Generalizations of Egyptian fractions

David A. Ross
Department of Mathematics
University of Hawai‘i

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Recall: An *Egyptian Fraction* is a sum of unitary fractions

\[
\frac{1}{m_1} + \cdots + \frac{1}{m_n}
\]

(where \( n, m_i \in \mathbb{N} \); for today \( 0 \notin \mathbb{N} \))

These have been studied for a very long time:

**Fibonacci/Leonardo of Pisa 1202**

Every rational number has a representation as an Egyptian fraction with distinct summands.

(In fact, any rational \( \alpha \) with one Egyptian fraction representation has infinitely many, eg

\[
\frac{3}{4} = \frac{1}{2} + \frac{1}{4} = \frac{1}{3} + \frac{1}{4} + \frac{1}{6} = \cdots
\]

where you can always replace \( \frac{1}{k} \) by \( \frac{1}{2k} + \frac{1}{2k+1} + \frac{1}{2k(k+1)} \))
**Kellogg 1921; Curtiss 1922**

Bounded the number of (positive) integer solutions to the Diophantine equation

\[ 1 = \frac{1}{x_1} + \cdots + \frac{1}{x_n} \]

(which is the same as counting the number of \(n\)-term representations of 1 as an Egyptian fraction.)

**Erdös 1932**

No integer is represented by a harmonic progression

\[ \frac{1}{n} + \frac{1}{n+d} + \frac{1}{n+2d} + \cdots + \frac{1}{n+kd} \]

**Erdös-Graham 1980; Croot 2003**

If we finitely-color \(\mathbb{N}\) then there is a monochrome finite set \(S\) such that

\[ 1 = \sum_{s \in S} \frac{1}{s} \]

e.tc.
Sierpinski 1956
Several results about the structure of the set of Egyptian fractions, eg:

1. The number of representations of a given number by $n$-term Egyptian fractions is finite.
2. No sequence of $n$-term Egyptian fractions is strictly increasing (Mycielski).
3. If $a \neq 0$ has a 3-term representation but no 1-term representations then $a$ has only finitely many representations (even if we allow negative terms).
4. The set of $n$-term Egyptian fractions is nowhere dense (even if we allow negative terms).
(Nathanson 2018) Extended results like these to more generalized sets of real numbers (*Weighted Real Egyptian Numbers*, below)

(R 2018) Nonstandard methods are natural tools for understanding and extending the Sierpinski results, also apply to related Diophantine equations (generalizations of Kellogg; Znám, Lagarias, . . . )

**Today:** Do Sierpinski-like results extend to general topological groups?
Fix a topological group \( \langle G, 0, + \rangle \) (not necessarily Abelian)

**Definition:** \( T \subset G \) is *locally cofinite at 0* if 
- \( 0 \notin T \)
- For every open neighborhood \( u \) of 0, \( T \setminus u \) is finite

**Note:** For such \( T \), if \( x \in ^*T \) then either \( x \in T \) or \( x \approx 0 \).

Fix \( T_1, T_2, \ldots \subset G \) locally cofinite at 0

Define \( E_n := T_1 + \cdots + T_n = \{ a_1 + a_2 + \cdots + a_n : a_i \in T_i \} \)
(n-term generalized Egyptian fractions)

**EG** \( \forall i \ T_i = \{ \frac{1}{m} : m \in \mathbb{N} \} \) (classical Egyptian fractions)

**EG** \( \forall i \ A_i \subset \mathbb{R}^+ \) finite, \( B_i \subset \mathbb{R}^+ \) discrete, \( T_i = \{ \frac{a}{b} : a \in A_i, b \in B_i \} \)
(Nathanson’s *Weighted Real Egyptian Numbers*)
Sierpinski somehow didn’t notice the following:

**Theorem 0.1.** For \( n > 0 \) the set \( E'_n := T'_1 + \cdots + T'_n \) is compact (where \( T'_i = T_i \cup \{0\} \))

In the case of \( G = \mathbb{R} \) this implies Sierpinski’s result:

**Corollary 0.1.** (Sierpinski) *The set of n-term Egyptian fractions is NWD in \( \mathbb{R} \).*

and Nathanson’s result

**Corollary 0.2.** *The set of n-term weighted Egyptian real numbers is NWD in \( \mathbb{R} \).*

*Proof of corollaries.* Otherwise \( E'_n \) would contain an interval (since it is closed), but it is countable from the definition. \( \square \)
Theorem 0.1 is trivial using Robinson’s compactness criterion:

**Proof.** Suppose

\[ x = x_1 + \cdots + x_n \]
\[ \in ^*(T'_1 + \cdots + T'_n) \]
\[ = ^*T'_1 + \cdots + ^*T'_n \]

Let

\[ y_i = \begin{cases} x_i, & x_i \text{ standard;} \\ 0, & \text{otherwise.} \end{cases} \]

Then \( ^o x = y_1 + \cdots + y_n \in E'_n \).
Note: The result about $E_n = T_1 + \cdots + T_n$ being NWD fails on arbitrary groups.

However:

**Theorem 0.2.** Suppose $T_1, \ldots, T_n$ are locally cofinite at 0 and NWD in $G$. Then $E_n = T_1 + \cdots + T_n$ is NWD in $G$.

The proof follows by induction and a lemma:

**Lemma 0.1.** Let $T, E$ be NWD subsets of $G$ with $T$ be locally cofinite at 0. Then $E + T$ is NWD in $G$.

**Outline of proof:** Let $u \subseteq G$ be nonempty open. To find: $v \subseteq u$ nonempty open missing $(E + T)$

Let $c \in u, \mu = \text{monad}(c) \subseteq *u$ with $\mu \cap *E = \emptyset$

Show: $\exists I \subset G$ finite s.t. $\mu \cap *(E + T) \subseteq *(E + I)$

(This is the meat of the argument, uses saturation and the structure of $T$)

Since $I$ is finite, $*E + I$ is *NWD, so $\exists$ open $v \subseteq \mu$ s.t. $(*E + I) \cap v = \emptyset$

Then $\exists$ open $v \subseteq *u$ s.t. $(*E + T) \cap v = \emptyset$

and apply transfer.
Other results that still hold (sometimes with slight modification):

**Theorem 0.3** Let $0 \neq a \in E_3 \setminus E_1$. Then $a$ has only finitely many representations.

**Theorem 0.4** Let $(G, +, 0, \leq)$ be a linearly ordered group, and $T_1, T_2, \ldots$ locally cofinite subsets of the positive cone of $G$. Then no sequence of elements of $E_n$ is strictly increasing.

**Theorem 0.5** Let $(G, +, 0, \leq)$ be a partially ordered abelian group, and $T_1, T_2, \ldots$ locally cofinite subsets of the positive cone of $G$. Then the number of $n$-term representations of any element of $E_n$ is finite.

*Proof of one of these:* On board, if there is time (and a board).
Some obvious questions:

In the case of $G = \mathbb{R}$ there are some interesting results this technology produces for Diophantine equations, eg

**Generalization of Znám equation** Suppose in the equation

$$\sum_I \frac{a_I}{\prod_{i \in I} x_i} = a_\emptyset$$

where the sum ranges over nonempty subsets $I$ of $\{1, \ldots, s\}$, and $a_I \in \mathbb{R}$ such that every $a_I \geq 0$ and $a_\emptyset > 0$.

The following are equivalent:

1. For every $i \leq s$ there is an $I$ with $i \in I$ and $a_I \neq 0$.
2. The equation has only finitely many solutions in $\mathbb{N}$.

**Lagarias 2013** Let $a, b, c \in \mathbb{Z} \setminus \{0\}$ with $c \geq 1$ and $gcd(b, c) = 1$. If the equation

$$c(1/x_1 + \cdots + 1/x_s) + b/x_1 x_2 \cdots x_s = a$$

has infinitely many integer solutions then $c = 1$, and either (a) $|a| = s - 1$ and $b = -(\text{sign}(a)^{s-1})$, or (b) $|a| < s - 1$ and $b$ is arbitrary

**Question 1:** Are there interesting variants of these for groups?
Question 2: How much nonstandard analysis do we actually need?

Facetious answer: None. Didn’t use any powerful tools like Loeb measures or nonstandard hulls.

But: Used saturation, transfer, nonstandard characterizations of ‘compact’ and ‘dense’ to get much shorter proofs of the results in $\mathbb{R}$. Is there some way to isolate fundamental principles to make these arguments accessible to standard number theorists?

Question 3: Is Theorem 0.4 true for partially-ordered groups?

Question 4: Is Theorem 0.5 true for non-Abelian groups?
References


[9] Ross, David, *Egyptian fractions, nonstandard extensions of $\mathbb{R}$, and some Diophantine equations without many solutions*, 2018