AMS Sectional Meeting at University of Hawaii: Special Session on Applications of Ultrafilters and Nonstandard Methods

Local weak*-Convergence, algebraic actions, and a max-min principle.

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- Discussion of algebraic actions.
- Sofic groups.
- Discussion of entropy.
- The appearance of the Loeb measure space.
- Order lattices associated to the Loeb measure space.

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Sofic Groups

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A group G is sofic if there is a homomorphism $\sigma \colon G \to \mathcal{S}$ so that $d_H \circ (\sigma \times \sigma)$ is the discrete metric.Write $\sigma(g) = (\sigma_n(g))_{n \to \mathcal{U}}$, we call $(\sigma_n)_n$ a sofic approximation.

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Given another pmp action $G \curvearrowright (X, \mu)$ the sofic entropy of $G \curvearrowright (X, \mu)$ measure "how many" factor maps $(Z_{\mathcal{U}}, u_{\mathcal{U}}) \to (X, \mu)$. If X is compact, and $G \curvearrowright X$, then the topological entropy of $G \curvearrowright X$ measures "how many" Borel equivariant maps $Z_{\mathcal{U}} \to X$ there are. (Due to Bowen, Kerr-Li).

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Previous results: Berg, Lind-Schmidt, Bowen + Kerr-Li, Bowen-Li, Gaboriau-Seward.

Local wk$*$-convergence

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$$\lim_{n\to\mathcal{U}} \frac{1}{n} \left| \left\{ j : \left| \int f\, d\mu_{n,j} - \int f\, d\mu \right| < \varepsilon \right\} \right| = 1.$$

Relation to Entropy

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If $G \curvearrowright (X, m_X)$ *is ergodic, and there exists* $\mu_n \to^{lw^*} m_X$*, then*

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- Convolve μ_n to make it have “better separation properties."

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- Hard analysis proofs. We'd like soft analysis proofs.

Finally, back to ultraproduct spaces.

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For $\mu \in \text{Prob}(X)$, $\mathcal{E}((\mu_n)_n) = \mu$ if and only if $\mu_n \to^{lw^*}_{n \to \mathcal{U}} \mu$.

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Proofs are easy: Łos's theorem!

A subgroup version

Let $Y \in \mathrm{Meas}(Z_{\mathcal{U}}, \mathrm{Sub}(X))$. We define $m_Y \in \mathrm{Meas}(Z_{\mathcal{U}}, \mathrm{Prob}(X))$ by $m_Y(z) = m_{Y(z)}$.

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Say that $Y \in \mathrm{Meas}(Z_{\mathcal{U}}, \mathrm{Sub}(X))$ *absorbs all topological microstates* if for every measurable, G-equivariant $\Theta \colon Z_{\mathcal{U}} \to X$ we have $\Theta(z) \in Y(z)$ a.e. z.

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- *Given* $Y_1, Y_2 \in \mathcal{S}_{\mathcal{U}}$, *set* $(Y_1 \vee Y_2)(z) = \overline{\langle Y_1(z), Y_2(z) \rangle}$. *The* $Y_1 \vee Y_2 \in \mathcal{S}_{\mathcal{U}}$.

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The Big Theorem

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- $\mathcal{S}_{\mathcal{U}}$ *is a complete join lattice.*
- *The maximal element of* $\mathcal{S}_{\mathcal{U}}$ *is the minimal element of* $\mathrm{Meas}(Z_{\mathcal{U}}, \mathrm{Sub}(X))$ *which absorbs all topological microstates.*

Comments on the proof

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These are proved by noncommutative Fourier analysis (the Peter-Weyl theorem+ the finite-dimensional spectral theorem).

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We may also view $\operatorname{Meas}(Z_{\mathcal{U}}, B(L^2(X))) \subseteq B(L^2(Z_{\mathcal{U}} \times X))$.

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This necessitates showing that $\lambda_*(\mathcal{L}_{\mathcal{U}})$ is strong operator topology closed. This follows from a continuous Łos's theorem.

Applications

Special case: Let G' be the centralizer of G in $\prod_{n \to \mathcal{U}}(S_n, d_H)$.

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Theorem

If G is residually finite, $G \curvearrowright (X, m_X)$ is ergodic, and there is a factor map $(Z_{\mathcal{U}}, u_{\mathcal{U}}) \to (X, m_X)$, then $h_{\text{top}}(G \curvearrowright X) = h_{m_X}(G \curvearrowright X)$.

More applications

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If G is residually finite, and $G \curvearrowright X$ is topologically cpe, then $h_{\text{top}}(G \curvearrowright X) = h_{m_X}(G \curvearrowright X)$, further $G \curvearrowright (X, m_X)$ is measure cpe.

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Can say the same in the general case, but need to make sense of $h^{lw^*}_{m_Y}(G \curvearrowright Y)$ for $Y \in \text{Meas}(Z_{\mathcal{U}}, \text{Sub}(X))$. This also leads naturally to G-invariant random subgroups of X.

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- Ergodic theorists tend to think of their subject as "combinatorics plus ε" and are heavily influenced by the legacy of Halmos (see: "A nonstandard analysis of Paul Halmos.")
- Because of this, these techniques are a fair bit out of the norm in the ergodic theory community.

Thanks for paying attention!