{N}})$ to $(B, (\mu_n)_{n\in\mathbb{N}})$.

5.2. **Abelian groups.**

Lemma 5.1. *Suppose $(G, (\nu_s)_{s\in\mathbb{N}})$ is an inductive limit of $(G_s, \phi_s)_{s\in\mathbb{N}}$ in the category of abelian groups.*

(1) *If $(G_s^\#, \phi_s)_{s\in\mathbb{N}}$ is an inductive sequence in the category of c.e. presentations of abelian groups, and if $(G^\#, (\nu_s)_{s\in\mathbb{N}})$ is a computable inductive upper limit of this sequence, then $(G^\#, (\nu_s)_{s\in\mathbb{N}})$ is a computable inductive limit of $(G^\#, (\nu_s)_{s\in\mathbb{N}})$ in the category of c.e. presentations of abelian groups.*
(2) *If $(G_s^\#, \phi_s)_{s\in\mathbb{N}}$ is an inductive sequence in the category of computable presentations of abelian groups, and if $(G^\#, (\nu_s)_{s\in\mathbb{N}})$ is a computable inductive upper limit of this sequence, then $(G^\#, (\nu_s)_{s\in\mathbb{N}})$ is a computable inductive limit of $(G^\#, (\nu_s)_{s\in\mathbb{N}})$ in the category of computable presentations of abelian groups.*

Proof. Suppose $(G_s^\#, \phi_s)_{s\in\mathbb{N}}$ is an inductive sequence in the category of c.e. presentations of abelian groups, and assume $(G^\#, (\nu_s)_{s\in\mathbb{N}})$ is a computable inductive upper limit of this sequence. Let $(H^\#, (\mu_s)_{s\in\mathbb{N}}$ be a computable inductive upper limit of $(G_s^\#, \phi_s)_{s\in\mathbb{N}}$ in the category of c.e. presentations of abelian groups. Thus, $(H, (\mu_s)_{s\in\mathbb{N}})$ is an inductive upper limit of $(G_s, \phi_s)_{s\in\mathbb{N}}$ in the category of abelian groups. Hence, there is a unique reduction λ of $(G, (\nu_s)_{s\in\mathbb{N}})$ to $(H, (\mu_s)_{s\in\mathbb{N}})$.

It only remains to show that λ is a computable map from $G^\#$ to $H^\#$. Suppose w is a $G^\#$ label of $g \in G$. Then, there exists $s_0 \in \mathbb{N}$ so that $g \in \mathrm{ran}(\nu_{s_0})$ and so there is a $\mathrm{ran}(\nu_{s_0})^\#$ label w' of g. Since the inclusion map is a computable map from $\mathrm{ran}(\nu_s)^\#$ to $G^\#$ uniformly in s, s_0 and w' can be found by a search procedure. Then, by a search procedure, we can compute a $G_{s_0}^\#$ label w'' of a $g' \in G_{s_0}$ so that $\nu_{s_0}(g') = g$ (since ν_s is a computable map from $G_s^\#$ to $G^\#$ uniformly in s). Since $\lambda(g) = \mu_{s_0}(g')$, it follows that we can now compute an $H^\#$ label of $\lambda(g)$ from w.

The proof of the second part is almost identical. □

Proposition 5.2. (1) *Every computable inductive sequence of c.e. presentations of abelian groups has a computable inductive limit.*
(2) *Every computable inductive sequence of computable presentations of abelian groups has a computable inductive limit.*

Proof. Suppose $(G_s^\#, \phi_s)_{s\in\mathbb{N}}$ is an inductive sequence of c.e. presentations of abelian groups. Let $(G, (\nu_s)_{s\in\mathbb{N}})$ be an inductive limit of $(G_s, \phi_s)_{s\in\mathbb{N}}$ in the category of abelian groups. We then construct, for each $s \in \mathbb{N}$, a c.e. presentation $\mathrm{ran}(\nu_s)^\#$ so that ν_s is a computable map from $\mathcal{G}_s^\#$ to $\mathrm{ran}(\nu_s)^\#$ and so that the inclusion map is a computable map from $\mathrm{ran}(\nu_s)^\#$ to $\mathrm{ran}(\nu_{s+1})^\#$; namely, we declare w to be a $\mathrm{ran}(\nu_s)^\#$ label of g if it is a label of a ν_s preimage of g. Since $G = \bigcup_{s\in\mathbb{N}} \mathrm{ran}(\nu_s)$, by standard techniques it is possible to construct a c.e. presentation $G^\#$ so that the inclusion map from $\mathrm{ran}(\nu_s)^\#$ to $G^\#$ is computable uniformly in s.

By construction, $(G^\#, (\nu_s)_{s\in\mathbb{N}})$ is a computable inductive upper limit of $(G_s^\#, \phi_s)_{s\in\mathbb{N}}$ in the category of c.e. presentations of abelian groups. It now follows from Lemma 5.1 that $(G^\#, (\nu_s)_{s\in\mathbb{N}})$ is a computable inductive limit of $(G_s^\#, \phi_s)_{s\in\mathbb{N}}$.

The proof of the second part is nearly identical. □

Lemma 5.3. *Suppose $(G^\#, (\nu_s)_{s\in\mathbb{N}})$ is a computable inductive limit of $(G_s^\#, \phi_s)_{s\in\mathbb{N}}$ in either the category of c.e. presentations of abelian groups in or in the category of computable presentations of abelian groups. Then, $(G, (\nu_s)_{s\in\mathbb{N}})$ is an inductive limit of $(G_s, \phi_s)_{s\in\mathbb{N}}$ in the category of abelian groups.*

Proof. For the moment, assume $(G^\#, (\nu_s)_{s\in\mathbb{N}})$ is a computable inductive limit of $(G_s^\#, \phi_s)_{s\in\mathbb{N}}$ in the category of c.e. presentations of abelian groups. By definition, $(G, (\nu_s)_{s\in\mathbb{N}})$ is an inductive upper limit of $(G_s, \phi_s)_{s\in\mathbb{N}}$. Suppose $(H, (\mu_s)_{s\in\mathbb{N}})$ is an inductive limit of this sequence, and let λ be the reduction of $(H, (\mu_s)_{s\in\mathbb{N}})$ to $(G, (\nu_s)_{s\in\mathbb{N}})$. As in the proof of Proposition 5.2, it is possible to define a c.e. presentation $H^\#$ of H so that $(H^\#, (\mu_s)_{s\in\mathbb{N}})$ is an inductive limit of $(G_s^\#, \phi_s)_{s\in\mathbb{N}}$ in the category of c.e. presentations of abelian groups. Hence, λ is an isomorphism. It follows that $(G, (\nu_s)_{s\in\mathbb{N}})$ is an inductive limit of $(G_s, \phi_s)_{s\in\mathbb{N}}$.

The case for the category of computable presentations of abelian groups is identical. □

5.3. **Ordered abelian groups.** We now discuss c.e. presentations of ordered abelian groups and related concepts and results.

5.3.1. *Presentations.*

Definition 5.4. Suppose $\mathcal{G} = (G, P)$ is an ordered abelian group.

(1) We say that $(G^\#, P)$ is a *presentation* of $\mathcal{G}$ if $G^\#$ is a presentation of G.
(2) We say that a presentation $(G^\#, P)$ of $\mathcal{G}$ is *c.e.* if $G^\#$ is c.e. and P is a c.e. set of $G^\#$.

If $\mathcal{G}^\# = (G^\#, P)$ is a c.e. presentation of an ordered abelian group, we say that $e \in \mathbb{N}$ is an *index* of $\mathcal{G}^\#$ if it is the code of a pair (e_0, e_1) that consists of an index of $G^\#$ and a $G^\#$ index of P. If $g \in G$ and $w \in F_\omega$, we say that w is a $\mathcal{G}^\#$ *label* of g if it is a $G^\#$ label of g.

Definition 5.5. Suppose $\mathcal{G}$ is an ordered abelian group and let u be a unit of $\mathcal{G}$.

(1) We say that $(\mathcal{G}^\#, u)$ is a *presentation* of $(\mathcal{G}, u)$ if $\mathcal{G}^\#$ is a presentation of $\mathcal{G}$.
(2) We say that a presentation $(\mathcal{G}^\#, u)$ of $(\mathcal{G}, u)$ is *c.e.* if $\mathcal{G}^\#$ is c.e.

If $(\mathcal{G}^{\#}, u)$ is a c.e. presentation of $(\mathcal{G}, u)$, then we say that $e \in \mathbb{N}$ is an *index* of $(\mathcal{G}^{\#}, u)$ if it codes a pair that consists of an index of $\mathcal{G}^{\#}$ and a code of a $\mathcal{G}^{\#}$ label of u. When $\mathbf{A}$ is stably finite and unital, we let $K_0^{\mathrm{sc}}(\mathbf{A}^{\#}) = ((K_0(\mathbf{A}^{\#}), K_0(\mathbf{A})^+), K_0(\mathbf{1_A}))$.

The following is a linchpin for our proof of Main Theorem 2.2.

Lemma 5.6. *If $\mathcal{H} = (H, H \cap \mathbb{Z}^n_{\geq 0})$ is the convex subgroup of $\mathbb{Z}^n$ generated by an element x of $\mathbb{Z}^n$, then H is a computable set of $\mathbb{Z}^n$.*

Proof. We first show that for every $g \in \mathbb{Z}^n$, $g \in H$ if and only if there is a positive integer n so that $-nx \leq g \leq nx$. Let $g \in \mathbb{Z}^n$. On the one hand, if such a positive integer exists, then the convexity of $\mathcal{H}$ ensures that $g \in H$. Conversely, since by Lemma 2.2 x is an order unit of $\mathcal{H}$, if $g \in H$, then such a positive integer n must exist.

Given $g \in \mathbb{Z}^n$, by inspection of components, it is possible to determine if there is a positive integer n so that $-nx \leq g \leq nx$. Thus, by what has just been shown, H is a computable set of $\mathbb{Z}^n$. ☐

We note that the proof of Lemma 5.6 is uniform.

5.3.2. *Inductive limits.* We now apply our framework for computable inductive limits to the category of ordered abelian groups. We have already discussed how we computably index this category. The following not only tell us that computable inductive sequences in this category have inductive limits, but also how to find them.

The following is an immediate consequence of Lemma 5.1.

Corollary 5.7. *Suppose $(\mathcal{G}_s^{\#}, \phi_s)_{s\in\mathbb{N}}$ is a computable inductive sequence of c.e. presentations of ordered abelian groups. Assume $(\mathcal{G}, (\nu_s)_{s\in\mathbb{N}})$ is an inductive limit of $(\mathcal{G}_s, \phi_s)_{s\in\mathbb{N}}$, and assume also that $\mathcal{G}^{\#}$ is a c.e. presentation of $\mathcal{G}$ so that $(\mathcal{G}^{\#}, (\nu_s)_{s\in\mathbb{N}})$ is a computable upper limit of $(\mathcal{G}_s^{\#}, \phi_s)_{s\in\mathbb{N}}$. Then $(\mathcal{G}^{\#}, (\nu_s)_{s\in\mathbb{N}})$ is a computable inductive limit of $(\mathcal{G}_s^{\#}, \phi_s)_{s\in\mathbb{N}}$.*

Corollary 5.8. *Every computable inductive sequence of c.e. presentations of ordered abelian groups has a computable inductive limit.*

Proof sketch. Suppose $(\mathcal{G}_s^{\#}, \phi_s)_{s\in\mathbb{N}}$ is a computable inductive sequence of c.e. presentations of ordered abelian groups. Let $\mathcal{G}_s^{\#} = (G_s^{\#}, P_s)$. By Proposition 5.2, $(G_s^{\#}, \phi_s)_{s\in\mathbb{N}}$ has a computable inductive limit $(G^{\#}, (\nu_s)_{s\in\mathbb{N}}$ in the category of c.e. presentations of abelian groups.

Set $P = \bigcup_{s\in\mathbb{N}} \nu_s[P_s]$. Let $\mathcal{G} = (G, P)$, and let $\mathcal{G}^{\#} = (G^{\#}, P)$. By Lemma 5.3, $(G, \nu_s)_{s\in\mathbb{N}}$ is an inductive limit of $(G_s, \phi_s)_{s\in\mathbb{N}}$. It then follows from Lemma 4.2 that $(\mathcal{G}, (\nu_s)_{s\in\mathbb{N}})$ is an inductive limit of $(\mathcal{G}_s, \phi_s)_{s\in\mathbb{N}}$.

We claim that $\mathcal{G}^{\#}$ is c.e. It is only required to prove that P is a c.e. set of $G^{\#}$. For each s, P_s is a c.e. set of $G_s^{\#}$ uniformly in s. At the same time, ν_s is a computable map from $G_s^{\#}$ to $G^{\#}$ uniformly in s. Hence, $\nu_s[P_s]$ is a c.e. set of $G^{\#}$ uniformly in s, and so P is a c.e. set of $G^{\#}$.

Finally, since ν_s is a computable map from $G_s^{\#}$ to $G^{\#}$ uniformly in s, it follows that $(\mathcal{G}^{\#}, (\nu_s)_{s\in\mathbb{N}})$ is a computable inductive upper limit of $(\mathcal{G}_s^{\#}, \phi_s)_{s\in\mathbb{N}}$. Hence, by Corollary 5.7, $(\mathcal{G}^{\#}, (\nu_s)_{s\in\mathbb{N}})$ is a computable inductive limit of $(\mathcal{G}_s^{\#}, \phi_s)_{s\in\mathbb{N}}$. ☐

5.3.3. *Dimension groups.* The material in this section will be used in our proof of Main Theorem 2.

Definition 5.9. Suppose $\mathcal{G}^\#$ is a presentation of an ordered abelian group. We say that $\mathcal{G}^\#$ has the *effective Shen property* if from a $(\mathbb{Z}^n, \mathcal{G}^\#)$ index of a homomorphism θ from $\mathbb{Z}^n$ to $\mathcal{G}^\#$ and an $\alpha \in \ker(\theta)$, it is possible to compute a positive integer p, an index of an ordered group homomorphism $\varphi : \mathbb{Z}^n \to \mathbb{Z}^p$, and a $(\mathbb{Z}^p, \mathcal{G}^\#)$ index of a homomorphism $\theta' : \mathbb{Z}^p \to \mathcal{G}$ so that (ϕ, θ') factors θ at α.

Lemma 5.10. *Every c.e. presentation of a dimension group has the effective Shen property.*

Proof sketch. Suppose $\mathcal{G}^\#$ is a c.e. presentation of a dimension group. By Proposition 2.3, $\mathcal{G}$ has the Shen property. Given a homomorphism $\theta : \mathbb{Z}^n \to \mathcal{G}$ and $\alpha \in \ker \theta$, the ordered homomorphisms required by the effective Shen property do in fact exist since $\mathcal{G}$ has the (ordinary) Shen property. As these maps are all completely determined by their values on the standard bases, they can now be found by a search procedure. □

5.4. C*-**algebras.** We now attend to some preliminary developments from the computability theory of C*-algebras. We first add to existing results on computable weak stability so as to compute matricial systems within c.e. presentations of AF algebras. These results will be crucial in our proof of the Effective Glimm Lemma. We then present some material on computing inductive limits of C*-algebras, and finally our formal definition of computable AF certificate.

5.4.1. *Computable weak stability results.*

Lemma 5.11. *Suppose $\mathcal{R}(x_1, \ldots, x_n)$ and $\mathcal{S}(x_1, \ldots, x_n)$ are finite sets of relations as in Definition 3.7. Assume further that $\mathcal{R}$ is computably weakly stable and that there is a computable $\Delta : \mathbb{N} \to \mathbb{N}$ so that for every* C*-*algebra $\mathbf{A}$ and all $a_1, \ldots, a_n \in \mathbf{A}$, if $\mathbf{A} \models \mathcal{R}(a_1, \ldots, a_n)$, and if $(\mathcal{R} \cup \mathcal{S})^{\mathbf{A}}(a_1, \ldots, a_n) \leq 2^{-\Delta(k)}$, then there exist $a'_1, \ldots, a'_n \in \mathbf{A}$ so that $\mathbf{A} \models (\mathcal{R} \cup \mathcal{S})(a'_1, \ldots, a'_n)$ and so that $\max_j \|a_j - a'_j\| < 2^{-k}$. Then, $\mathcal{R} \cup \mathcal{S}$ is effectively weakly stable.*

Proof. Let Δ_1 be a computable modulus of weak stability for $\mathcal{R}$. Let M denote the maximum of the constants that appear in $\mathcal{S}$ (that is, the C_j's). It follows that there is a computable $g : \mathbb{N} \to \mathbb{N}$ so that for every C*-algebra $\mathbf{A}$, g is a modulus of uniform continuity for $\mathcal{S}^{\mathbf{A}}$ on $B(\mathbf{0}; M)^n$. Set $\Delta_2(k) = \Delta_1(\max\{k+1, g(\Delta(k+1))\})$.

Suppose $c_1, \ldots, c_n \in \mathbf{A}$ and $(\mathcal{R} \cup \mathcal{S})^{\mathbf{A}}(c_1, \ldots, c_n) < 2^{-\Delta_2(k)}$. Let

$$a = \max\{1, g(\Delta(k+1)) - k\}.$$

Since $\Delta_2(k) \geq \Delta_1(k+a)$, there exist $a_1, \ldots, a_n \in A$ so that $\mathbf{A} \models \mathcal{R}(a_1, \ldots, a_n)$ and so that $\max_j \|a_j - c_j\| < 2^{-(k+a)}$. Since $\mathbf{A} \models \mathcal{R}(a_1, \ldots, a_n)$, $(\mathcal{R} \cup \mathcal{S})^{\mathbf{A}}(a_1, \ldots, a_n) = \mathcal{S}^{\mathbf{A}}(a_1, \ldots, a_n)$. As $k + a \geq g(\Delta(k+1))$, it follows that $\mathcal{S}^{\mathbf{A}}(a_1, \ldots, a_n) < 2^{-\Delta(k+1)}$. Hence, there exist $a'_1, \ldots, a'_n \in \mathbf{A}$ so that $\mathbf{A} \models (\mathcal{R} \cup \mathcal{S})(a'_1, \ldots, a'_n)$ and $\max_j \|a_j - a'_j\| < 2^{-(k+1)}$. Therefore, $\max_j \|c_j - a_j\| < 2^{-k}$. □

We note that the proof of Lemma 5.11 is uniform in that an index of a modulus of weak stability for $\mathcal{R} \cup \mathcal{S}$ can be computed from an index of Δ and an index of a modulus of weak stability for $\mathcal{R}$.

Proposition 5.12. *The following are defined by computably weakly stable sets of relations.*

(1) *For each $n \in \mathbb{N}$, the property of being an n-tuple of mutually orthogonal projections.*
(2) *For each $m, n_1, \ldots, n_m \in \mathbb{N}$, the property of being a (unital) matricial system of type $(n_1, \ldots, n_m)$.*

Proof. (1): The property of being a projection is effectively weakly stable (see [13, Examples 12]). Define $\Delta(0, k)$ arbitarily, and for positive n let $\Delta(n, k) = \lceil \log_2(\delta_1(2^{-k}, n)) \rceil$. Apply Lemmas 2.9 and 5.11.

(2): We use Lemma 5.11. However, some extra steps must be taken (compared to (1)) to show that the required function Δ exists.

$\Delta(0, k)$ can be defined arbitrarily. Suppose $n, k \in \mathbb{N}$, and assume n is positive. Let $k_0 = 1 - \lfloor \log_2(\delta_2(2^{-k} n^{-1}, n)) \rfloor$, and set $\Delta(n, k) = 1 - \lfloor \log_2(n^{-2} \delta_1(2^{-k_0}, n)) \rfloor$.

Assume $n_1, \ldots, n_m$ are positive integers so that $n_1^2 + \ldots + n_m^2 = n$. Suppose that for each $s \in \{1, \ldots, m\}$, $(e^s_{i,j})_{i,j}$ is an $n_s \times n_s$ system of matrix units of $\mathbf{A}$, and assume also that $\max_{i,j,j',j'} \left\| e^s_{i,j} e^{s'}_{i',j'} \right\| \leq 2^{-\Delta(n,k)}$ when $s \neq s'$. Set $p_s = \sum_i e^s_{i,i}$. Then, $p_1, \ldots, p_m$ are projections, and $\max_{s \neq s'} \|p_s p_{s'}\| \leq n^2 2^{-\Delta(n,k)}$. By definition, $n^2 2^{-\Delta(n,k)} < \delta_1(2^{-k_0}, n)$. Thus, by Lemma 2.9, there exist mutually orthogonal projections $q_1, \ldots, q_n \in \mathbf{A}$ so that $\max_j \|p_j - q_j\| < 2^{-k_0}$.

Let $\mathbf{A}_s = q_s \mathbf{A} q_s$. Since $\|p_s - q_s\| < 2^{-k_0}$,

$$\left\| e^s_{i,j} - q_s e^s_{i,j} q_s \right\| = \left\| p_s e^s_{i,j} p_s - q_s e^s_{i,j} q_s \right\| < 2^{-k_0}.$$

Thus, $\max_{s,i,j} d(e^s_{i,j}, \mathbf{A}_s) < 2^{-k_0}$. As $2^{-k_0} < \delta_2(2^{-k}, n)$, it follows from Lemma 2.10 that $\mathbf{A}_s$ contains an $n_s \times n_s$ system $(f^s_{i,j})_{s,i,j}$ of matrix units so that $\left\| e^s_{i,j} - f^s_{i,j} \right\| < n^{-1} 2^{-k}$. And, since $q_1, \ldots, q_m$ are mutually orthogonal, it follows that $(f^s_{i,j})_{s,i,j}$ is a matricial system. Moreover, if $\sum_{s,i} e^s_{i,i} = \mathbf{1_A}$, then $\left\| \mathbf{1_A} - \sum_{s,i} f^s_{i,i} \right\| < 1$. Hence, since each $f^s_{i,i}$ is a projection $\sum_{s,i} f^s_{i,i} = \mathbf{1_A}$. ☐

5.4.2. *Inductive limits.* The c.e. presentations of C*-algebras form a category in which the morphisms are the computable $*$-homomorphims. That is, ψ is a morphism from $\mathbf{A}^\#$ to $\mathbf{B}^\#$ if it is a computable $*$-homomorphism from $\mathbf{A}^\#$ to $\mathbf{B}^\#$. There is a natural indexing of this category, but note that a number e will always index several presentations. However, there is no harm in identifying all presentations indexed by a number e even if they are presentations of different algebras as there is an obvious computable unital $*$-isomorphism between any two of them; namely map the n-th special point to the n-th special point.

We present two results on computable inductive limits of C*-algebras that parallel the results in Section 5.3.2.

Corollary 5.13. *Suppose $(\mathbf{A}^\#_s, \phi_s)_{s \in \mathbb{N}}$ is a computable inductive sequence in the category of c.e. C*-algebra presentations, and let $(\mathbf{A}, (\nu_s)_{s \in \mathbb{N}})$ be an inductive limit of $(\mathbf{A}_s, \phi_s)_{s \in \mathbb{N}}$. If $\mathbf{A}^\#$ is a c.e. presentation so that $(\mathbf{A}^\#, (\nu_s)_{s \in \mathbb{N}})$ is a computable inductive upper limit of $(\mathbf{A}^\#_s, \phi_s)_{s \in \mathbb{N}}$, then $(\mathbf{A}^\#, (\nu_s)_{s \in \mathbb{N}})$ is a computable inductive limit of $(\mathbf{A}^\#_s, \phi_s)_{s \in \mathbb{N}}$.*

Proof sketch. Suppose $(\mathbf{B}^\#, (\mu_s)_{s \in \mathbb{N}})$ is a computable inductive upper limit of $(\mathbf{A}^\#_s, \phi_s)_{s \in \mathbb{N}}$, and let λ be the reduction of $(\mathbf{A}, (\nu_s)_{s \in \mathbb{N}})$ to $(\mathbf{B}, (\mu_s)_{s \in \mathbb{N}})$. Let g be a generated point of $\mathbf{A}^\#$. Wait for $s_0 \in \mathbb{N}$ and a generated point g_0 of $\mathbf{A}^\#_{s_0}$ so that $\nu_{s_0}(g_0)$ is sufficiently close to g. By Lemma 4.4, such a number and generating point must exist. Since λ is 1-Lipschitz, $\lambda(g)$ is sufficiently close to $\lambda(\nu_{s_0}(g_0)) = \mu_{s_0}(g_0)$ which can be computed with arbitrarily good precision. □

The following is due to the second author [16].

Corollary 5.14. *Suppose* $(\mathbf{A}_s^{\#}, \phi_s)_{s\in\mathbb{N}}$ *is a computable inductive sequence in the category of c.e.* C^**-algebra presentations.*

1. $(\mathbf{A}_s^{\#}, \phi_s)_{s\in\mathbb{N}}$ *has a computable inductive limit.*
2. *If* $(\mathbf{A}_s^{\#}, \phi_s)_{s\in\mathbb{N}}$ *is a computable inductive sequence in the category of computable* C^**-algebra presentations, and if each* ϕ_s *is injective, then* $(\mathbf{A}_s^{\#}, \phi_s)_{s\in\mathbb{N}}$ *has a computable inductive limit in the category of computable* C^**-algebra presentations.*

Proof sketch. We use a technique similar to that used in the proof of Corollary 5.8 to construct $\mathbf{A}^{\#}$. Part (2) follows from Lemma 4.4.2. □

5.4.3. *Computable AF certificates.* We conclude this section by formally defining computable AF certificates.

Definition 5.15. Suppose $(F_j, \psi_j)_{j\in\mathbb{N}}$ is an AF certificate for $\mathbf{A}$, and assume $\mathbf{A}^{\#}$ is a presentation of $\mathbf{A}$. We say that $(F_j, \psi_j)_{j\in\mathbb{N}}$ is a *computable AF certificate of* $\mathbf{A}^{\#}$ if $(F_j)_{j\in\mathbb{N}}$ is computable and if each ψ_j is a computable map from $\bigoplus_{n\in F_j} M_n(\mathbb{C})$ to $\mathbf{A}^{\#}$.

5.5. **Bratteli diagrams.** We set forth our framework for the computability of Bratteli diagrams.

Definition 5.16. Suppose $\mathcal{D}$ is a Bratteli diagram.

1. A *presentation* of $\mathcal{D}$ is a pair $(\mathcal{D}, \nu)$ where ν is a surjection of $\mathbb{N}$ onto $V_{\mathcal{D}}$.
2. The *kernel* of a presentation $(\mathcal{D}, \nu)$ is $\{(m,n) \ : \ \nu(m) = \nu(n)\}$.

Definition 5.17. Suppose $\mathcal{D}^{\#} = (\mathcal{D}, \nu)$ is a presentation of a Bratteli diagram. We say that $\mathcal{D}^{\#}$ is *computable* if it satisfies the following:

1. The kernel of $\mathcal{D}^{\#}$ is computable.
2. $L_{\mathcal{D}} \circ \nu$ is computable.
3. The map $(m,n) \mapsto E_{\mathcal{D}}(\nu(m), \nu(n))$ is computable.
4. If $\mathcal{D}$ is labeled, then $\Lambda_{\mathcal{D}} \circ \nu$ is computable.
5. The sequence $(\# L_{\mathcal{D}}^{-1}[\{n\}])_{n\in\mathbb{N}}$ is computable.

If $\mathcal{D}$ and $\mathcal{D}'$ are Bratteli diagrams, then a *computable telescoping* of $\mathcal{D}'$ to $\mathcal{D}$ is a computable sequence of integers that satisfies the conditions of Definition 2.14.

Definition 5.18. Suppose $\mathcal{D}$ and $\mathcal{D}'$ are Bratteli diagrams. We say that $\mathcal{D}$ and $\mathcal{D}'$ are *computably equivalent* if there is a sequence $\mathcal{D}_0 = \mathcal{D}$, ..., $\mathcal{D}_m = \mathcal{D}'$ so that for each $j < m$, either $\mathcal{D}_j$ and $\mathcal{D}_{j+1}$ are isomorphic or one is a computable telescoping of the other.

This concludes our discussion of preliminary developments. We now attend to formally stating and proving our Effective Glimm Lemma.

6. An effective Glimm Lemma

Throughout this section, $\mathbf{A}$ denotes a stably finite unital C*-algebra and $\mathbf{A}^\#$ denotes a c.e. presentation of $\mathbf{A}$.

Suppose $\mathbf{B}$ is a c.e. closed $*$-subalgebra of $\mathbf{A}^\#$. Then from an $\mathbf{A}^\#$ index of $\mathbf{B}$ and an index of $\mathbf{A}^\#$, it is possible to compute an $\mathbf{A}^\#$ index of a sequence $(c_n)_{n\in\mathbb{N}}$ that is dense in $\mathbf{B}$. We then define $\mathbf{B}^\#$ to be the presentation of $\mathbf{B}$ whose n-th special point is c_n. Since $(c_n)_{n\in\mathbb{N}}$ is a computable sequence of $\mathbf{A}^\#$, it follows that $\mathbf{B}^\#$ is a c.e. presentation of $\mathbf{B}$. Furthermore, the inclusion map is a computable map from $\mathbf{B}^\#$ to $\mathbf{A}^\#$.

Theorem 6.1 (Effective Glimm Lemma)**.** *There is a computable function $\Delta : \mathbb{N}^2 \to \mathbb{N}$ so that for every c.e.-closed $*$-subalgebra $\mathbf{B}$ of $\mathbf{A}^\#$, all $n, k \in \mathbb{N}$, and every c.e.-closed $*$-subalgebra $\mathbf{F}$ of $\mathbf{A}^\#$ for which $\mathbf{1_A} \in \mathbf{F}$, if $\dim(\mathbf{F}) = n$, and if $B(\mathbf{B}; 2^{-\Delta(n,k)})$ contains a matricial generating system of $\mathbf{F}$, then there is a computable unitary u of $\mathbf{A}^\#$ so that $\|u - \mathbf{1_A}\| < 2^{-k}$ and $u^*\mathbf{F}u \subseteq \mathbf{B}$. Furthermore, an $\mathbf{A}^\#$ index of u can be computed from an index of $\mathbf{A}^\#$ and $\mathbf{A}^\#$ indexes of $\mathbf{F}$ and $\mathbf{B}$.*

We approach this proof through a sequence of lemmas. For the sake of the first lemma, we define a function $\Delta_1 : \mathbb{N}^2 \to \mathbb{N}$ as follows. Define $\Delta_1(0, k)$ arbitrarily. When $n \geq 1$, define $\Delta_1(n, k)$ to be the least natural number so that $2^{-\Delta_1(n,k)} < \delta_0(2^{-(k+1)}, n)$. We note that by induction, $\delta_0(2^{-(k+1)}, n)$ is a positive rational number when $n > 0$. Thus, $\Delta_1(n, k)$ is defined for all $n, k \in \mathbb{N}$. Furthermore, the recursive definition of δ_0 ensures that Δ_1 is computable.

Lemma 6.2. *For all $n, k \in \mathbb{N}$, if $p_1, \ldots, p_n \in B(\mathrm{Proj}(\mathbf{B}); 2^{-\Delta_1(n,k)})$ are mutually orthogonal computable projections of $\mathbf{A}^\#$, then $\mathbf{B}$ contains mutually orthogonal computable projections $q_1, \ldots, q_n$ of $\mathbf{A}^\#$ so that $\max_j \|p_j - q_j\| < 2^{-k}$. Furthermore, if $\sum_j p_j = \mathbf{1_A}$, then $\sum_j q_j = \mathbf{1_A}$.*

Proof. Suppose $p_1, \ldots, p_n \in B(\mathrm{Proj}(\mathbf{B}); 2^{-\Delta_1(n,k)})$ are mutually orthogonal computable projections of $\mathbf{A}^\#$. It follows from Lemma 2.8 that there exist mutually orthogonal projections $q_1, \ldots, q_n \in \mathbf{B}$ so that $\max_j \|q_j - p_j\| < 2^{-k}$. Let Ω denote the set of all mutually orthogonal $(a_1, \ldots, a_n) \in \mathbf{B}^n$. By Proposition 5.12, $\mathrm{Proj}(\mathbf{B})^n \cap \Omega$ is a c.e. closed set of $(\mathbf{B}^\#)^n$. Since the inclusion map from $\mathbf{B}^\#$ to $\mathbf{A}^\#$ is computable, $\mathrm{Proj}(\mathbf{B}) \cap \Omega$ is a c.e. closed set of $(\mathbf{A}^\#)^n$. Then, since $p_1, \ldots, p_n$ are computable points of $\mathbf{A}^\#$, $B((p_1, \ldots, p_n); 2^{-k})$ is a c.e. open set of $(\mathbf{A}^\#)^n$. Thus, by [9, Lemma 1.11], $\mathrm{Proj}(B) \cap \Omega$ contains a computable point $(q_1, \ldots, q_n)$ of $(\mathbf{A}^\#)^n$. Hence, $q_1, \ldots, q_n$ are computable points of $\mathbf{A}^\#$.

Finally, the definitions of δ_0 and Δ_1 ensure that if $\sum_j p_j = \mathbf{1_A}$, then $\sum_j q_j = \mathbf{1_A}$. (See proof of Lemma 2.8.) ∎

Set $\Delta_2(n, k) = n + k + 2$.

Lemma 6.3. *Suppose $p_1, \ldots, p_n, q_1, \ldots, q_n$ are computable projections of $\mathbf{A}^\#$ so that $p_i p_j = q_i q_j = \mathbf{0}$ whenever $i \neq j$. Assume also that $\max_j \|p_j - q_j\| < 2^{-\Delta_2(n,k)}$. Then, $\mathrm{C}^*(\{p_1, \ldots, p_n, q_1, \ldots, q_n, \mathbf{1_A}\})$ contains a computable partial isometry v of $\mathbf{A}^\#$ so that for all $j \in \{1, \ldots, n\}$, $v^* p_j v = q_j$ and $\max_j \|p_j - p_j v q_j\| < 2^{-k}$. Furthermore, if $\sum_j p_j = \mathbf{1_A}$, then v is unitary and $\|v - \mathbf{1_A}\| < 2^{-k}$.*

Proof. For each $j \in \{1, \ldots, n\}$, let $u_j = \mathbf{1_A} - p_j - q_j - 2q_j p_j$. By [21, Lemma 8.3.6], $u_j q_j u_j^* = p_j$ and $\|\mathbf{1_A} - u_j\| \leq \sqrt{2}\,\|p_j - q_j\|$. Set $v = \sum_j p_j u_j q_j$. It is shown in the proof of [21, Lemma 8.4.3] that v is a partial isometry and that $v^* p_j v = q_j$ for all $j \in \{1, \ldots, n\}$. We also have that

$$\begin{aligned} \|p_j - p_j v q_j\| &= \|p_j - p_j u_j q_j\| \\ &= \|p_j - p_j u_j p_j + p_j u_j p_j - p_j u_j q_j\| \\ &\leq \|p_j\| \, \|\mathbf{1}_A - u_j\| + \|p_j\| \, \|u_j\| \, \|p_j - q_j\| \\ &\leq \sqrt{2} \, \|p_j - q_j\| + \|p_j - q_j\| \\ &= (1 + \sqrt{2}) \, \|p_j - q_j\| \\ &\leq (1 + \sqrt{2}) 2^{-(n+k+2)} < 2^{-n} 2^{-k} < n^{-1} 2^{-k} \leq 2^{-k}. \end{aligned}$$

Finally, suppose $\sum_j p_j = \mathbf{1_A}$. Then,

$$\left\| \mathbf{1_A} - \sum_j q_j \right\| = \left\| \sum_j (q_j - p_j) \right\| \leq n \cdot 2^{-n} < 1.$$

Hence, $\sum_j q_j = \mathbf{1_A}$. Moreover, since $v^* p_j v = q_j$ for all $j \in \{1, \ldots, n\}$, it also follows that v is unitary. In addition,

$$\begin{aligned} \|\mathbf{1_A} - v\| &= \left\| \sum_{j=1}^n (p_j - p_j u_j q_j) \right\| \\ &\leq n \max_j \|p_j - p_j u_j q_j\| \\ &< 2^{-k}. \end{aligned}$$

□

When $n, k \in \mathbb{N}$ and $n > 0$, define $\Delta_3(n, k)$ to be the smallest natural number so that $2^{-\Delta_3(n,k)} < \delta_2(2^{-(k+1)}, n)$; define $\Delta_3(0, k)$ arbitrarily.

Lemma 6.4. *Suppose $(e_{i,j})_{i,j}$ is a computable $n \times n$ system of matrix units of $\mathbf{A}^\#$, and assume $\max_{i,j} d(e_{i,j}, \mathbf{B}) \leq 2^{-\Delta_3(n,k)}$. Then, $\mathbf{B}$ contains a computable $n \times n$ system $(f_{i,j})_{i,j}$ of matrix units so that $\max_{i,j} \|e_{i,j} - f_{i,j}\| < 2^{-k}$.*

Proof. By definition, $2^{-\Delta_3(n,k)} < \delta_2(2^{-(k+1)}, n)$. It then follows from Lemma 2.10 that $\mathbf{B}$ contains an $n \times n$ system $(g_{i,j})_{i,j}$ of matrix units so that $\max_{i,j} \|e_{i,j} - g_{i,j}\| < 2^{-k}$. However, the property of being an $n \times n$ system of matrix units is defined by a computably weakly stable set of relations (see [13, Examples 12]). Since $e_{i,j}$ is a computable point of $\mathbf{A}^\#$, $B((e_{i,j})_{i,j}; 2^{-k}) \cap \mathbf{B}^{n^2}$ is a c.e. open set of $(\mathbf{B}^\#)^{n^2}$. Therefore, by [9, Lemma 1.11], there is a computable $n \times n$ system $(f_{i,j})_{i,j}$ of matrix units of $\mathbf{B}^\#$ so that $\max_{i,j} \|e_{i,j} - f_{i,j}\| < 2^{-k}$. Again, since the inclusion map from $\mathbf{B}^\#$ to $\mathbf{A}^\#$ is computable, $(f_{i,j})_{i,j}$ is also a computable $n \times n$ system of matrix units of $\mathbf{A}^\#$. □

For every $n, k \in \mathbb{N}$, let

$$\Delta_4(n, k) = \max\{\max_{m \leq k} \Delta_1(m, \max_{n' \leq n} \Delta_3(n', k) + 2), 1 + \max_{n' \leq n} \Delta_3(n', k)\}.$$

Lemma 6.5. *Suppose $(e^s_{i,j})_{s,i,j}$ is a type-$(n_1, \ldots, n_m)$ computable matricial system of $\mathbf{A}^\#$ so that $\max_{i,j,s} d(e^s_{i,j}, \mathbf{B}) \leq 2^{-\Delta_3(\sum_j n_j^2, k)}$. Then, $\mathbf{B}$ contains a computable type-$(n_1, \ldots, n_m)$ matricial system $(f^s_{i,j})_{s,i,j}$ of $\mathbf{A}^\#$ so that $\max_{s,i,j} \left\|e^s_{i,j} - f^s_{i,s}\right\| < 2^{-k}$.*

Proof. For each $s \in \{1, \ldots, m\}$, let $p_s = \sum_i e^s_{i,i}$. Set $n = n_1^2 + \ldots, n_m^2$, and set $k_0 = \max_{n' \leq n} \Delta_3(n', k)+2$. By definition, $\Delta_4(n, k) \geq \Delta_1(m, k_0)$. Hence, by Lemma 6.2, $\mathbf{B}$ contains computable and mutually orthogonal projections $q_1, \ldots, q_m$ of $\mathbf{A}^\#$ so that $\max_i \|q_i - p_i\| < 2^{-k_0}$.

For each $s \in \{1, \ldots, m\}$, let $\mathbf{B}_s = q_s \mathbf{B} q_s$. Fix $b \in \mathbf{B}$. Then:

$$\begin{aligned}
\left\|e^s_{i,j} - q_s b q_s\right\| &= \left\|p_s e^s_{i,j} p_s - q_s b q_s\right\| \\
&\leq \left\|p_s e^s_{i,i} p_s - p_s e^s_{i,j} q_s\right\| + \left\|p_s e^s_{i,j} q_s - q_s b q_s\right\| \\
&\leq \|p_s - q_s\| + \left\|p_s e^s_{i,j} - q_s b\right\| \\
&\leq \|p_s - q_s\| + \left\|p_s e^s_{i,j} - q_s e^s_{i,j}\right\| + \left\|q_s e^s_{i,j} - q_s b\right\| \\
&\leq 2\|p_s - q_s\| + \left\|e^s_{i,j} - b\right\|.
\end{aligned}$$

Hence,

$$d(e^s_{i,j}, \mathbf{B}_s) \leq 2^{-k_0+1} + 2^{-\Delta_4(n,k)}.$$

However, by the definitions of k_0 and Δ_4, $2^{-k_0+1} + 2^{-\Delta_4(n,k)} < 2^{-\Delta_3(n_s,k)}$. Hence, by Lemma 6.4, $\mathbf{B}_s$ contains a computable $n_s \times n_s$ system $(f^s_{i,j})_{i,j}$ of matrix units of $\mathbf{A}^\#$ so that $\max_{i,j} \left\|e^s_{i,j} - f^s_{i,j}\right\| < 2^{-k}$ and so that $\sum_i f^s_{i,i} = q_s$. Since $q_1, \ldots, q_m$ are mutually orthogonal, it follows that $(f^s_{i,j})_{s,i,j}$ is a matricial system. $\square$

We are now ready to prove the Effective Glimm Lemma.

Proof of Theorem 6.1. Let $k \in \mathbb{N}$. We may define $\Delta(0, k)$ arbitrarily. So, assume n is a positive integer. Let P denote the set of all square partitions of n; that is, P is the set of all sequences $(n_1, \ldots, n_m)$ of positive integers so that $\sum_j n_j^2 = n$. We note that $P \neq \emptyset$ (set each $n_j = 1$). Set:

$$\begin{aligned}
k_0 &= \lceil \log_2(n 2^{k+1}) \rceil \\
\Delta(n, k) &= \max\{\Delta_5(n, \Delta_2(\sum_j n_j, k_0)) \ : \ (n_1, \ldots, n_m) \in P\}.
\end{aligned}$$

We remark that $\Delta_2(\sum_j n_j, k_0) \geq k_0$.

Suppose $\mathbf{F}$ is a c.e. closed $*$-subalgebra of $\mathbf{A}^\#$ that contains $\mathbf{1_A}$, and assume $\dim(\mathbf{F}) = n$. Further, assume $B(\mathbf{B}; 2^{-\Delta(n,k)})$ contains a matricial generating system of $\mathbf{F}$.

We first show that there is a unital matricial generating system $(g^s_{i,j})_{s,i,j}$ of $\mathbf{F}$ so that $g^s_{i,s}$ is a computable point of $\mathbf{A}^\#$ uniformly in s, i, j and so that $\max_{s,i,j} d(g^s_{i,j}, \mathbf{B}) < 2^{-\Delta(n,k)}$. In order to demonstrate this, when $(n_1, \ldots, n_m) \in P$, let $\mathcal{G}_{n_1,\ldots,n_m}$ denote the set of all type-$(n_1, \ldots, n_m)$ unital matricial systems $(e^s_{i,j})_{s,i,j}$ of $\mathbf{F}$. By Proposition 5.12, $\mathcal{G}_{n_1,\ldots,n_m}$ is a c.e. closed set of $(\mathbf{F}^\#)^n$ and hence of $(\mathbf{A}^\#)^n$. Since $\mathbf{B}$ is a c.e. closed set of $\mathbf{A}^\#$, $B(\mathbf{B}; 2^{-\Delta(n,k)})$ is a c.e. open set of $\mathbf{A}^\#$. It follows that the set of all $(n_1, \ldots, n_m) \in P$ so that $\mathcal{G}_{n_1,\ldots,n_m} \cap B(\mathbf{B}; 2^{-\Delta(n,k)}) \neq \emptyset$ is c.e. uniformly in n, k. Furthermore, there is a unique $(n_1, \ldots, n_m) \in P$ so that $\mathcal{G}_{n_1,\ldots,n_m} \cap B(\mathbf{B}; 2^{-\Delta(n,k)}) \neq \emptyset$. By [9, Lemma 1.11], this intersection contains a computable point of $(\mathbf{F}^\#)^n$ which is also a computable point of $(\mathbf{A}^\#)^n$.

We now construct u. Firstly, since $\Delta(n, k) \geq \Delta_4(n, \Delta_2(\sum_j n_j; k_0))$, by Lemma 6.5, $\mathbf{B}$ contains a computable type-$(n_1, \ldots, n_m)$ matricial generating system $(h^s_{i,j})_{s,i,j}$ of $\mathbf{A}^\#$ so that $\max_{s,i,j} \left\|g^s_{i,j} - h^s_{i,j}\right\| < \min\{2^{-k_0}, 2^{-\Delta_2(\sum_j n_j, k_0)}\}$. Hence, by Lemma

6.3, there is a computable unitary v of $\mathbf{A}^\#$ so that $v^* g^s_{i,i} v = h^s_{i,j}$ for all i, s and so that $\|v - \mathbf{1}_\mathbf{A}\| < 2^{-k_0}$. We then set $u = \sum_{i,s} g^s_{i,1} v h^s_{1,i}$.

By direct computation, u is unitary, and $u^* g^s_{i,j} u = h^s_{i,j}$. It only remains to show that $\|u - \mathbf{1}_\mathbf{A}\| < 2^{-k}$. To begin, we observe that

$$\left\|g^s_{i,j} - g^s_{i,1} v h^s_{i,1}\right\| \leq \left\|g^s_{i,1} - g^s_{i,1} v g^s_{1,i}\right\| + \left\|g^s_{i,1} v g^s_{1,i} - g^s_{i,1} v h^s_{1,i}\right\|.$$

Next, we obtain an upper bound on the first term in the above sum as follows.

$$\begin{aligned}\left\|g^s_{i,i} - g^s_{i,1} v g^s_{1,i}\right\| &\leq \left\|g^s_{1,i} - g^s_{i,1}\mathbf{1}_\mathbf{A} g^s_{1,i}\right\| + \left\|g^s_{i,1}\mathbf{1}_\mathbf{A} g^s_{1,i} - g^s_{i,1} v g^s_{1,i}\right\| \\ &\leq \|\mathbf{1}_\mathbf{A} - v\| \\ &< 2^{-k_0}.\end{aligned}$$

We then note that

$$\left\|g^s_{i,1} v g^s_{1,i} - g^s_{i,1} v h^s_{1,i}\right\| \leq \left\|g^s_{1,i} - h^s_{1,i}\right\| < 2^{-k_0}.$$

Hence, since $\sum_{s,i} g^s_{i,i} = \mathbf{1}_\mathbf{A}$,

$$\begin{aligned}\|\mathbf{1}_\mathbf{A} - u\| &< \sum_{s,i} 2^{-k_0+1} \\ &= n2^{-k_0+1} \\ &\leq 2^{-k}.\end{aligned}$$

□

We can now advance to prove the first of our main theorems.

7. Proof of Main Theorem 1

We begin with two lemmas concerning finite-dimensional algebras. The first was already proved by A. Fox [12]; we give an alternate proof based on our results on computable weak stability.

Lemma 7.1. *If $\mathbf{A}^\#$ is a c.e. presentation of a finite-dimensional and unital* C*-*algebra, then there is a finite multiset F of positive integers and a computable unital $*$-isomorphism ψ from $\bigoplus_{n\in F} M_n(\mathbb{C})$ to $\mathbf{A}^\#$. Moreover, F and an $(\bigoplus_{n\in F} M_n(\mathbb{C}), \mathbf{A}^\#)$ index of ψ can be computed from the dimension of $\mathbf{A}$ and an index of $\mathbf{A}^\#$.*

Proof. By following the proof of Theorem 6.1, it is possible to compute positive integers $m, n_1, \ldots, n_m$ and a computable type-$(n_1, \ldots, n_m)$ unital matricial generating system $(g^s_{r,t})_{s,r,t}$ of $\mathbf{A}^\#$. Let F denote the multiset whose elements are $n_1, \ldots, n_m$ and each element is repeated according to the number of times it appears in $(n_1, \ldots, n_m)$. Let $(E^s_{r,t})_{s,r,t}$ denote the standard matricial generating system for $\bigoplus_{n\in F} M_n(\mathbb{C})$. Let ψ be the unique $*$-isomorphism of $\bigoplus_{n\in F} M_n(\mathbb{C})$ onto $\mathbf{A}$ so that $\psi(E^s_{r,t}) = g^s_{r,t}$. It follows that ψ is a computable map from $\bigoplus_{n\in F} M_n(\mathbb{C})$ to $\mathbf{A}^\#$. □

Lemma 7.2. *Assume $\mathbf{A}^\#$ is a c.e. presentation of an AF algebra. Suppose $p_0, \ldots, p_n$ are computable points of $\mathbf{A}^\#$, and let $k \in \mathbb{N}$. Then $\mathbf{A}^\#$ has a c.e. closed and finite-dimensional $*$-subalgebra $\mathbf{F}$ so that $\mathbf{1}_\mathbf{A} \in \mathbf{F}$ and $\max_j d(p_j, \mathbf{F}) < 2^{-k}$. Furthermore, an $\mathbf{A}^\#$ index of $\mathbf{F}$ can be computed from k and $\mathbf{A}^\#$ indexes of p_0, $\ldots$, p_n.*

Proof. Fix a sequence $(n_1, \ldots, n_m)$ of positive integers and let $\ell = \sum_j n_j^2$. Define $\mathcal{M}_{n_1,\ldots,n_m}$ to be the set of all type-$(n_1, \ldots, n_m)$ matricial systems of $\mathbf{A}$. For each $\vec{a} = (a_1, \ldots, a_\ell) \in \mathbf{A}^\ell$ and every $\vec{q} = (q_1, \ldots, q_n) \in \mathbb{Q}(i)^\ell$, let $\langle \vec{q}, \vec{a} \rangle = \sum_j q_j a_j$. Then let $U_{n_1,\ldots,n_m}$ denote the set of all $\vec{a} \in \mathbf{A}^\ell$ for which there exists $\vec{q} \in \mathbb{Q}(i)^\ell$ so that $\max_j \|p_j - \langle \vec{a}, \vec{q} \rangle\| < 2^{-k}$.

It follows from Proposition 5.12 and [9, Theorem 1.14] that $\mathcal{M}_{n_1,\ldots,n_m}$ is a c.e. closed set of $(\mathbf{A}^\#)^\ell$ uniformly in $n_1, \ldots, n_m$. We claim that $U_{n_1,\ldots,n_m}$ is a c.e. open set of $(\mathbf{A}^\#)^\ell$ uniformly in $n_1, \ldots, n_m$. To see this, for each $\vec{q} \in \mathbb{Q}(i)^\ell$, let $f_{j,\vec{q}} : \mathbf{A}^\ell \to \mathbb{C}$ be defined by setting $f_{j,\vec{q}}(\vec{a}) = p_j - \langle \vec{q}, \vec{a} \rangle$. It follows that $f_{j,\vec{q}}$ is a computable map from $(\mathbf{A}^\#)^\ell$ to $\mathbf{A}^\#$ uniformly in $j, \vec{q}$. Thus, by Proposition 3.6, $f_{j,\vec{q}}^{-1}[B(\mathbf{0}; 2^{-k})]$ is a c.e. open set of $(\mathbf{A}^\#)^\ell$ uniformly in $j, \vec{q}$. Hence, $\bigcap_j f_{j,\vec{q}}^{-1}[B(\mathbf{0}; 2^{-k})]$ is a c.e. open set of $(\mathbf{A}^\#)^\ell$ uniformly in $\vec{q}$. However, $U_{n_1,\ldots,n_m} = \bigcup_{\vec{q}} \bigcap_j f_{j,\vec{q}}^{-1}[B(\mathbf{0}; 2^{-k})]$. It follows that $U_{n_1,\ldots,n_m}$ is c.e. open uniformly in $n_1, \ldots, n_m$.

It now follows that the set of all $(n_1, \ldots, n_m)$ so that $U_{n_1,\ldots,n_m} \cap \mathcal{M}_{n_1,\ldots,n_m} \neq \emptyset$ is c.e. At the same time, since $\mathbf{A}$ is an AF algebra, there must exist $n_1, \ldots, n_m$ so that $U_{n_1,\ldots,n_m} \cap \mathcal{M}_{n_1,\ldots,n_m} \neq \emptyset$. Therefore, it is possible to compute positive integers $m, n_1, \ldots, n_m$ so that $U_{n_1,\ldots,n_m} \cap \mathcal{M}_{n_1,\ldots,n_m} \neq \emptyset$ via a search procedure. It then follows from [9, Lemma 1.12] that it is possible to compute a type-$(n_1, \ldots, n_m)$ compound generating matricial system $(f^s_{r,t})_{s,r,t}$ of $\mathbf{A}^\#$ that belongs to $U_{n_1,\ldots,n_m}$. Taking $\mathbf{F}$ to be the $*$-algebra generated by this system yields the desired result. $\square$

Proof of Main Theorem 1. Let $\Delta : \mathbb{N}^2 \to \mathbb{N}$ denote the function whose existence is guaranteed by the Effective Glimm Lemma (Theorem 6.1). Without loss of generality, we assume Δ is increasing in each variable.

Suppose $\mathbf{A}^\#$ is a c.e. presentation of an AF algebra. Fix an effective enumeration $(g_n)_{n\in\mathbb{N}}$ of the generated points of $\mathbf{A}^\#$.

We construct a monotonically non-decreasing sequence $(\mathbf{A}_k)_{k\in\mathbb{N}}$ of finite-dimensional and c.e. closed $*$-subalgebras of $\mathbf{A}^\#$ so that $d(g_k, \mathbf{A}_k) < 2^{-k}$ for all $k \in \mathbb{N}$ and so that $\mathbf{1}_\mathbf{A} \in \mathbf{A}_0$. Furthermore, we ensure $(\dim(\mathbf{A}_k))_{k\in\mathbb{N}}$ is computable. The existence of a computable AF certificate for $\mathbf{A}^\#$ then follows from Lemma 7.1.

To begin, set $\mathbf{A}_0 = \{\mathbf{1}_\mathbf{A}\}$. Let $k \in \mathbb{N}$, and suppose $\mathbf{A}_k$ has been defined. Assume also that $n := \dim(\mathbf{A}_k)$ has been computed. It follows from Lemma 7.1 that it is possible to compute a matricial generating system $(e^s_{r,t})_{s,r,t}$ for $\mathbf{A}_k$; let $(n_1, \ldots, n_m)$ denote the type of this system. Then, by Lemma 7.2, it is possible to compute an $\mathbf{A}^\#$ index of a c.e. closed and finite-dimensional unital $*$-subalgebra $\mathbf{F}$ of $\mathbf{A}^\#$ so that $d(a, \mathbf{F}) < 2^{-\Delta(n,k+2)}$ whenever

$$a \in \{g_{k+1}\} \ \cup \ \{e^s_{r,t} \ : \ s \in \{1, \ldots, m\} \ \wedge \ r, t \in \{1, \ldots, n_s\}\}.$$

By the Effective Glimm Theorem, it is now possible to compute an $\mathbf{A}^\#$ index of a unitary u so that $u^*\mathbf{A}_k u \subseteq \mathbf{F}$ and so that $\|u - \mathbf{1}_\mathbf{A}\| < 2^{-(k+2)}$. We then set $\mathbf{A}_{k+1} = u\mathbf{F}u^*$. We now observe that

$$\begin{aligned} d(g_{k+1}, \mathbf{A}_{k+1}) &< \|g_{k+1} - u^* g_{k+1} u\| + 2^{-\Delta(n,k+2)} \\ &\leq \|g_{k+1} - u^* g_{k+1}\| + \|u^* g_{k+1} - u^* g_{k+1} u\| + 2^{-\Delta(n,k+2)} \\ &< 2^{-(k+1)} + 2^{-\Delta(n,k+2)}. \end{aligned}$$

As noted above, Δ is increasing in each variable. Thus, in particular, $\Delta(n, k+2) \geq k+2$, and so $d(g_{k+1}, \mathbf{A}_{k+1}) < 2^{-(k+1)}$. $\square$

We note that our proof of Main Theorem 1 is uniform. The following is an immediate, and somewhat surprising, consequence.

Corollary 7.3. *Every c.e. presentation of a unital AF algebra is computable.*

Proof. Apply Main Theorem 1 and Corollary 5.14. □

We now proceed to prove our second main theorem.

8. Proof of Main Theorem 2

We begin with two lemmas concerning computable certificates of dimensionality.

Lemma 8.1. *If $\mathcal{G}^\#$ is a c.e. presentation of a dimension group, then $\mathcal{G}^\#$ has a computable certificate of dimensionality.*

Proof. Suppose $\mathcal{G}^\#$ is a c.e. presentation of a dimension group and let $(g_n)_{n\in\mathbb{N}}$ be an effective enumeration of the positive cone of $\mathcal{G}$. By Lemma 5.10, $\mathcal{G}^\#$ has the Effective Shen Property.

For each $s \in \mathbb{N}$, we construct the following:

- A positive integer n_s.
- An ordered group homomorphism $\phi_s : \mathbb{Z}^{n_s} \to \mathbb{Z}^{n_{s+1}}$.
- A homomorphism θ_s from $\mathbb{Z}^{n_s}$ to $\mathcal{G}^\#$.

We ensure the following requirements.

(R-1) $(\mathcal{G}, (\theta_s)_{s\in\mathbb{N}})$ is an inductive upper limit of $(\mathbb{Z}^{n_s}, \phi_s)_{s\in\mathbb{N}}$.
(R-2) $\mathcal{G}^+ = \bigcup_{s\in\mathbb{N}} \theta_s[\mathbb{Z}^{n_s}_{\geq 0}]$.
(R-3) For every $s \in \mathbb{N}$, $\ker(\theta_s) = \bigcup_{k\in\mathbb{N}} \ker(\phi_{s,s+k})$.

It follows from Lemma 4.2 that if these requirements are satisfied, then $(\mathcal{G}^\#, (\theta_s)_{s\in\mathbb{N}})$ is an inductive limit of $(\mathbb{Z}^{n_s}, \phi_s)_{s\in\mathbb{N}}$.

We divide the construction into stages. At stage s, we define n_s and θ_s. At stage $s+1$, we define ϕ_s.

We let $\ker_s(\theta_k)$ denote the set of all vectors that have been enumerated into $\ker(\theta_k)$ after s steps of computation. We control the enumeration of these kernels so as to satisfy the following for every $t \in \mathbb{N}$.

(C-1) If $t = 0$, then $\ker_t(\theta_k) = \emptyset$ for all $k \in \mathbb{N}$.
(C-2) There is at most one $k \in \mathbb{N}$ so that $\ker_{t+1}(\theta_k) \setminus \ker_t(\theta_k) \neq \emptyset$.
(C-3) For all $k \in \mathbb{N}$, if $\ker_{t+1}(\theta_k) \setminus \ker_t(\theta_k) \neq \emptyset$, then $k \leq t$ and $\#(\ker_{t+1}(\theta_k) \setminus \ker_t(\theta_k)) = 1$.

In addition, we assume these enumerations are produced so that from k, t it is possible to compute an index of $\ker_t(\theta_k)$. We note that $\#\ker_t(\theta_k) \leq t$ for all k, t (since it takes at least t steps of computation to produce an output of size at least t).

During the construction, we ensure the following invariants.

(I-1) $\theta_{s+1} \circ \phi_s = \theta_s$.
(I-2) If $\alpha \in \ker_{t+1}(\theta_s) \setminus \ker_t(\theta_s)$ then, $\alpha \in \ker(\phi_{s,t})$.
(I-3) $g_s \in \theta_k[(\mathbb{Z}^{n_s})^+]$.

It follows that if these invariants are ensured, then the above requirements are satisfied.

Stage 0: Let $n_0 = 1$ and let $\theta_0(n) = ng_0$.

Stage $s+1$: Assume $n_0, \ldots, n_s$ and $\theta_0, \ldots, \theta_s$ have been defined. If $s > 0$, then assume that $\phi_0, \ldots, \phi_{s-1}$ have been defined. Assume all invariants have been maintained at previous stages.

Let ι_s denote the natural injection of $\mathbb{Z}^{n_s}$ into $\mathbb{Z}^{n_s} \bigoplus \mathbb{Z}$. That is, $\iota_s(\vec{n}) = (\vec{n}, 0)$. Define $\zeta_s : \mathbb{Z}^{n_s} \bigoplus \mathbb{Z} \to G$ by setting $\zeta_s(\vec{n}, m) = \theta_s(\vec{n}) + mg_{s+1}$.

Case 1: $s = 0$ or $\ker_s(\theta_k) \subseteq \ker_{s-1}(\theta_k)$ for all $k < s$ (that is, for $k < s$ nothing new was added to our approximation of the kernel of θ_k at stage s).

We set:

$$\begin{aligned} n_{s+1} &= n_s + 1 \\ \phi_s &= \iota_s \\ \theta_{s+1} &= \zeta_s. \end{aligned}$$

Case 2: $s > 0$ and there exists $k < s$ so that $\ker_s(\theta_k) \setminus \ker_{s-1}(\theta_k) \neq \emptyset$.

Because of our conditions on the enumeration of the kernels, the number k is unique, and we set $k_{s+1} = k$. We also define α_{s+1} to be the unique element in $\ker_s(\theta_k) \setminus \ker_{s-1}(\theta_k)$. We seek to define ϕ_s so that $\phi_{s-1} \circ \cdots \circ \phi_k(\alpha_{s+1}) \in \ker(\phi_s)$. To this end, set $\beta_{s+1} = \phi_{s-1} \circ \cdots \circ \phi_k(\alpha_{s+1})$. We note that $\beta_{s+1} \in \ker(\theta_s)$ (since all invariants have been maintained at prior stages). Thus, by definition, $\beta_{s+1} \in \ker(\zeta_s)$. We apply the Effective Shen Property to n_s, ζ_s, and β_{s+1} to obtain a positive integer p, a homomorphism $\phi'_s : \mathbb{Z}^{n_s} \bigoplus \mathbb{Z} \to \mathbb{Z}^p$, and an ordered group homomorphism $\theta'_s : \mathbb{Z}^p \to \mathcal{G}$ so that $\beta_{s+1} \in \ker(\phi'_s)$ and so that $\zeta_s = \theta'_s \circ \phi'_s$. We then set:

$$\begin{aligned} n_{s+1} &= p \\ \theta_{s+1} &= \theta'_s \\ \phi_s &= \phi'_s \circ \iota_s. \end{aligned}$$

This completes the construction.

We now demonstrate that all invariants are maintained at every stage of the construction. (I-2) and (I-3) follow directly from the construction. We proceed to verify (I-1). We first note that by construction, $\zeta_s \circ \iota_s = \theta_s$ for all $s \in \mathbb{N}$. Let $s \in \mathbb{N}$.

Suppose $s = 0$. By construction, $\phi_0 = \iota_0$ and $\theta_1 = \zeta_0$. By definition, for each $n \in \mathbb{N}$, $\zeta_0(\iota_0(n)) = \zeta_0(n, 0) = \theta_0(n)$. Hence, $\theta_1 \circ \phi_0 = \theta_0$.

Now, suppose $s > 0$. We first consider the case where $\ker_s(\theta_k) \subseteq \ker_{s-1}(\theta_k)$ for all $k < s$ (that is, Case 1 holds at stage $s+1$). By construction, $n_{s+1} = n_s + 1$, $\phi_s = \iota_s$ and $\theta_{s+1} = \zeta_s$. Thus, for all $\vec{n} \in \mathbb{Z}^{n_{s+1}}$, $\zeta_s(\iota_s(\vec{n})) = \zeta_s(\vec{n}, 0) = \theta_s(\vec{n})$; that is, $\theta_{s+1} \circ \phi_s = \theta_s$.

Now, we consider the case in which there exists $k < s$ so that $\ker_s(\theta_k) \setminus \ker_{s-1}(\theta_k) \neq \emptyset$ (that is, Case 2 holds at stage $s+1$). By construction, $\phi_s = \phi'_s \circ \iota_s$ and $\theta_{s+1} = \theta'_s$. Furthermore, $\zeta_s = \theta'_s \circ \phi'_s$. Hence, $\theta_s = \zeta_s \circ \iota_s = \theta_{s+1} \circ \phi_s$. □

The proof given above is an adaptation of the classical proof as given in, for example, [8]. The main difference between our proof and the classical proof is that in the latter, one can simply consider a basis for $\ker(\theta_k)$ and apply the Shen property a finite number of times, once to each element of the basis. In the effective setting, one cannot, in general, know when one has found a basis for a finitely generated free abelian group; for this reason, we have to "revisit" each kernel infinitely often.

Lemma 8.2. *Every c.e. presentation of a unital dimension group has a unital certificate of dimensionality.*

Proof. Suppose $(\mathcal{G}, u)^\#$ is a c.e. presentation of a unital dimension group. By Lemma 8.1, $\mathcal{G}^\#$ has a computable certificate of dimensionality $(n_s, \nu_s, \phi_s)_{s\in\mathbb{N}}$. By the proof of Lemma 8.1, we may assume $n_0 = 1$ and $\nu_0(n) = nu$. Set $u_s = \phi_{0,s}(1)$.

We now construct an inductive sequence $((\mathbb{Z}^{m_s}, v_s), \gamma_s)_{s\in\mathbb{N}}$ of unital ordered abelian groups. To begin, for each s, let $\mathcal{H}_s$ denote the convex ordered subgroup of $\mathbb{Z}^{n_s}$ generated by u_s, and assume $\mathcal{H}_s = (H_s, H_s \cap \mathbb{Z}^{n_s}_{\geq 0})$. By the uniformity of Lemma 5.6, H_s is computable uniformly in s. Define B_s to be the intersection of the standard basis of $\mathbb{Z}^{n_s}$ and H_s, and set $m_s =$ the cardinality of B_s. By the proof of [18, Proposition 3.8], B_s is a basis for $\mathcal{H}_s$.

For each $s \in \mathbb{N}$, there is a unique isomorphism τ_s from $\mathbb{Z}^{m_s}$ to $\mathcal{H}_s$ so that for each $j \in \{1, \ldots, m_s\}$, $\tau_s(e_j^{m_s}) = e_k^{n_s}$ where k is the j-th element of $\{k' : e_{k'}^{n_s} \in B_s\}$. Let $v_s = \tau_s^{-1}(u_s)$. Thus, v_s is an order unit of $\mathbb{Z}^{m_s}$. Let $\psi_s = \phi_s|_{H_s}$. Since $\phi_s(u_s) = u_{s+1}$, it follows that $\phi_s[H_s] \subseteq H_{s+1}$. We now set $\gamma_s = \tau_{s+1}^{-1}\psi_s\tau_s$.

Let ι_s denote the inclusion map of $\mathcal{H}_s$ into $\mathbb{Z}^{n_s}$, and define μ_s to be $\nu_s\iota_s\tau_s$. It follows that $((\mathcal{G}, u)^\#, (\mu_s)_{s\in\mathbb{N}})$ is a computable upper inductive limit of $((\mathbb{Z}^{m_s}, v_s), \gamma_s)_{s\in\mathbb{N}}$; we claim it is in fact a computable inductive limit. It suffices to show that $(\mathcal{G}, (\mu_s)_{s\in\mathbb{N}})$ is an inductive limit of $(\mathbb{Z}^{m_s}, \gamma_s)_{s\in\mathbb{N}}$. To this end, suppose $x \in \mathcal{G}^+$. By Lemma 4.2, there exists $s \in \mathbb{N}$ and $y \in \mathbb{Z}^{n_s}_{\geq 0}$ so that $\nu_s(y) = x$. Since u is an order unit of $\mathcal{G}$, there exists a positive integer t so that $-tu \leq x \leq tu$. Thus, $-t\nu_s(u_s) \leq \nu_s(y) \leq t\nu_s(u_s)$. Since $\ker(\nu_s) = \bigcup_{k\in\mathbb{N}} \ker(\phi_{s,s+k})$, it follows that there exists $k \in \mathbb{N}$ so that $-t\phi_{s,s+k}(u_s) \leq \phi_{s,s+k}(y) \leq t\phi_{s,s+k}(u_s)$. However, $\phi_{s,s+k}(u_s) = u_{s+k}$ which belongs to $\mathcal{H}_{s+k}$. Since $\mathcal{H}_{s+k}$ is convex, $\phi_{s,s+k}(y) \in \mathcal{H}_{s+k}$. Moreover, $\phi_{s,s+k}(y) \in \mathcal{H}^+_{s+k}$, and so $x = \nu_{s+k}(\phi_{s,s+k}(y)) \in \nu_{s+k}[\mathcal{H}^+_{s+k}]$. Hence, $\mathcal{G}^+ = \bigcup_{s\in\mathbb{N}} \mu_s[\mathcal{H}^+_s]$. Since τ_s and ι_s are injective, it now follows that $\ker(\mu_s) = \bigcup_{k\in\mathbb{N}} \ker(\gamma_{s,s+k})$. Thus, by Lemma 4.2, $(\mathcal{G}, (\mu_s)_{s\in\mathbb{N}})$ is an inductive limit of $(\mathbb{Z}^{m_s}, \gamma_s)_{s\in\mathbb{N}}$.

It now follows that $(m_s, v_s, \mu_s, \gamma_s)_{s\in\mathbb{N}}$ is a computable unital certificate of dimensionality of $(\mathcal{G}, u)^\#$. □

We note that the proofs of the above lemmas are uniform.

Proof of Main Theorem 2.

(1): Suppose $\mathcal{G}^\#$ is a c.e. presentation of a dimension group. By Lemma 8.1, $\mathcal{G}^\#$ has a computable certificate of dimensionality $(n_s, \nu_s, \phi_s)_{s\in\mathbb{N}}$.

Define F_0 to be the multiset whose only element is 1 repeated n_0 times. Set $u_0 = v_{F_0}$ (which is $(1, \ldots, 1) \in \mathbb{Z}^{n_0}$), and let $\mathbf{A}_0 = \bigoplus_{n\in F_0} M_n(\mathbb{C})$.

Assume F_s has been defined, $u_s = v_{F_s}$, and $\mathbf{A}_s = \bigoplus_{n\in F_s} M_n(\mathbb{C})$. We compute an order unit $u_{s+1} \in \mathbb{Z}^{n_{s+1}}$ so that $u_{s+1} \geq \phi_0(u_s)$. We let F_{s+1} be the multiset that consists of the integers that appear in u_s and the number of times an integer is repeated in F_{s+1} is the number of times it is repeated in u_s. Set $u_{s+1} = v_{F_{s+1}}$, and set $\mathbf{A}_{s+1} = \bigoplus_{n\in F_{s+1}} M_n(\mathbb{C})$. Since $\phi_s(u_s) \leq u_{s+1}$, it follows from the discussion in Section 2.3.4 that there exists a $*$-homomorphism $\psi_s : \mathbf{A}_s \to \mathbf{A}_{s+1}$ so that $\eta_{F_s,F_{s+1}}(\psi_s)) = \phi_s$. Furthermore, ψ_s can be computed from the information in ϕ_s.

By Corollary 5.14, $(\mathbf{A}_s, \psi_s)_{s\in\mathbb{N}}$ sequence has a computable inductive limit $(\mathbf{A}^\#, (\mu_s)_{s\in\mathbb{N}})$ in the category of c.e. presentations of C*-algebras. As noted in Section 2.3.5, $\mathbf{A}$ is an AF algebra.

We now show that $(K_0(\mathbf{A}^\#), K_0(\mathbf{A})^+)$ is computably isomorphic to $\mathcal{G}^\#$. It follows from [20, Theorem 6.3.2] and Lemma 4.2 that $((K_0(\mathbf{A}^\#), K_0(\mathbf{A})^+), (K_0(\mu_s))_{s\in\mathbb{N}})$ is an inductive limit of $((K_0(\mathbf{A}_s), K_0(\mathbf{A}_s)^+), K_0(\psi_s))_{s\in\mathbb{N}}$ in the category of c.e. presentations of ordered abelian groups. It then follows that $((K_0(\mathbf{A}^\#), K_0(\mathbf{A})^+), (K_0(\mu_s)\circ \zeta_{F_s}^{-1})_{s\in\mathbb{N}}$ is an inductive limit of $(\mathbb{Z}^{n_s}, \phi_s)_{s\in\mathbb{N}}$ in the category of c.e. presentations of ordered abelian groups. Thus, the reduction of $(\mathcal{G}, (\nu_s)_{s\in\mathbb{N}})$ to $((K_0(\mathbf{A}^\#), K_0(\mathbf{A})^+), (K_0(\mu_s)\circ \zeta_{F_s}^{-1})_{s\in\mathbb{N}}$ is an isomorphism.

(2): Suppose u is an order unit of $\mathcal{G}$. By Lemma 8.2, $(\mathcal{G}, u)^\#$ has a computable unital certificate of dimensionality $(n_s, u_s, \nu_s, \phi_s)_{s\in\mathbb{N}}$. Thus, by Definition 4.3 u_s is an order unit for $\mathbb{Z}^{n_s}$ and $\nu_s(u_s) = u$. We modify the above construction so that $\eta_{F_s}(K_0(\mathbf{1}_{\mathbf{A}_s})) = u_s$. □

Corollary 8.3. *Suppose $\mathcal{G}^\# = (G^\#, P)$ is a c.e. presentation of a dimension group. Then, $G^\#$ is computable.*

Proof. Let $(n_s, \nu_s, \phi_s)_{s\in\mathbb{N}}$ be a computable certificate of dimensionality for $\mathcal{G}^\#$. Thus, $(G^\#, (\nu_s)_{s\in\mathbb{N}})$ is a computable inductive limit of $(\mathbb{Z}^{n_s}, \phi_s)_{s\in\mathbb{N}}$ in the category of c.e. presentations of abelian groups. However, by Proposition 5.2, $(\mathbb{Z}^{n_s}, \phi_s)_{s\in\mathbb{N}}$ has a computable inductive limit $(H^\#, (\mu_s)_{s\in\mathbb{N}})$ in the category of computable presentations of abelian groups. Since every computable presentation is also c.e., $(H^\#, (\mu_s)_{s\in\mathbb{N}})$ is also a computable inductive upper limit of $(\mathbb{Z}^{n_s}, \phi_s)_{s\in\mathbb{N}}$ in the category of c.e. presentations of abelian groups. Let λ be the reduction of $(G^\#, (\nu_s)_{s\in\mathbb{N}})$ to $(H^\#, (\mu_s)_{s\in\mathbb{N}}$. Thus, by definition of computable inductive limit, λ is a computable map from $G^\#$ to $H^\#$.

On the other hand, by Lemma 5.3 $(H, (\mu_s)_{s\in\mathbb{N}})$ is an inductive limit of $(\mathbb{Z}^{n_s}, \phi_s)_{s\in\mathbb{N}}$ and $(G, (\nu_s)_{s\in\mathbb{N}})$ is as well. Thus, λ is the reduction of $(G, (\nu_s)_{s\in\mathbb{N}})$ to $(H, (\mu_s)_{s\in\mathbb{N}})$ (since reductions are unique). However, it then follows that λ is an isomorphism.

Thus, λ is a computable isomorphism from $G^\#$ to $H^\#$. Since $H^\#$ is computable, it follows that $G^\#$ is computable. □

Corollary 8.4. *If $\mathbf{A}^\#$ is a c.e. presentation of a unital AF algebra, then $K_0(\mathbf{A}^\#)$ is computable.*

We now turn to the proof of our third main theorem.

9. Proof of Main Theorem 3

(1) ⇒ (2): This is a straightforward application of Main Theorem 1 and Corollary 4.5.

(2) ⇒ (3): Suppose $(\bigoplus_{n\in F_s} M_n(\mathbb{C}), \psi_s)_{s\in\mathbb{N}}$ is a computable inductive sequence whose inductive limit is $*$-isomorphic to $\mathbf{A}$. Furthermore, assume each ψ_s is unital. By Corollary 5.14, $(\bigoplus_{n\in F_s} M_n(\mathbb{C}), \psi_s)_{s\in\mathbb{N}}$ has a computable inductive limit $(\mathbf{B}^\#, (\nu_s)_{s\in\mathbb{N}})$ in the category of computable presentations of C*-algebras. Thus, $\mathbf{B}$ is $*$-isomorphic to $\mathbf{A}$. As remarked in Section 3.2, the class of computably

presentable C*-algebras is closed under isomorphism. Hence, **A** is computably presentable.

(3) ⇒ (4): This follows from Corollary 8.4.

(4) ⇒ (5): Let $(\mathcal{G}, u) = K_0^{\mathrm{sc}}(\mathbf{A})$, and suppose $(\mathcal{G}, u)^\#$ is a c.e. presentation. By Main Theorem 2, $(\mathcal{G}, u)^\#$ has a computable unital certificate of dimensionality $(n_s, u_s, \nu_s, \gamma_s)_{s\in\mathbb{N}}$.

We first show that **A** has a c.e. presentation.

Let F_s denote the multiset that consists of the integers that appear in u_s where the number of times an integer is repeated in F_s is the number of times it is repeated in u_s. Set $\mathbf{B}_s = \bigoplus_{n\in F_s} M_n(\mathbb{C})$. For each s, it is fairly straightforward to compute a unital $*$-homomorphism $\psi_s : \mathbf{B}_s \to \mathbf{B}_{s+1}$ so that $\eta_{F_s,F_{s+1}}(K_0(\psi_s)) = \phi_s$. By Corollary 5.14, $(B_s, \psi_s)_{s\in\mathbb{N}}$ has a computable inductive limit $(\mathcal{B}^\#, (\nu_s)_{s\in\mathbb{N}})$ in the category of C*-algebras with c.e. presentations.

It follows that $(K_0^{\mathrm{sc}}(\mathbf{A}), (\nu_s \circ \zeta_{F_s}^{-1})_{s\in\mathbb{N}})$ is an inductive limit of $(K_0^{\mathrm{sc}}(\mathbf{B}_s), K_0(\psi_s))_{s\in\mathbb{N}}$ as is $(K_0^{\mathrm{sc}}(\mathbf{B}), (K_0(\mu_s))_{s\in\mathbb{N}})$. Thus, $K_0^{\mathrm{sc}}(\mathbf{A})$ and $K_0^{\mathrm{sc}}(\mathbf{B})$ are isomorphic. Hence, by Elliott's Theorem, **B** is $*$-isomorphic to **A**. Therefore, **A** has a c.e. presentation $\mathbf{A}^\#$.

By Main Theorem 1, $\mathbf{A}^\#$ has a computable AF certificate $(G_s, \gamma_s)_{s\in\mathbb{N}}$. It is straightforward to construct a computably presented labeled Bratteli diagram of this certificate.

(5) ⇒ (1): Suppose $\mathcal{D}^\#$ is a computable presentation of labeled Bratteli diagram of an AF certificate $(F_s, \phi_s)_{s\in\mathbb{N}}$ of **A**. By means of the labeling, it is possible to compute F_s from s. Set $\mathbf{A}_s = \bigoplus_{n\in F_s} M_n(\mathbb{C})$.

By means of the information provided by the edge function of $\mathcal{D}$, it is possible to compute from $s \in \mathbb{N}$ a unital $*$-monomorphism $\tau_s : \mathbf{A}_s \to \mathbf{A}_{s+1}$ so that $K_0(\phi_{s+1}^{-1} \circ \phi_s) = K_0(\tau_s)$. By Corollary 5.14, $(\mathbf{A}_s, \tau_s)_{s\in\mathbb{N}}$ has a computable inductive limit $(\mathbf{B}^\#, (\nu_s)_{s\in\mathbb{N}})$ in the category of computably presentable C*-algebras. However, by construction, $K_0^{\mathrm{sc}}(\mathbf{A}) = K_0^{\mathrm{sc}}(\mathbf{B})$. By Elliott's Theorem, **A** and **B** are $*$-isomorphic. Thus, **A** is computably presentable.

10. Proof of Main Theorem 4

Our proof relies on two very technical lemmas: one for AF algebras and one for groups.

Lemma 10.1. *Suppose $(F_s, \phi_s)_{s\in\mathbb{N}}$ is an AF certificate for **A**, and let $\mathbf{G} = \bigoplus_{n\in G} M_n(\mathbb{C})$ where G is a finite multiset of positive integers. Furthermore, assume $\omega : K_0^{\mathrm{sc}}(\bigoplus_{s\in F_0} M_n(\mathbb{C})) \to K_0^{\mathrm{sc}}(\mathbf{G})$ and $\chi : K_0^{\mathrm{sc}}(\mathbf{G}) \to K_0^{\mathrm{sc}}(\mathbf{A})$ are homomorphisms so that $\chi \circ \omega = K_0(\phi_0)$. Then there exists $s_0 \in \mathbb{N}$ and a homomorphism $h : K_0^{\mathrm{sc}}(\mathbf{G}) \to K_0^{\mathrm{sc}}(\bigoplus_{n\in F_{s_0}} M_n(\mathbb{C}))$ so that $h \circ \omega = K_0(\phi_{s_0}^{-1} \circ \phi_0)$ and $K_0(\phi_{s_0}) \circ h = \chi$.*

Proof. Set $\mathbf{A}_s = \bigoplus_{n\in F_s} M_n(\mathbb{C})$, and let $\psi_s = \phi_{s+1}^{-1} \circ \phi_s$. By Corollary 4.5, $(\mathbf{A}, (\phi_s)_{s\in\mathbb{N}})$ is an inductive limit of $(\mathbf{A}_s, \phi_s)_{s\in\mathbb{N}}$. We now apply [20, Lemma 7.3.3]. □

Lemma 10.2. *Suppose $(n_s, \nu_s, \phi_s)_{s\in\mathbb{N}}$ is a computable certificate of dimensionality of $\mathcal{G}^\#$, and assume $(m_s, \mu_s, \psi_s)_{s\in\mathbb{N}}$ is a computable certificate of dimensionality of*

$\mathcal{G}^\#$. Suppose $s, r, m_s, m_r \in \mathbb{N}$ are such that $s > r$, $m_s > n_r$, and let $\gamma : \mathbb{Z}^{n_r} \to \mathbb{Z}^{m_s}$ be an ordered group homomorphism so that $\nu_r = \mu_s \gamma$. Then there exists $t > s$ so that $n_t > m_s$ and so that there exists an ordered group homomorphism $\delta : \mathbb{Z}^{m_s} \to \mathbb{Z}_{n_t}$ so that $\delta\gamma = \phi_{r,t}$ and $\mu_s = \nu_t \delta$.

Proof. Compute $n_{t'} > m_s$ and a positive group homomorphism $\delta' : \mathbb{Z}^{m_s} \to \mathbb{Z}^{n_{t'}}$ so that $\nu_{t'}\delta' = \mu_s$.

Fix a g in the standard basis for $\mathbb{Z}^{n_r}$. Then,

$$\nu_r(g) = \nu_r(g) = \mu_s\gamma(g) = \nu_{t'}\delta'\gamma(g).$$

Thus, $\nu_{t'}\phi_{r,t'}(g) = \nu_{t'}\delta'\gamma(g)$. Thus, $\phi_{r,t'}(g) - \delta'\gamma(g) \in \ker(\nu_{t'})$. Therefore, there exists $k_g \in \mathbb{N}$ so that $\phi_{r,t'}(g) - \delta'\gamma(g) \in \ker(\phi_{t',t'+k_g})$. Therefore, $\phi_{r,t'+k_g}(g) = \phi_{t',t'+k_g}\delta'\gamma(g)$. Set $t = t' + \max_g k_g$ and $\delta = \phi_{t',t}\delta'$. Thus, $\phi_{r,t} = \delta\gamma$, and $\nu_t\delta = \nu_t\phi_{t',t}\delta' = \nu_{t'}\delta' = \mu_s$. □

Proof of Main Theorem 4. We first show that (1) and (2) are equivalent.

To begin, suppose $\mathbf{A}^\#$ and $\mathbf{B}^\#$ are c.e. presentations of AF algebras, and let ψ be a computable unital $*$-isomorphism from $\mathbf{A}^\#$ to $\mathbf{B}^\#$. Thus, as discussed in Section 3.2.2 $K_0(\psi)$ is a computable isomorphism from $K_0^{\mathrm{sc}}(\mathbf{A}^\#)$ to $K_0^{\mathrm{sc}}(\mathbf{B}^\#)$.

We note that if the AF certificate in Lemma 10.1 is a computable AF certificate of a c.e. presentation $\mathbf{A}^\#$, then s_0, h can be computed from G, ω, χ as h is determined by a finite amount of data (see the discussion in Section 2.3.4).

Suppose α is a computable isomorphism from $K_0^{\mathrm{sc}}(\mathbf{A}^\#)$ to $K_0^{\mathrm{sc}}(\mathbf{B}^\#)$. By Main Theorem 1, $\mathbf{A}^\#$ has a computable AF certificate $(F_s, \phi_s)_{s\in\mathbb{N}}$, and $\mathbf{B}^\#$ has a computable AF certificate $(G_s, \psi_s)_{s\in\mathbb{N}}$. Set $\mathbf{A}_s = \bigoplus_{n\in F_s} M_n(\mathbb{C})$, and set $\mathbf{B}_s = \bigoplus_{n\in G_s} M_n(\mathbb{C})$. Without loss of generality, we assume $\mathbf{A}_0 = \mathbf{B}_0 = \mathbb{C}$. Set $f_s = \phi_{s+1}^{-1} \circ \phi_s$, and set $g_s = \psi_{s+1}^{-1} \circ \psi_s$. By Corollary 4.5, $(\mathbf{A}^\#, (\phi_s)_{s\in\mathbb{N}})$ is an inductive limit of $(\mathbf{A}_s, f_s)_{s\in\mathbb{N}}$ and $(\mathbf{B}^\#, (\psi_s)_{s\in\mathbb{N}})$ is an inductive limit of $(\mathbf{B}_s, g_s)_{s\in\mathbb{N}}$.

By iterating Lemma 10.1, we compute increasing sequences $(n_s)_{s\in\mathbb{N}}$ and $(m_s)_{s\in\mathbb{N}}$ of natural numbers along with sequences $(\alpha_s)_{s\in\mathbb{N}}$ and $(\beta_s)_{s\in\mathbb{N}}$ that satisfy the following.

(1) $n_0 = 0$.
(2) α_s is a computable homomorphism from $K_0^{\mathrm{sc}}(\mathbf{A}^\#_{n_s})$ to $K_0^{\mathrm{sc}}(\mathbf{B}^\#_{m_s})$ uniformly in s.
(3) β_s is a computable homomorphism from $K_0^{\mathrm{sc}}(\mathbf{B}^\#_{m_s})$ to $K_0^{\mathrm{sc}}(\mathbf{A}^\#_{n_{s+1}})$.
(4) $\beta_s \circ \alpha_s = f_{n_s,n_{s+1}}$ and $\alpha_{s+1} \circ \beta_s = g_{m_s,m_{s+1}}$.

It is now possible to compute from s, a unital $*$-homomorphism $\sigma'_s : \mathbf{A}_{n_s} \to \mathbf{B}_{m_s}$ so that $K_0(\sigma'_s) = \alpha_s$ (see the material in Section 2.3.4), and a unital $*$-homomorphism $\tau'_s : \mathbf{B}_{m_s} \to \mathbf{A}_{n_{s+1}}$ so that $K_0(\tau'_s) = \beta_s$.

It follows from the discussion of multiplicities in Section 2.3.3, that there exist, and we can compute, sequences $(u_s)_{s\in\mathbb{N}}$ and $(v_s)_{s\in\mathbb{N}}$ that satisfy the following.

(1) v_s is a unitary of $\mathbf{A}_{n_{s+1}}$ and u_s is a unitary of $\mathbf{B}_{m_s}$.
(2) $f_{n_s,n_{s+1}} = \mathrm{Ad}_{v_s} \circ \tau'_s \circ \mathrm{Ad}_{u_s} \circ \sigma'_s$ and $g_{m_s,m_{s+1}} = \mathrm{Ad}_{u_{s+1}} \circ \sigma'_{s+1} \circ \mathrm{Ad}_{v_s} \circ \tau'_s$.

Set $\sigma_s = \mathrm{Ad}_{u_s} \circ \sigma'_s$ and set $\tau_s = \mathrm{Ad}_{v_s} \circ \tau'_s$. Thus, for each $s \in \mathbb{N}$, $K_0(\sigma_s) = \alpha_s$ and $K_0(\tau_s) = \beta_s$. Furthermore, $f_{n_s,n_{s+1}} = \tau_s \circ \sigma_s$ and $g_{m_s,m_{s+1}} = \sigma_{s+1} \circ \tau_s$.

It now follows that $(\mathbf{A}^\#, (\phi_{n_{s+1}} \circ \tau_s)_{s\in\mathbb{N}})$ is an inductive upper limit of $(\mathbf{B}_{m_s}, g_{m_s,m_{s+1}})_{s\in\mathbb{N}}$ in the category of c.e. presentations of unital C*-algebras. It follows from Lemma

4.4 that $(\mathbf{B}^\#, (\psi_{m_s})_{s\in\mathbb{N}})$ is an inductive limit of $(\mathbf{B}_{m_s}, g_{m_s,m_{s+1}})_{s\in\mathbb{N}}$ in this category. So, let λ be the reduction of $(\mathbf{B}^\#, (\psi_{m_s})_{s\in\mathbb{N}})$ to $(\mathbf{A}^\#, (\phi_{n_{s+1}} \circ \tau_s)_{s\in\mathbb{N}})$. Hence, $\lambda \circ \psi_{m_s} = \phi_{n_{s+1}} \circ \tau_s$.

At the same time, $(\mathbf{B}^\#, (\psi_{m_s} \circ \sigma_s)_{s\in\mathbb{N}})$ is an upper inductive limit of $(\mathbf{A}_s, f_{n_s,n_{s+1}})_{s\in\mathbb{N}}$, and $(\mathbf{A}^\#, (\phi_{n_s})_{s\in\mathbb{N}})$ is an inductive limit of this inductive sequence. So, let λ' be the reduction from $(\mathbf{A}^\#, (\phi_{n_s})_{s\in\mathbb{N}})$ to $(\mathbf{B}^\#, (\psi_{m_s} \circ \sigma_s)_{s\in\mathbb{N}})$. Hence, $\lambda' \circ \phi_{n_s} = \psi_{m_s} \circ \sigma_s$. Therefore,

$$\lambda\lambda'\phi_{n_s} = \lambda\psi_{m_s}\sigma_s = \phi_{n_{s+1}}\tau_s\sigma_s = \phi_{n_{s+1}} f_{n_s,n_{s+1}} = \phi_{n_s}.$$

Thus, $\lambda\lambda'$ is the identity on $\bigcup_{s\in\mathbb{N}} \operatorname{ran}(\phi_{n_s})$; hence it is the identity on $\mathbf{A}$ as well. Similarly, $\lambda'\lambda$ is the identity on $\mathbf{B}$, and λ is an isomorphism. By the continuity of K_0 (see, e.g. [20, Section 6.3]), $\lambda = K_0(\alpha)$.

Now we show that (1) and (3) are equivalent. For one direction, it suffices to show that if $\mathcal{C}_0 = (F_s, \phi_s)_{s\in\mathbb{N}}$ and $\mathcal{C}_1 = (G_s, \psi_s)_{sin\mathbb{N}}$ are computable AF certificates for $\mathbf{A}^\#$, then every Bratteli diagram of $\mathcal{C}_0$ is computably equivalent to every Bratteli diagram of $\mathcal{C}_1$. Set $\mathbf{A}_s = \bigoplus_{n\in F_s} M_n(\mathbb{C})$, and set $\mathbf{B}_s = \bigoplus_{n\in F_s} M_n(\mathbb{C})$. In addition, set $\sigma_s = \phi_{s+1}^{-1} \circ \phi_s$, and set $\tau_s = \psi_{s+1}^{-1} \circ \psi_s$. It suffices to show that the standard Bratteli diagram of $(\mathbf{A}_s, \sigma_s)_{s\in\mathbb{N}}$ is computably equivalent to the standard Bratteli diagram of $(\mathbf{B}_s, \tau_s)$; denote these Bratteli diagrams by $\mathcal{D}_0$ and $\mathcal{D}_1$ respectively.

Let $(\mathbb{Z}^{m_s}, u_s)$ denote the co-domain of ζ_{F_s}, and let $(\mathbb{Z}^{n_s}, v_s)$ denote the co-domain of ζ_{G_s}. Set $\gamma_s = \eta_{F_s,F_{s+1}}(K_0(\sigma_s))$, and set $\tau_s = \eta_{G_s,G_{s+1}}(\tau_s))$. Let $\mu_s = K_0(\phi_s) \circ \zeta_{F_s}^{-1}$, and let $\nu_s = K_0(\psi_s) \circ \zeta_{G_s}^{-1}$. Hence, $(m_s, u_s, \mu_s, \gamma_s)_{s\in\mathbb{N}}$ is a computable unital certificate of dimensionality of $K_0^{\mathrm{sc}}(\mathbf{A}^\#)$ as is $(n_s, v_s, \nu_s, \kappa_s)_{S\in\ N}$.

We now construct computable increasing sequences $(k_s)_{s\in\mathbb{N}}$ and $(\ell_s)_{s\in\mathbb{N}}$ of natural numbers and an inductive sequence $((\mathbb{Z}^{x_s}, w_s), \rho_s)_{s\in\mathbb{N}}$ that satisfy the following.

(1) $x_{2s} = m_{k_s}$ and $x_{2s+1} = n_{\ell_s}$.
(2) $w_{2s} = u_{k_s}$ and $w_{2s+1} = v_{\ell_s}$.
(3) $\sigma_{m_{k_s},m_{k_{s+1}}} = \rho_{2s,2(s+1)}$.
(4) $\tau_{n_{\ell_s},n_{\ell_{s+1}}} = \rho_{2s+1,2s+3}$.

Without loss of generality, we assume $m_0 = n_0 = u_0 = v_0 = 1$. Thus, $u_s = \sigma_{0,s}(1)$ and $v_s = \tau_{0,s}(1)$. Set $k_0 = 0$. Let $\ell_0 = 1$ and set $x_0 = 1$. Let $x_1 = n_1$. Let $\delta_0(n) = nv_1$. We now define the remaining terms of the desired sequences by iterating Lemma 10.2. Since σ_s and ψ_s are unit-preserving, the constructed maps $\rho_0, \rho_1, \ldots$ are as well.

It follows fairly straightforwardly that $\mathcal{D}_0$ is a Bratteli diagram of $((\mathbb{Z}^{m_s}, u_s), \gamma_s)_{s\in\mathbb{N}}$ and that $\mathcal{D}_1$ is a Bratteli diagram of $((\mathbb{Z}^{n_s}, v_s), \kappa_s)_{s\in\mathbb{N}}$. The sequences we have constructed witness that the standard Bratteli diagrams of $((\mathbb{Z}^{m_s}, u_s), \gamma_s)_{s\in\mathbb{N}}$, $((\mathbb{Z}^{x_s}, w_s), \rho_s)_{s\in\mathbb{N}}$, and $((\mathbb{Z}^{n_s}, v_s), \kappa_s)_{s\in\mathbb{N}}$ are computably equivalent. Hence, $\mathcal{D}_0$ and $\mathcal{D}_1$ are computably equivalent.

Conversely, suppose $\mathcal{D}_{\mathbf{A}}$ is a labeled Bratteli diagram for a computable AF certificate $(F_s, \phi_s)_{s\in\mathbb{N}}$ of $\mathbf{A}^\#$, and assume $\mathcal{D}_{\mathbf{B}}$ is a labeled Bratteli diagram for a computable AF certificate $(G_s, \psi_s)_{s\in\mathbb{N}}$ of $\mathbf{B}^\#$. Furthermore, assume $\mathcal{D}_{\mathbf{A}}$ and $\mathcal{D}_{\mathbf{B}}$ are computably equivalent.

By Definition 5.18, there is a finite sequence $\mathcal{D}_0 = \mathcal{D}_{\mathbf{A}}, \ldots, \mathcal{D}_m = \mathcal{D}_{\mathbf{B}}$ of labeled Bratteli diagrams so that for each $j < m$, one of $\mathcal{D}_j$ and $\mathcal{D}_{j+1}$ is a computable reduction of the other or they are isomorphic. Starting with $\mathcal{D}_0$, each of these reductions yields another computable AF certificate for $\mathbf{A}^\#$, and isomorphisms yield the same AF certificate; let $\mathcal{C}_0, \ldots, \mathcal{C}_m$ be the resulting sequence of computable AF certificates for $\mathbf{A}^\#$ with $\mathcal{C}_0 = \mathcal{C}_A$. Since $\mathcal{D}_m = \mathcal{D}_\mathbf{B}$, it follows that $\mathcal{C}_m$ has the form $(G_s, \gamma_s)_{s\in\mathbb{N}}$. Set $C_s = \bigoplus_{n\in G_s} M_n(\mathbb{C})$. Since $\mathcal{D}_m = \mathcal{D}_\mathbf{B}$, it then follows that $K_0(\gamma_{s+1}^{-1} \circ \gamma_s) = K_0(\psi_{s+1}^{-1} \circ \psi_s)$. Hence, it follows that $K_0^{\mathrm{sc}}(\mathbf{A}^\#)$ is computably isomorphic to $K_0^{\mathrm{sc}}(\mathbf{B}^\#)$. Thus, by what has just been shown, $\mathbf{A}^\#$ is computably $*$-isomorphic to $\mathbf{B}^\#$. □

We note that all parts of the above proof are uniform which we can exploit as follows. Recall that a structure is *computably categorical* if all of its computable presentations are computably isomorphic. The following stands in contrast to our result in [9] that all UHF algebras are computably categorical.

Corollary 10.3.
(1) *The AF algebra $C(\omega + 1)$ is not computably categorical.*
(2) *There is a unital dimension group that has two c.e. presentations that are not computably isomorphic.*
(3) *There exist two computably presentable Bratteli diagrams that are equivalent but not computably equivalent.*

Proof sketch. Only the first item requires proof. It is well-known that the linear order $(\mathbb{N}, <)$ is not computably categorical. By elaborating on the proof of this result, we obtain that there are computably compact presentations of $\omega + 1$ that are not computably homeomorphic. It then follows from the results in [19] that $C(\omega+1)$ is not computably categorical. As $\omega+1$ is totally disconnected, $C(\omega+1)$ is an AF algebra. □

11. Proof of Main Theorem 5

11.1. **A preliminary lemma.** We first prove the following which is made possible by Main Theorem 1.

Lemma 11.1. *Suppose $\mathbf{A}^\#$ and $\mathbf{B}^\#$ are c.e. presentations of unital AF algebras, and let ρ be a computable homomorphism from $K_0^{\mathrm{sc}}(\mathbf{A}^\#)$ to $K_0^{\mathrm{sc}}(\mathbf{B}^\#)$. Then there is a computable unital $*$-homomorphism τ from $\mathbf{A}^\#$ to $\mathbf{B}^\#$ so that $K_0(\tau) = \rho$.*

Proof. Let $(F_s, \phi_s)_{s\in\mathbb{N}}$ be a computable AF certificate for $\mathbf{A}^\#$, and let $(G_s, \sigma_s)_{s\in\mathbb{N}}$ be a computable AF certificate for $\mathbf{B}^\#$. Set $\mathbf{A}_s = \bigoplus_{n\in F_s} M_n(\mathbb{C})$, and set $\mathbf{B}_s = \bigoplus_{n\in G_s} M_n(\mathbb{C})$. Let $\psi_s = \phi_{s+1}^{-1} \circ \phi_s$.

Fix $s \in \mathbb{N}$. By Lemma 4.2, there exists $s_0 \in \mathbb{N}$ so that $\mathrm{ran}(\rho \circ K_0(\phi_s)) \subseteq \mathrm{ran}(K_0(\sigma_{s_0}))$. It is possible to compute s_0 from s. Set $\mathbf{C} = \mathrm{ran}(\sigma_{s_0})$, and let $\kappa = \rho \circ K_0(\phi_s)$. Thus, κ is a homomorphism from $K_0(\mathbf{A}_s)$ to $K_0(\mathbf{C})$. Since σ_{s_0} is injective, by Lemma 7.1, it is possible to compute a matricial generating system for $\mathbf{C}$. It then follows from the material in Section 2.3.3 that we can now compute a unital $*$-homomorphism τ_s so that $K_0(\tau_s) = \kappa$.

It now follows that $(\mathbf{B}^\#, (\tau_s)_{s\in\mathbb{N}})$ is an inductive upper limit of $(\mathbf{A}_s, \psi_s)_{s\in\mathbb{N}}$. Hence, we can compute the reduction τ of $(\mathbf{A}^\#, (\phi_s)_{s\in\mathbb{N}})$ to $(\mathbf{B}^\#, (\tau_s)_{s\in\mathbb{N}})$. Thus, $\tau \circ \phi_s = \tau_s$. Since $\rho \circ K_0(\phi_s) = K_0(\tau_s)$, it now follows that $K_0(\tau) \circ K_0(\phi_s) = \rho \circ K_0(\phi_s)$, and so $K_0(\tau) = \rho$. □

11.2. **Computable functors and equivalences.** In order to prove our final main theorem, we lay out what we mean by a computable equivalence of categories. Suppose $\mathbf{Cat}_0$ and $\mathbf{Cat}_1$ are categories for which we have fixed computable indexings.

Definition 11.2. Let F be a multi-valued functor from $\mathbf{Cat}_0$ to $\mathbf{Cat}_1$. We say that F is *computable* if there exist computable functions $\alpha : \mathbb{N} \to \mathbb{N}$ and $\alpha' : \mathbb{N}^3 \to \mathbb{N}$ that satisfy the following:

(1) If $e \in \mathbb{N}$ indexes an object A of $\mathbf{Cat}_0$, then $\alpha(e)$ indexes a value of $F(A)$.
(2) If (e_0, e_1, e) indexes a morphism $f : A \to B$ of $\mathbf{Cat}_0$, then $(\alpha(e_0), \alpha(e_1), \alpha'(e_0, e_1, e))$ indexes a value of $F(f)$.

We say that (α, α') *witnesses* F is computable.

We note that every computable multi-valued functor is effectively essentially single-valued in the sense that from indexes e, e' of an object A it is possible to compute an index of a computable isomorphism between the objects indexed by $\alpha(e)$ and $\alpha(e')$. Hence, we occasionally treat them as single-valued.

Definition 11.3. A *computable equivalence of* $\mathbf{Cat}_0$ *and* $\mathbf{Cat}_1$ is a quadruple $(F, G, \mathcal{E}_0, \mathcal{E}_1)$ for which there exist α, α', β, β', γ_0, γ_1 that satisfy the following:

(1) (α, α') witnesses that F is a computable functor from $\mathbf{Cat}_0$ to $\mathbf{Cat}_1$.
(2) (β, β') witnesses that G is a a computable functor from $\mathbf{Cat}_1$ to $\mathbf{Cat}_0$.
(3) $\mathcal{E}_0$ is a natural multi-valued isomorphism of GF and $\mathrm{Id}_{\mathbf{Cat}_0}$, and for each $e \in \mathbb{N}$, if e indexes an object A of $\mathbf{Cat}_0$, then $(e, \beta(\alpha(e)), \gamma_0(e))$ indexes a value of $\mathcal{E}_0(A)$.
(4) $\mathcal{E}_1$ is a natural multi-valued isomorphism of FG and $\mathrm{Id}_{\mathbf{Cat}_1}$, and for each $e \in \mathbb{N}$, if e indexes an object B of $\mathbf{Cat}_1$, then $(e, \alpha(\beta(e)), \gamma_1(e))$ indexes a value of $\mathcal{E}_1(B)$.

If these conditions are satisfied, then we say that $(\alpha, \alpha', \beta, \beta', \gamma_0, \gamma_1)$ *witnesses* that $(F, G, \mathcal{E}_0, \mathcal{E}_1)$ is a computable equivalence of categories.

Again, it follows that these computable natural isomorphisms are effectively essentially single-valued.

11.3. **The categories.** We now set forth formal definitions of the categories considered in Main Theorem 5 and their indexings.

11.3.1. *The category of c.e. presentations of unital AF algebras.* The objects in this category are c.e. presentations of unital AF algebras; we have alredy defined the indexing of these objects.

Suppose $\mathbf{A}_0^\#$ and $\mathbf{A}_1^\#$ are c.e. presentations of unital AF algebras. A *morphism* from $\mathbf{A}_0^\#$ to $\mathbf{A}_1^\#$ in this category is an approximately unitarily equivalence class of a computable unital $*$-homomorphism from $\mathbf{A}_0^\#$ to $\mathbf{A}_1^\#$. We say that (e_0, e_1, e) indexes a morphism Ψ from $\mathbf{A}_0^\#$ to $\mathbf{A}_1^\#$ if e_j indexes $\mathbf{A}_j^\#$ and e is an $(\mathbf{A}_0^\#, \mathbf{A}_1^\#)$ index of an element of Ψ.

11.3.2. *The category of c.e. presentations of unital dimension groups.* The objects in this category are c.e. presentations of unital dimension groups. The morphisms are the computable homomorphisms. We have already described the indexing of the objects in this category.

11.4. **Proof of computable equivalence.** Throughout the rest of this section, $\mathbf{Cat}_0$ denotes the category of c.e. presentations of unital dimension groups, and $\mathbf{Cat}_1$ denotes the category of c.e. presentations of unital AF algebras.

We set $G = K_0^{\mathrm{sc}}$. There exists (β, β') that witnesses G is a computable functor.

We construct F, $\mathcal{E}_0$, $\mathcal{E}_1$ so that $(F, G, \mathcal{E}_0, \mathcal{E}_1)$ is a computable equivalence of $\mathbf{Cat}_0$ and $\mathbf{Cat}_1$. We simultaneously construct α, α', γ_0, γ_1 so that $(\alpha, \alpha', \beta, \beta', \gamma_0, \gamma_1)$ witnesses $(F, G, \mathcal{E}_0, \mathcal{E}_1)$ is a computable equivalence of categories.

11.4.1. *Construction of F, α, α'.* Suppose $A = (\mathcal{G}, u)^\#$ is an object of $\mathbf{Cat}_0$, and let x be an index of A. By the uniformity of Lemma 8.2, it is possible to compute an index $\alpha_0(x)$ of a computable unital certificate of dimensionality $(m_s, u_s, \epsilon_s, \phi_s)_{s\in\mathbb{N}}$ of A.

For each $s \in \mathbb{N}$, let H_s be the multiset of positive integers so that the number of times an integer appears in H_s is the number of times it appears in u_s. Set $\mathbf{A}_s = \bigoplus_{n\in H_s} M_n(\mathbb{C})$. It is now possible to compute for each s a unital $*$-homomorphism $\sigma_s : \mathbf{A}_s \to \mathbf{A}_{s+1}$ so that $\eta_{H_s,H_{s+1}}(K_0(\sigma_s)) = \phi_s$. From x, it is possible to compute an index $\alpha_1(x)$ of $(\mathbf{A}_s, \sigma_s)_{s\in\mathbb{N}}$. By the uniformity of Corollary 5.14, it is now possible to compute an index $\alpha_2(x)$ of a computable inductive limit $(\mathbf{A}^\dagger, (\psi_s)_{s\in\mathbb{N}})$ of $(\mathbf{A}_s, \sigma_s)_{s\in\mathbb{N}}$. From $\alpha_2(x)$, we can compute an index $\alpha(x)$ of $\mathbf{A}^\dagger$. We declare $\mathbf{A}^\dagger$ to be a value of $F(A)$.

Now, for each $j \in \{0, 1\}$, suppose $A_j = (\mathcal{G}_j, u_j)^\#$ is an object of $\mathbf{Cat}_0$, and let x_j be an index of A_j. Assume (x_0, x_1, y) is an index of a morphism δ from A_0 to A_1. Thus, x_j is an index of A_j, and y is a (A_0, A_1) index of δ. Assume that for $j \in \{0, 1\}$, $\alpha_0(x_j)$ indexes $(m_{j,s}, u_{j,s}, \epsilon_{j,s}, \phi_{j,s})_{s\in\mathbb{N}}$. Thus, $((\mathcal{G}_j, u_j)^\#, (\epsilon_{j,s})_{s\in\mathbb{N}})$ is a computable inductive limit of $((\mathbb{Z}^{m_{j,s}}, u_{j,s}), \phi_{j,s})_{s\in\mathbb{N}}$. Assume $\alpha_1(x_j)$ indexes $(\mathbf{A}_{j,s}, \sigma_{j,s})_{s\in\ N}$, and assume $\alpha_2(x_j)$ indexes $(\mathbf{A}_j^\dagger, (\psi_{j,s})_{s\in\mathbb{N}})$. Thus, $\mathbf{A}_j^\dagger$ is indexed by $\alpha(x_j)$ for $j \in \{0, 1\}$.

For the moment, fix $j \in \{0, 1\}$. We observe that $((\mathcal{G}_j, u_j)^\#, (\epsilon_{j,s} \circ \zeta_s)_{s\in\mathbb{N}})$ is a computable inductive limit of $(K_0^{\rm sc}(\mathbf{A}_{j,s}), K_0(\sigma_{j,s}))_{s\in\mathbb{N}}$ as is $(K_0^{\rm sc}(\mathbf{A}_j^\dagger), (K_0(\psi_{j,s}))_{s\in\mathbb{N}})$. Hence, it is now possible to compute an index $\gamma_0(x_j)$ of the reduction λ_j from $(K_0^{\rm sc}(\mathbf{A}_j^\dagger), (K_0(\psi_{j,s}))_{s\in\mathbb{N}}$ to $((\mathcal{G}_j, u_j)^\#, (\epsilon_{j,s} \circ \zeta_s)_{s\in\mathbb{N}})$ which is an isomorphism from $K_0^{\rm sc}(\mathbf{A}^\dagger)$ to $(\mathcal{G}_j, u_j)^\#$. We can now compute a $(K_0^{\rm sc}(\mathbf{A}_0^\dagger), K_0^{\rm sc}(\mathbf{A}_1^\dagger))$ index $\alpha_0'(x_0, x_1, y)$ of $\rho := \lambda^{-2}\delta\lambda_1$. By Lemma 11.1, it is now possible to compute an $(\mathbf{A}_0^\dagger, \mathbf{A}_1^\dagger)$ index $\alpha'(x_0, x_1, y)$ of a unital $*$-homomorphism τ so that $K_0(\tau) = \rho$. We declare τ to be a value of $F(\delta)$.

11.4.2. *Construction of $\mathcal{E}_j$ and γ_j.* In the course of the above process, we constructed γ_0 and $\mathcal{E}_0$ so that $\mathcal{E}_0$ is a natural isomorphism from GF to $\mathrm{Id}_{\mathbf{Cat}_0}$; namely declare λ_j to be a value of $\mathcal{E}_0(A_j)$.

We now construct γ_1 and $\mathcal{E}_1$. Assume the notation of Section 11.4.1. To begin, suppose z is a unital index of a c.e. presentation $\mathbf{A}^\#$ of a unital AF algebra. Let $B = \mathbf{A}^\#$. Set $x = \beta(z)$, and set $A = (\mathcal{G}, u)^\# = G(B) = K_0^{\rm sc}(\mathbf{A}^\#)$. We now consider $\mathbf{A}^\dagger = F(A)$ which is indexed by $\alpha(x) = \alpha(\beta(z))$. We make two observations. First, by construction $(A, (\epsilon_s)_{s\in\mathbb{N}})$ is a computable inductive limit of $((\mathbb{Z}^{m_s}, u_s), \phi_s)_{s\in\mathbb{N}}$. Second, also by construction, $(K_0^{\rm sc}(\mathbf{A}^\dagger), (K_0(\psi_s) \circ \zeta_{F_s}^{-1})_{s\in\mathbb{N}})$ is also a computable inductive limit of this inductive sequence. Thus, we can compute the reduction λ of $(K_0^{\rm sc}(\mathbf{A}^\dagger), (K_0(\psi_s) \circ \zeta_{F_s}^{-1})_{s\in\mathbb{N}})$ to $(A, (\epsilon_s)_{s\in\mathbb{N}})$ which is also an isomorphism from $K_0^{\rm sc}(\mathbf{A}^\dagger)$ to $K_0^{\rm sc}(\mathbf{A}^\#)$. By the uniformity of Main Theorem 4, it is now possible to compute an index $\gamma_1(z)$ of a unital $*$-isomorphism τ from $\mathbf{A}^\dagger$ to $\mathbf{A}^\#$, and we declare τ to be a value of $\mathcal{E}_1(B)$.

As the morphisms of $\mathbf{Cat}_1$ are approximate unitary equivalence classes, it follows that $\mathcal{E}_1$ is a natural isomorphism of FG and $\mathrm{Id}_{\mathbf{Cat}_1}$ (see, for example, [20, Exercise 7.6]).

12. Applications to index set and isomorphism problems

We now use our main theorems to resolve some questions concerning index set and isomorphism problems for AF and UHF algebras. We first discuss the meaning of these kinds of problems and how they relate to classification problems in mathematics.

Suppose $\mathcal{C}$ is some class of mathematical structures. The *index set problem* for $\mathcal{C}$ is the complexity of determining if a natural number indexes a computable (or c.e.) presentation of a structure in $\mathcal{C}$. This models how hard it is to discern if a given structure belongs to $\mathcal{C}$. Here, complexity of a set is measured by its position in the arithmetical hierarchy, which measures the number of alternating quantifiers required to define the set. For example, a Π^0_2 set is one that can be defined by an expression of the form $\forall y \exists z R(x, y, z)$ where R is a computable predicate. Π^0_2 complete sets are Π^0_2 sets that are 'as hard as possible'. We refer the reader to standard sources such as [5] for a more precise definition (which can also be gleaned from the proofs below).

We let $(\phi_e)_{e\in\mathbb{N}}$ be an effective enumeration of all computable partial functions from $\mathbb{N}$ into $\mathbb{N}$. The set **Tot** is defined to be $\{e \in \mathbb{N} : \mathrm{dom}(\phi_e) = \mathbb{N}\}$. It is well-known that **Tot** is Π^0_2 complete. Let $\phi_{e,s}(x) = \phi_e(x)$ if the computation of ϕ_e on x halts in at most s steps; otherwise $\phi_{e,s}(x)$ is undefined.

We begin with the index problem for all C*-algebras. The proof of the following is very similar to the proof of [3, Theorem 2.6]; we leave the details to the reader.

Theorem 12.1. *The set of all indexes for c.e. presentation of* C**-algebras is* Π^0_2 *complete as is the set of all indexes of computable presentations of* C**-algebras.*

Theorem 12.2. *The index set problem for AF algebras is* Π^0_2 *complete.*

Proof. We say that $(a^s_{i,j})_{s,i,j}$ is a *type-*$(n_1, \ldots, n_m)$ *system* if for each $s \in \{1, \ldots, m\}$, $(a^s_{i,j})_{i,j}$ is an $n_s \times n_s$ array.

Suppose e is an index of a c.e. presentation $\mathbf{A}^\#$ of a C*-algebra. Let m, n_1, …, n_m be positive integers. When $m, n_1, \ldots, n_m$ are positive integers, let $\mathcal{S}_{(k,n_1,\ldots,n_m)}[\mathbf{A}^\#]$ denote the set of all type-$(n_1, \ldots, n_m)$ arrays $(g^s_{i,j})_{s,i,j}$ of generated points of $\mathbf{A}^\#$ for which there exists a type-$(n_1, \ldots, n_m)$ matricial generating system of $\mathbf{A}$ so that $\left\|g^s_{i,j} - f^s_{i,j}\right\| < 2^{-k}$ for all s, i, j. It follows from Proposition 5.12 that $\mathcal{S}_{(k,n_1,\ldots,n_m)}[\mathbf{A}^\#]$ is c.e. uniformly in $e, k, m, n_1, \ldots, n_m$.

Now, when $M \in \mathbb{N}$, let $\mathcal{S}'_{k,M,n_1,\ldots,n_m}[\mathbf{A}^\#]$ denote the set of all finite sets F of generated points of $\mathbf{A}^\#$ so that for each $a \in F$, there is a type-$(n_1, \ldots, n_m)$ system $(\alpha^s_{i,j})_{s,i,j}$ of rational scalars so that $\sum_{s,i,j} |\alpha^s_{i,j}| < 2^{-k}$ and so that $\left\|a - \sum_{s,i,j} \alpha^s_{i,j} g^s_{i,j}\right\| < 2^{-k}$. It follows that $\mathcal{S}'_{k,M,n_1,\ldots,n_m}[\mathbf{A}^\#]$ is c.e. uniformly in $e, k, M, n_1, \ldots, n_m$.

It now follows from [20, Proposition 7.2.2] that $\mathbf{A}^\#$ is AF if and only if for every finite set F of generated points of $\mathbf{A}^\#$ and every $k \in \mathbb{N}$, there exists $M, k_1 \in \mathbb{N}$ so that $F \in \mathcal{S}'_{k_1,M,n_1,\ldots,n_m}[\mathbf{A}^\#]$ and so that $2^{-k_1}(1 + M) < 2^{-k}$. Thus, the index set problem for AF algebras is a Π^0_2 set.

For the lower bound, we create a computable reduction from **Tot** to the index set problem for AF algebras. Let f be a partial computable function. For each

$n \in \mathbb{N}$, let I_n be the union of the intervals removed in step n of the middle-thirds construction of the Cantor set. We now carry out the following procedure. Begin with $A_0 = [0,1]$. At step n, begin the computation of $f(n)$ and for each $m < n$ advance the computation of $f(m)$ by one step. If computations $m_1, \ldots, m_k$ terminate at step n, then set $A_n = A_{n-1} \setminus \bigcup_{j=1}^{k} I_{m_j}$ (if no computations terminate at step N, set $A_n = A_{n-1}$). Let $A = \bigcap_{n=1}^{\infty} A_n$. Then A is a computably compact space (see [7, Definition 2.5]), so the abelian C*-algebra $C(A)$ has a computable presentation [12]. It is fairly easy to see (on the basis of, say, [20, Proposition 7.2.2]) that $C(A)$ is an AF algebra if and only if A is totally disconnected. By construction, A is totally disconnected if and only if f is a total function. □

The *CAR algebra* is the C*-algebra obtained via the inductive limit of $(M_{2^n}(\mathbb{C}), \phi_n)_{n\in\mathbb{N}}$, where ϕ_n is the diagonal embedding of $M_{2^n}(\mathbb{C})$ into $M_{2^{n+1}}(\mathbb{C})$. A C*-algebra is *simple* if it has no non-trivial proper closed two-sided ideals.

Lemma 12.3. *From $e \in \mathbb{N}$, it is possible to compute an index of a c.e. presentation $\mathbf{A}^\#$ of an AF algebra so that the following are equivalent:*

(1) $e \in \mathbf{Tot}$.
(2) $\mathbf{A}$ *is isomorphic to the CAR algebra.*
(3) $\mathbf{A}$ *is UHF.*
(4) $\mathbf{A}$ *is simple.*

Proof. Let $e \in \mathbb{N}$. For every $s \in \mathbb{N}$, let $m_{e,s} = \max\{y \in \mathbb{N} : \forall x < y\ x \in \mathrm{dom}(\phi_{e,s})\}$. Thus, $m_{e,0} = 0$. Let:

$$\begin{array}{rcl} \mathfrak{p}_0 & = & \{0\} \\ \mathfrak{p}_{s+1} & = & \begin{cases} \{0\} & m_{e,s} < m_{e,s+1} \\ \{0,1\} & m_{e,s} = m_{e,s+1} \end{cases} \end{array}$$

We define a labeled Bratteli diagram $\mathcal{D}_e$ as follows. Set $V_{\mathcal{D}_e} = \bigcup_{s\in\mathbb{N}}\{s\} \times \mathfrak{p}_s$, and let $L_{\mathcal{D}_e}(s,t) = s$. If $(s,1) \notin V_{\mathcal{D}_e}$, then set $E_{\mathcal{D}_e}((s,0),v) = 1$ for all $v \in L^{-1}[\{s+1\}]$. Suppose $(s,1) \in V_{\mathcal{D}_e}$. Set $E_{\mathcal{D}_e}((s,0),(s+1,0)) = E_{\mathcal{D}_e}((s,1),(s+1,j) = 1$ where j is the largest number so that $(s+1,j) \in V_{\mathcal{D}_e}$. For all other pairs (v,v') of vertices we set $E_{\mathcal{D}_e}(v,v') = 0$. We set $\Lambda_{\mathcal{D}_e}(0,0) = 1$. For all other vertices v, we set $\Lambda_{\mathcal{D}_e}(v) = \sum_{v'} \Lambda_{\mathcal{D}_e}(v')$ where v' ranges over all vertices so that $E_{\mathcal{D}_e}(v',v) > 0$.

It is now straightforward to construct a computable presentation $\mathcal{D}_e^\#$ of $\mathcal{D}_e$. Furthermore, it is straightforward to construct an AF algebra $\mathbf{A}$ whose labeled Bratteli diagram is $\mathcal{D}_e$. By the uniformity of Main Theorem 3, it is now possible to compute an index of a c.e. presentation $\mathbf{A}^\#$ of $\mathbf{A}$.

Suppose ϕ_e is total. In this case, by appropriately telescoping the diagram $\mathcal{D}_e$ we obtain a diagram where every level has exactly one vertex and between each pair of adjacent levels there are exactly two edges. This Bratteli diagram corresponds to the CAR algebra (see [6, Example III.2.4]), which is UHF and, therefore, simple ([6, Corollary III.5.3]).

On the other hand, suppose ϕ_e is not total, and let s_0 be the least number so that $m_{e,s_0} = m_{e,s}$ for all $s \geq s_0$. This entails that for all $s > s_0$, $P_{\mathcal{D}_e}((s_0+1,0),(s,1)) = 0$. This implies that the associated AF algebra is not simple (see [6, Corollary III.4.3]) and therefore is not a UHF algebra (and hence is not $*$-isomorphic to the CAR algebra). □

Theorem 12.4. *The index set problem for UHF algebras is Π_2^0 complete.*

Proof. To see that the index set problem for UHF algebras is a Π^0_2-set, argue as in the proof of Theorem 12.2, using that a separable unital C*-algebra is UHF if and only if it is locally matricial (see [15, Theorem 1.13]).

For the lower bound, Lemma 12.3 gives a many-one reduction from **Tot** to the index set problem for UHF algebras. □

Theorem 12.5. *The index set problem for simple AF algebras is Π^0_2 complete.*

Proof. For the upper bound, by Theorem 12.2, expressing that an algebra is AF is Π^0_2. Suppose e is an index of a c.e. presentation $\mathbf{A}^\#$ of a C*-algebra. By Main Theorem 3, from e it is possible to compute a computable presentation $\mathcal{D}^\#$ of a labelled Bratteli diagram of $\mathbf{A}^\#$. By [6, Corollary III.4.3] $\mathbf{A}$ is simple if and only if for every $n \in \mathbb{N}$ and every $v \in L^{-1}[\{n\}]$, there is an $m > n$ such that there is a path from v to every vertex in $L^{-1}[\{m\}$. This is clearly a Π^0_2 assertion.

For the lower bound, Lemma 12.3 gives a computable reduction from **Tot** to the index set problem for simple AF algebras. □

We now turn to isomorphism problems. The isomorphism problem for a class $\mathcal{C}$ of structures is the problem of determining if two natural numbers index presentations of isomorphic structures in the class. Isomorphism problems thus correspond to sets of pairs of natural numbers, and their complexity is still measured by their position in the arithmetical hierarchy. These problems model the complexity of the classification problem for $\mathcal{C}$. We begin with the isomorphism problem for UHF algebras.

Theorem 12.6. *The isomorphism problem for UHF algebras is Π^0_2 complete.*

Proof. Suppose $\mathbf{A}^\#$ and $\mathbf{B}^\#$ are computable presentations of UHF algebras. For the upper bound, first compute c.e. presentations of their K_0 groups. Then compute certificates of dimensionality for their K_0 groups. Then lower semicompute their supernatural numbers from these certificates. Testing equality of the supernatural numbers is easily seen to be Π^0_2.

For the lower bound, we show that there is a many-one reduction from **Tot** to the isomorphism problem for UHF algebras. Let $(p_n)_{n\in\mathbb{N}}$ be the increasing enumeration of the prime numbers. For each $n \in \mathrm{dom}(\phi_e)$, let $\mathcal{E}_0(n) = \mathcal{E}_1(n) =$ the least $s \in \mathbb{N}$ so that $n \in \mathrm{dom}(\phi_{e,s})$. When $n \notin \mathrm{dom}(\phi_e)$, let $\mathcal{E}_0(n) = \infty$ and let $\mathcal{E}_1(n) = 0$. It follows that $\mathcal{E}_0$, $\mathcal{E}_1$ are lower semi-computable uniformly in e. By the results in [9], it is now possible to compute for each $j \in \{0, 1\}$ a c.e. presentation $\mathbf{A}_j^\#$ of a UHF algebra whose supernatural number is $\mathcal{E}_j$. We now note that $\mathbf{A}_0$ is $*$-isomorphic to $\mathbf{A}_1$ if and only if $\mathcal{E}_0 = \mathcal{E}_1$ (see, for example, [20, Section 7.4]). However, by construction, $\mathcal{E}_0 = \mathcal{E}_1$ if and only if ϕ_e is total. □

While the complexity of the index set problem for AF algebras is fairly low, the complexity of the isomorphism problem for this class is as high as possible, namely, it is Σ^1_1 complete. The Σ^1_1 sets are those that can be defined by expressions of the form $\exists f \forall y R(x, f, y)$ where f ranges over all functions from $\mathbb{N}$ into $\mathbb{N}$ (that is, over Baire space) and R is a computable predicate. Every isomorphism problem is at worst Σ^1_1 (search through all possible functions and check if any are isomorphisms). If an isomorphism problem is Σ^1_1 complete, this is interpretated as saying that no effective classification is possible.

Theorem 12.7. *The isomorphism problem for commutative AF algebras is Σ^1_1 complete.*

Proof. For each Boolean algebra B, let X_B denote the Stone space of B. From an index of a computable presentation $B^\#$ of a Boolean algebra, it is possible to compute an index of a computably compact presentation X_B of the Stone space of B (see [1]). From this index, it is possible to compute an index of a presentation of $C(X_B)$ (see [4], [12]). By Gelfand Duality, $C(X_{B_0})$ is $*$-isomorphic to $C(X_{B_1})$ if and only if X_{B_0} is homeomorphic to X_{B_1}. By the Stone Duality Theorem, X_{B_0} is homeomorphic to X_{B_1} if and only if B_0 is isomorphic to B_1.

Thus, we have defined a many-one reduction from the isomorphism problem for Boolean algebras to the isomorphism problem for commutative AF algebras; since the former problem is known to be Σ^1_1 complete (see [17, Theorem 4.4(d)]), so is the latter. □

Corollary 12.8. *The isomorphism problem for unital dimension groups is Σ^1_1 complete.*

Corollary 12.9. *The equivalence problem for labeled Bratteli diagrams is Σ^1_1 complete.*

Proof. Suppose e is an index of a c.e. presentation $(\mathcal{G}, u)^\#$ of a unital dimension group. By the uniformity of Lemma 8.2, it is possible to compute from e an index of a unital certificate of dimensionality $(n_s, u_s, \mu_s, \phi_s)_{s\in\mathbb{N}}$ of $(\mathcal{G}, u)^\#$. Define a Bratteli diagram $\mathcal{D}_e$ by setting:

$$\begin{aligned}
V_{\mathcal{D}_e} &= \{(s+1, j) \;:\; s \in \mathbb{N} \;\wedge\; j \in \{1, \ldots, n_s\}\} \cup \{(0,1)\}\\
L_{\mathcal{D}_e}(s, j) &= s\\
E_{\mathcal{D}_e}((s+1, j), (s+2, j')) &= \phi_s(e_j^{n_s})(j')\\
E_{\mathcal{D}_e}((0,1), (1, j)) &= u_1(j)
\end{aligned}$$

It is straightforward to compute an index of a computable presentation of $\mathcal{D}_e^\#$ from e.

Now, suppose that for each $j \in \{0, 1\}$, e_j is an index of a c.e. presentation $(\mathcal{G}_j, u_j)^\#$ of a unital dimension group. It is well-known that $(\mathcal{G}_0, u_0)$ is isomorphic to $(\mathcal{G}_1, u_1)$ if and only if $\mathcal{D}_{e_0}$ is equivalent to $\mathcal{D}_{e_1}$ (see [8]). The corollary now follows from Corollary 12.8. □

The previous two theorems show that the isomorphism problem for AF algebras is substantially more difficult than the isomorphism problem for UHF algebras, confirming the intuition that the former class is indeed more complicated than the latter. That being said, since UHF algebras are simple while commutative ones are not, a more satisfying confirmation of this aforementioned intuition would be to show that the isomorphism problem for the class of *simple* AF algebras is also Σ^1_1 complete. This raises the following question:

Question 12.10. *What is the complexity of the isomorphism problem for simple AF algebras?*

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References

1. Nikolay Bazhenov, Matthew Harrison-Trainor, and Alexander Melnikov, *Computable Stone spaces*, Ann. Pure Appl. Logic **174** (2023), no. 9, Paper No. 103304, 25.
2. Ola Bratteli, *Inductive limits of finite dimensional C^*-algebras*, Trans. Amer. Math. Soc. **171** (1972), 195–234.
3. Tyler A. Brown, Timothy H. McNicholl, and Alexander G. Melnikov, *On the complexity of classifying Lebesgue spaces*, J. Symb. Log. **85** (2020), no. 3, 1254–1288.
4. Peter Burton, Christopher J Eagle, Alec Fox, Isaac Goldbring, Matthew Harrison-Trainor, Timothy H McNicholl, Alexander Melnikov, and Teerawat Thewmorakot, *Computable Gelfand duality*, To appear in Proceedings of the American Mathematical Society; arXiv preprint arXiv:2402.16672.
5. S. Barry Cooper, *Computability theory*, Chapman & Hall/CRC, Boca Raton, FL, 2004.
6. Kenneth R. Davidson, C^**-algebras by example*, Fields Institute Monographs, vol. 6, American Mathematical Society, Providence, RI, 1996.
7. Rodney G. Downey and Alexander G. Melnikov, *Computably compact metric spaces*, Bull. Symb. Log. **29** (2023), no. 2, 170–263. MR 4616053
8. Fabien Durand and Dominique Perrin, *Dimension groups and dynamical systems—substitutions, Bratteli diagrams and Cantor systems*, Cambridge Studies in Advanced Mathematics, Cambridge University Press, Cambridge, 2022.
9. C.J. Eagle, I. Goldbring, T.H. McNicholl, and R. Miller, *Computable K-theory for C^*-algebras: UHF algebras*, To appear in Transactions of the American Mathematical Society. Preprint available at https://arxiv.org/abs/2501.08526.
10. George A. Elliott, *On the classification of inductive limits of sequences of semisimple finite-dimensional algebras*, J. Algebra **38** (1976), no. 1, 29–44.
11. Ilijas Farah, Bradd Hart, Martino Lupini, Leonel Robert, Aaron Tikuisis, Alessandro Vignati, and Wilhelm Winter, *Model theory of* C^**-algebras*, Mem. Amer. Math. Soc. **271** (2021), no. 1324, viii+127.
12. Alec Fox, *Computable presentations of* C^**-algebras*, J. Symb. Log. **89** (2024), no. 3, 1313–1338.
13. Alec Fox, Isaac Goldbring, and Bradd Hart, *Locally universal* C^**-algebras with computable presentations*, Preprint available at https://arxiv.org/pdf/2303.02301.
14. Johanna F. Franklin, Isaac. Goldbring, and Timothy H. McNicholl, *Effective metric structure theory*, To appear.
15. James G. Glimm, *On a certain class of operator algebras*, Trans. Amer. Math. Soc. **95** (1960), 318–340.
16. Isaac Goldbring, *Computably strongly self-absorbing* C^**-algebras*, arXiv preprint arXiv:2409.18834, 2024.
17. Sergey S. Goncharov and Julia F. Knight, *Computable structure and non-structure theorems*, Algebra and Logic **41** (2002), 351–373.
18. K. R. Goodearl, *Partially ordered abelian groups with interpolation*, Mathematical Surveys and Monographs, vol. 20, American Mathematical Society, Providence, RI, 1986.
19. Timothy H. McNicholl, *Evaluative presentations*, J. Logic Comput. **35** (2025), no. 5, Paper No. exaf036, 11.
20. M. Rørdam, F. Larsen, and N. Laustsen, *An introduction to K-theory for* C^**-algebras*, London Mathematical Society Student Texts, vol. 49, Cambridge University Press, Cambridge, 2000.
21. Karen R. Strung, *An introduction to* C^**-algebras and the classification program*, Advanced Courses in Mathematics. CRM Barcelona, Birkhäuser/Springer, Cham, [2021] ©2021, Edited and with a foreword by Francesc Perera.

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