Generalizations of Egyptian fractions

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Recall: An *Egyptian Fraction* is a sum of unitary fractions

$$\frac{1}{m_1} + \cdots + \frac{1}{m_n}$$

(where $n, m_i \in \mathbb{N}$; for today $0 \notin \mathbb{N}$)

These have been studied for a *very* long time:

Fibonacci/Leonardo of Pisa 1202

Every rational number has a representation as an Egyptian fraction with distinct summands.

(In fact, any rational α with one Egyptian fraction representation has infinitely many, eg

$$\begin{aligned} \frac{3}{4} &= \frac{1}{2} + \frac{1}{4} \\ &= \frac{1}{3} + \frac{1}{4} + \frac{1}{6} \\ &= \cdots \end{aligned}$$

where you can always replace $\frac{1}{k}$ by $\frac{1}{2k} + \frac{1}{2k+1} + \frac{1}{2k(k+1)}$)

Kellogg 1921; Curtiss 1922

Bounded the number of (positive) integer solutions to the Diophantine equation

$$1 = \frac{1}{x_1} + \cdots + \frac{1}{x_n}$$

(which is the same as counting the number of n-term representations of 1 as an Egyptian fraction.)

Erdös 1932

No integer is represented by a harmonic progression

$$\frac{1}{n} + \frac{1}{n+d} + \frac{1}{n+2d} + \cdots + \frac{1}{n+kd}$$

Erdös-Graham 1980; Croot 2003

If we finitely-color $\mathbb{N}$ then there is a monochrome finite set S such that

$$1 = \sum_{s \in S} \frac{1}{s}$$

etc.

Sierpinski 1956

Several results about the structure of the set of Egyptian fractions, eg:

1. The number of representations of a given number by n-term Egyptian fractions is finite.
2. No sequence of n-term Egyptian fractions is strictly increasing (Mycielski)
3. If $a \neq 0$ has a 3-term representation but no 1-term representations then a has only finitely many representations (even if we allow negative terms).
4. The set of n-term Egyptian fractions is nowhere dense (even if we allow negative terms).

(Nathanson 2018) Extended results like these to more generalized sets of real numbers (*Weighted Real Egyptian Numbers*, below)

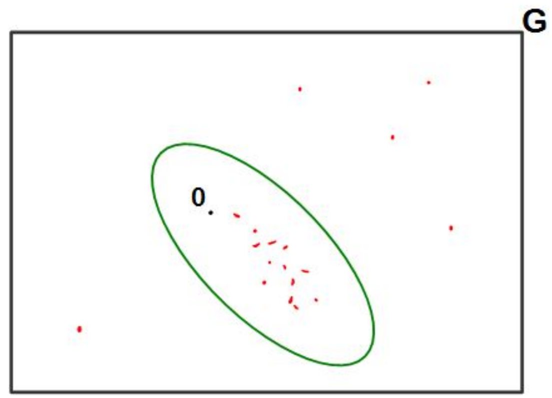
(R 2018) Nonstandard methods are natural tools for understanding and extending the Sierpinski results, also apply to related Diophantine equations (generalizations of Kellogg; Znám, Lagarias, . . .)

Today: Do Sierpinski-like results extend to general topological groups?

Fix a topological group $\langle G, 0, +\rangle$ (not necessarily Abelian)

Definition: $T \subset G$ is *locally cofinite at* 0 if

- $0 \notin T$
- For every open neighborhood u of 0, $T \setminus u$ is finite



Note: For such T, if $x \in {}^*T$ then either $x \in T$ or $x \approx 0$.

Fix $T_1, T_2, \cdots \subseteq G$ locally cofinite at 0

Define $E_n := T_1 + \cdots + T_n = \{a_1 + a_2 + \cdots + a_n \ : \ a_i \in T_i\}$
(n-term generalized Egyptian fractions)

EG $\forall i\ T_i = \{\frac{1}{m} \ : \ m \in \mathbb{N}\}$ (classical Egyptian fractions)

EG $\forall i\ A_i \subset \mathbb{R}^+$ finite, $B_i \subset \mathbb{R}^+$ discrete, $T_i = \{\frac{a}{b} \ : \ a \in A_i, b \in B_i\}$
(Nathanson's *Weighted Real Egyptian Numbers*)

Sierpinski somehow didn't notice the following:

Theorem 0.1. *For $n > 0$ the set $E'_n := T'_1 + \cdots + T'_n$ is compact (where $T'_i = T_i \cup \{0\}$)*

In the case of $G = \mathbb{R}$ this implies Sierpinski's result:

Corollary 0.1. *(Sierpinski) The set of n-term Egyptian fractions is NWD in $\mathbb{R}$.*

and Nathanson's result

Corollary 0.2. *The set of n-term weighted Egyptian real numbers is NWD in $\mathbb{R}$.*

Proof of corollaries. Otherwise E'_n would contain an interval (since it is closed), but it is countable from the definition. $\dashv$

Theorem 0.1 is trivial using Robinson's compactness criterion:

Proof. Suppose

$$\begin{aligned} x &= x_1 + \cdots + x_n \\ &\in {}^*(T'_1 + \cdots + T'_n) \\ &= {}^*T'_1 + \cdots + {}^*T'_n \end{aligned}$$

Let

$$y_i = \begin{cases} x_i, & x_i \text{ standard;} \\ 0, & \text{otherwise.} \end{cases}$$

Then $^\circ x = y_1 + \cdots + y_n \in E'_n$.

⊣

Note: The result about $E_n = T_1 + \cdots + T_n$ being NWD fails on arbitrary groups.

However:

Theorem 0.2. *Suppose $T_1, \ldots, T_n$ are locally cofinite at* 0 *and NWD in G. Then $E_n = T_1 + \cdots + T_n$ is NWD in G.*

The proof follows by induction and a lemma:

Lemma 0.1. *Let T, E be NWD subsets of G with T be locally cofinite at* 0*. Then $E + T$ is NWD in G.*

Outline of proof: Let $u \subseteq G$ be nonempty open. To find: $v \subseteq u$ nonempty open missing $(E + T)$

Let $c \in u, \mu =$monad$(c) \subseteq {}^*u$ with $\mu \cap {}^*E = \emptyset$

Show: $\exists\, I \subset G$ finite s.t. $\mu \cap {}^*(E + T) \subseteq {}^*(E + I)$

(This is the meat of the argument, uses saturation and the structure of T)

Since I is finite, ${}^*E + I$ is *NWD, so $\exists^*$open $v \subseteq \mu$ s.t. $({}^*E + I) \cap v = \emptyset$

Then $\exists^*$open $v \subseteq {}^*u$ s.t. $({}^*E + T) \cap v = \emptyset$

and apply transfer.

$\dashv$

Other results that still hold (sometimes with slight modification):

Theorem 0.3 Let $0 \neq a \in E_3 \setminus E_1$. Then a has only finitely many representations

Theorem 0.4 Let $\langle G, +, 0, \leq \rangle$ be a linearly ordered group, and $T_1, T_2, \ldots$ locally cofinite subsets of the positive cone of G. Then no sequence of elements of E_n is strictly increasing.

Theorem 0.5 Let $\langle G, +, 0, \leq \rangle$ be a partially ordered abelian group, and $T_1, T_2, \ldots$ locally cofinite subsets of the positive cone of G. Then the number of n-term representations of any element of E_n is finite.

Proof of one of these: On board, if there is time (and a board). ⊣

Some obvious questions:

In the case of $G = \mathbb{R}$ there are some interesting results this technology produces for Diophantine equations, eg

Generalization of Znám equation Suppose in the equation

$$\sum_{I} \frac{a_I}{\prod_{i \in I} x_i} = a_\varnothing$$

where the sum ranges over nonempty subsets I of $\{1, \ldots, s\}$, and $a_I \in \mathbb{R}$ such that every $a_I \geq 0$ and $a_\varnothing > 0$.

The following are equivalent:

1. For every $i \leq s$ there is an I with $i \in I$ and $a_I \neq 0$.
2. The equation has only finitely many solutions in $\mathbb{N}$.

Lagarias 2013 Let $a, b, c \in \mathbb{Z} \setminus \{0\}$ with $c \geq 1$ and $gcd(b, c) = 1$. If the equation

$$c(1/x_1 + \cdots + 1/x_s) + b/x_1 x_2 \cdots x_s = a$$

has infinitely many integer solutions then $c = 1$, and either (a) $|a| = s - 1$ and $b = -(\text{sign}(a)^{s-1})$, or (b) $|a| < s - 1$ and b is arbitrary

Question 1: Are there interesting variants of these for groups?

Question 2: How much nonstandard analysis do we actually need?

Facetious answer: None. Didn't use any powerful tools like Loeb measures or nonstandard hulls.

But: Used saturation, transfer, nonstandard characterizations of 'compact' and 'dense' to get *much* shorter proofs of the results in $\mathbb{R}$. Is there some way to isolate fundamental principles to make these arguments accessible to standard number theorists?

Question 3: Is Theorem 0.4 true for partially-ordered groups?

Question 4: Is Theorem 0.5 true for non-Abelian groups?

References

[1] Brenton, L., and Vasiliu, A., *Znam's Problem*, Mathematics Magazine, 2002, 3–11.

[2] Curtiss, D. R.,*On Kellogg's Diophantine Problem*, Amer. Math. Monthly, **29**, 1922, 380–387.

[3] Erdös, P.,*On a Diophantine equation*, Matematikai Lapok., 1950, 192–210.

[4] Graham, R. L., *On finite sums of unit fractions*, Proc. London Math. Soc., 1964, 193–207.

[5] Kellogg, O. D., *On a Diophantine problem*, Amer. Math. Monthly **28**, 1921, 300–303.

[6] Lagarias, J. C., *Cyclic systems of simultaneous congruences*, Int. J. Number Theory **6**, 2010, no. 2, 219–245.

[7] Nathanson, M. B., *Weighted real Egyptian numbers*, Functiones et Approximatio Commentarii Mathematici, 2018

[8] Pisano, Leonardo, *Scritti*, vol. 1, B. Boncompagni, Rome (1857) 77-83.

[9] Ross, David, *Egyptian fractions, nonstandard extensions of* $\mathbb{R}$*, and some Diophantine equations without many solutions*, 2018

[10] Sierpinski, W., *Sur les décompositions de nombres rationnels en fractions primaires*, 1956.