MIP* = RE

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1. Nonlocal games

2. A quantum detour

3. $\text{MIP}^* = \text{RE}$

4. A few words about the proof of $\text{MIP}^* = \text{RE}$
Alice and Bob against the world

- Alice and Bob are two cooperating but *noncommunicating* players playing a game against a “referee.”
- They are each asked a question \( x, y \in [k] := \{1, \ldots, k\} \) randomly according to some probability distribution \( \pi \) on \([k] \times [k]\).
- Somehow they return answers \( a, b \in [n] \) respectively.
- There is a function \( D : [k]^2 \times [n]^2 \to \{0, 1\} \), called the decision predicate, which determines if they win this round of the game, that is, they win if and only if \( D(x, y, a, b) = 1 \).
- This describes a nonlocal game \( \mathcal{G} := (\pi, D) \) with \( k \) questions and \( n \) answers.
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Strategies for nonlocal games

- Alice and Bob can meet before the game to decide on a **strategy** for playing $\mathcal{G}$ that they will use before the game.
- For us, a strategy will simply be a matrix $p(a, b| x, y) \in [0, 1]^{k^2n^2}$ describing the conditional probability they respond with answers $(a, b) \in [n]^2$ given that they are asked questions $(x, y) \in [k]^2$.
- Given a strategy $p$, the **value of the game $\mathcal{G}$ with respect to** $p$ is the quantity

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\text{val}(\mathcal{G}, p) := \sum_{(x, y) \in [k]^2} \pi(x, y) \sum_{(a, b) \in [n]^2} p(a, b| x, y) D(a, b, x, y).
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- $\text{val}(\mathcal{G}, p)$ measures the expected probability of winning the game if they play according to the strategy $p$. 

A **deterministic** strategy is given by a pair of functions $A, B : [k] \to [n]$ such that

$$p(A(x), B(y) \mid x, y) = 1 \text{ for all } (x, y) \in [k]^2.$$ 

A **classical** (or **local**) strategy is given by a probability space $(\Omega, \mu)$ together with pairs of functions $A_\omega, B_\omega : [k] \to [n]$ such that

$$p(a, b \mid x, y) = \mu(\{\omega \in \Omega : A_\omega(x) = a \text{ and } B_\omega(y) = b\}).$$

$C_{\text{loc}}(k, n) \subseteq [0, 1]^{k^2n^2}$ denotes the set of classical strategies. It is the convex hull of the set $C_{\text{det}}(k, n)$ of deterministic strategies.

The **classical value** of $\mathcal{G}$ is the quantity

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Classical strategies for nonlocal games

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The CHSH game

The CHSH game (named after Clauser, Horne, Shimony, and Holt) is the game $G_{\text{CHSH}}$ with $k = n = 2$ and such that:

- If $x = 1$ or $y = 1$, then Alice and Bob win if and only if their answers agree.
- If $x = y = 2$, then Alice and Bob win if and only if their answers disagree.

By inspecting all deterministic strategies, one sees that

$$\text{val}(G_{\text{CHSH}}) = \frac{3}{4}.$$
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4 A few words about the proof of \( \text{MIP}^* = \text{RE} \)
The spin of an electron

- An electron can have one of two spins: “up” or “down.”
- At any given moment, however, it does not have a definite spin and instead is in a **superposition** of the two spins, as represented by the linear combination $\alpha |\text{up}\rangle + \beta |\text{down}\rangle \in \mathbb{C}^2$, where $|\text{up}\rangle$ and $|\text{down}\rangle$ are two orthogonal vectors in $\mathbb{C}^2$ and $\alpha, \beta \in \mathbb{C}$ are such that $|\alpha|^2 + |\beta|^2 = 1$.
- If it is not disturbed, its state evolves linearly according to the Shrödinger equation.
- However, **when it is measured**, its state randomly and discontinuously jumps to one of the two definite spin states $|\text{up}\rangle$ or $|\text{down}\rangle$ with probabilities $|\alpha|^2$ and $|\beta|^2$ respectively.
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- If it is not disturbed, its state evolves linearly according to the **Shrödinger equation**.
- However, when it is measured, its state randomly and **discontinuously** jumps to one of the two definite spin states $|\text{up}\rangle$ or $|\text{down}\rangle$ with probabilities $|\alpha|^2$ and $|\beta|^2$ respectively.
Recommended summer reading

**WHAT IS REAL?**

"A thorough, illuminating exploration of the most consequential controversies raging in modern science."
—NEW YORK TIMES BOOK REVIEW

_Adam Becker_

**HELGOLAND**

 MAKING SENSE OF THE QUANTUM REVOLUTION

_Carlo Rovelli_

**THE CONCEPTUAL FOUNDATIONS OF QUANTUM MECHANICS**

_Jeffrey A. Barrett_

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More summer reading (shameless plug)
General quantum systems

- Associated to a quantum system is its **state space**, which is a complex Hilbert space $H$.
- The **state** of the system at any given moment is described by a unit vector $\xi \in H$, which evolves linearly until it is measured.
- A **measurement** with $n$ outcomes is a tuple $M_1, \ldots, M_n \in B(H)$ such that, upon measurement, the probability of outcome $i$ occurring is given by $\|M_i\xi\|^2$, in which case the state of the system jumps to $\frac{M_i\xi}{\|M_i\xi\|}$. (**Born rule**)
- For these to determine legitimate probabilities, for all unit vectors $\xi \in H$, one must have

$$1 = \sum_{i=1}^{n} \|M_i\xi\|^2 = \sum_{i=1}^{n} \langle M_i^* M_i \xi, \xi \rangle$$

and thus $\sum_{i=1}^{n} M_i^* M_i = I_H$. 
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POVMs and PVMs

- If one only cares about the statistics of the outcomes of a measurement (like us!), then we can simplify matters by assuming that each measurement operator is positive.

- A POVM (positive operator-valued measure) of length $n$ is a collection $A_1, \ldots, A_n$ of positive operators on $H$ such that $\sum_{i=1}^{n} A_i = I_H$.

- On state $\xi$, the probability outcome $i$ occurs is given by $\langle A_i \xi, \xi \rangle$.

- If each $A_i$ is actually a projection, we speak of PVMs (projection-valued measures). This is the same as an orthogonal decomposition of $H$ into $n$ orthogonal subspaces.

- The case of the spin of an electron was a PVM corresponding to the one-dimensional subspaces spanned by $|\text{up}\rangle$ and $|\text{down}\rangle$. 
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The EPR state

Another axiom of quantum mechanics is that if $H_A$ and $H_B$ are the state spaces for two quantum systems, then the state space for their composite system is given by $H_A \otimes H_B$.

Thus, the state space for two electrons is given by $\mathbb{C}^2 \otimes \mathbb{C}^2 \cong \mathbb{C}^4$.

The EPR state is given by $\psi_{\text{EPR}} = \frac{1}{\sqrt{2}} |\text{up}\rangle |\text{up}\rangle + \frac{1}{\sqrt{2}} |\text{down}\rangle |\text{down}\rangle$.

It was used by Einstein, Podolsky, and Rosen in their famous paper arguing that quantum mechanics was incomplete!

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Quantum strategies for nonlocal games

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This time, when playing the game, Alice and Bob have quantum systems $H_A$ and $H_B$ and share a state $\xi \in H_A \otimes H_B$.

Upon receiving question $x \in [k]$, Alice will perform a POVM $A^x = (A^x_1, \ldots, A^x_n)$ on her part of $\xi$ to decide which answer to give.

Bob similarly has a POVM $B^y = (B^y_1, \ldots, B^y_n)$ for measuring on his part of $\xi$.

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Quantum strategies for nonlocal games

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The entangled value of a nonlocal game

- $C_q(k, n)$ denotes the set of strategies for which there are:
  - finite-dimensional Hilbert spaces $H_A$ and $H_B$,
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- We also consider $C_{qa}(k, n) := \overline{C_q(k, n)}$.

- If $\mathcal{G}$ is a nonlocal game with $k$ questions and $n$ answers, the entangled value of $\mathcal{G}$ is

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- $C_{loc}(k, n) \subseteq C_q(k, n)$ so $\text{val}(\mathcal{G}) \leq \text{val}^*(\mathcal{G})$. 
Recall $\text{val}(\mathcal{G}_{\text{CHSH}}) = \frac{3}{4}$.

However, there is an entangled strategy $p$, based on the EPR state $\psi_{\text{EPR}}$, such that $\text{val}(\mathcal{G}, p) = \cos^2\left(\frac{\pi}{8}\right) \approx 0.85$ (which equals $\text{val}^*(\mathcal{G}_{\text{CHSH}})$ by a result of Tsirelson).

This inequality showed that EPR were wrong!
How hard is it to compute $\text{val}^*(\mathcal{G})$?

- One can effectively compute *lower bounds* for $\text{val}^*(\mathcal{G})$ uniformly in $\mathcal{G}$.

- Given some dimension $d$, you can enumerate a computable sequence of finite nets $N_1^d \subseteq N_2^d \subseteq \cdots$ over all states and POVMs in dimension $d$ with $|N_m^d| = m^{O(d^2)}$ such that for any $p \in C_q(k,n)$ based on a $d$-dimensional strategy and any $m$, there is $q \in N_m^d$ with $|\text{val}(\mathcal{G}, p) - \text{val}(\mathcal{G}, q)| < \frac{1}{m}$.

- Set

$$\text{val}^n(\mathcal{G}, p) = \max_{d,m \leq n} \max_{p \in N_m^d} \text{val}(\mathcal{G}, p).$$

- Then $\text{val}^n(\mathcal{G}, p)$ is computable and $\text{val}^n(\mathcal{G}, p) \nearrow \text{val}(\mathcal{G})$.

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Theorem (Ji, Natarajan, Vidick, Wright, Yuen (2020))

There is an effective mapping $\mathcal{M} \mapsto \mathcal{G}_\mathcal{M}$ from Turing machines to nonlocal games such that:

- If $\mathcal{M}$ halts, then $\text{val}^*(\mathcal{G}_\mathcal{M}) = 1$.
- If $\mathcal{M}$ does not halt, then $\text{val}^*(\mathcal{G}_\mathcal{M}) \leq \frac{1}{2}$. 
1. Nonlocal games

2. A quantum detour

3. MIP* = RE

4. A few words about the proof of MIP* = RE
A **uniform game sequence** (UGS) is an infinite sequence \( \bar{G} := (G_1, G_2, \ldots, ) \) of nonlocal games for which there is a single Turing machine \( V \) which computes in time \( \text{poly}(\log n) \):

- The number of questions and answers in \( G_n \).
- A Turing machine which specifies the probability distribution for \( G_n \).
- A Turing machine which specifies the decision predicate for \( G_n \).
Entanglement lower bound for nonlocal games

**Definition**

Given a nonlocal game $\mathcal{G}$ and $r \in [0, 1]$, we set $\mathcal{E}(\mathcal{G}, r)$ to be the minimum dimension $d$ for which there exists a strategy $p \in C_q$ based on $d$-dimensional Hilbert spaces so that $\text{val}(\mathcal{G}, p) \geq r$.

**Example**

1. $\mathcal{E}(\mathcal{G}_{\text{CHSH}}, \frac{3}{4}) = 0$
2. $\mathcal{E}(\mathcal{G}_{\text{CHSH}}, \cos^2(\frac{\pi}{8})) = 2$
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Theorem

There exists an algorithm $C$ such that upon input a Turing machine $V$ describing a UGS $\mathcal{G}$ with each $\mathcal{G}_n$ of “complexity” at most $O(n^2)$ outputs a Turing machine $V'$ describing a UGS $\mathcal{G}'$ of polynomial-time computable games such that:

- If $\text{val}^*(\mathcal{G}_n) = 1$, then $\text{val}^*(\mathcal{G}'_n) = 1$.
- $\varepsilon(\mathcal{G}'_n, \frac{1}{2}) \geq \max\{\varepsilon(\mathcal{G}_n, \frac{1}{2}), n\}$.
- The time complexity of $\mathcal{G}'_n$ is $\text{poly}(\log n)$.
Compression theorem for nonlocal games

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A few words about the proof of $\text{MIP}^* = \text{RE}$

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Given $\mathcal{M}$, we define a Turing machine $V^\mathcal{M}$ which computes a UGS $\bar{\mathcal{G}}^\mathcal{M} = (\mathcal{G}_1, \mathcal{G}_2, \ldots)$.

Here is how $\mathcal{G}_n$ looks:

- Run $\mathcal{M}$ on the empty input for $n$ time steps. If $\mathcal{M}$ halts, then victory!
- If not, run $C$ on $V^\mathcal{M}$ to get $V' := (V^\mathcal{M})'$ which computes the UGS $\bar{\mathcal{G}}'$.
- Then play $\mathcal{G}'_{n+1}$.

This is self-referential, but we are used to that :)

The compression algorithm is indeed applicable (check execution times of the various steps...)

Define $\mathcal{G}_{\mathcal{M}} := \mathcal{G}_1$.

Why does this work?
Given $\mathcal{M}$, we define a Turing machine $V^\mathcal{M}$ which computes a UGS $\tilde{G}^\mathcal{M} = (\mathcal{G}_1, \mathcal{G}_2, \ldots)$.

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MIP* = RE from Compression: Part II

- **Case 1:** $\mathcal{M}$ halts, say in $T$ steps.
  - Then $\text{val}^*(\mathcal{G}_n) = 1$ for all $n \geq T$.
  - What about $n < T$?
  - For $n < T$, $\text{val}^*(\mathcal{G}_n) = \text{val}^*(\mathcal{G}'_{n+1})$.
  - So $\text{val}^*(\mathcal{G}_{T-1}) = \text{val}^*(\mathcal{G}'_T) = 1$ since $\text{val}^*(\mathcal{G}_T) = 1$ (preservation of perfect completeness).
  - By induction, we get that $\text{val}^*(\mathcal{G}_M) = \text{val}^*(\mathcal{G}_1) = 1$. 
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Then \( \text{val}^*(\mathcal{G}_n) = \text{val}^*(\mathcal{G}'_{n+1}) \) and \( \mathcal{E}(\mathcal{G}_n, r) = \mathcal{E}(\mathcal{G}'_{n+1}, r) \) for all \( n \in \mathbb{N} \) and \( r \in [0, 1] \).

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\mathcal{E}(\mathcal{G}'_{n+1}, \frac{1}{2}) \geq \mathcal{E}(\mathcal{G}_{n+1}, \frac{1}{2}) = \mathcal{E}(\mathcal{G}'_{n+2}, \frac{1}{2}) \geq \mathcal{E}(\mathcal{G}_{n+2}, \frac{1}{2}) \ldots
\]

\[
\therefore \mathcal{E}(\mathcal{G}_n, \frac{1}{2}) \geq \mathcal{E}(\mathcal{G}'_m, \frac{1}{2}) \text{ for all } m > n.
\]

OTOH \( \mathcal{E}(\mathcal{G}'_m, \frac{1}{2}) \geq m \) for all \( m \in \mathbb{N} \).

Therefore \( \mathcal{E}(\mathcal{G}_n, \frac{1}{2}) = \infty \) for all \( n \in \mathbb{N} \) and thus

\[
\text{val}^*(\mathcal{G}_M) = \text{val}^*(\mathcal{G}_1) < \frac{1}{2}.
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Now suppose that $\mathcal{M}$ does not halt.

Then $\text{val}^*(\mathcal{G}_n) = \text{val}^*(\mathcal{G}_{n+1})$ and $\mathcal{E}(\mathcal{G}_n, r) = \mathcal{E}(\mathcal{G}_{n+1}, r)$ for all $n \in \mathbb{N}$ and $r \in [0, 1]$.

- $\mathcal{E}(\mathcal{G}_{n+1}, \frac{1}{2}) \geq \mathcal{E}(\mathcal{G}_{n+1}, \frac{1}{2}) = \mathcal{E}(\mathcal{G}_{n+2}, \frac{1}{2}) \geq \mathcal{E}(\mathcal{G}_{n+2}, \frac{1}{2}) \cdots$

- $\therefore \mathcal{E}(\mathcal{G}_n, \frac{1}{2}) \geq \mathcal{E}(\mathcal{G}_m, \frac{1}{2})$ for all $m > n$.

OTOH $\mathcal{E}(\mathcal{G}_m, \frac{1}{2}) \geq m$ for all $m \in \mathbb{N}$.

Therefore $\mathcal{E}(\mathcal{G}_n, \frac{1}{2}) = \infty$ for all $n \in \mathbb{N}$ and thus

$$\text{val}^*(\mathcal{G}_M) = \text{val}^*(\mathcal{G}_1) < \frac{1}{2}.$$
Now suppose that $\mathcal{M}$ does not halt.

Then $\text{val}^*(\mathcal{G}_n) = \text{val}^*(\mathcal{G}'_{n+1})$ and $\mathcal{E}(\mathcal{G}_n, r) = \mathcal{E}(\mathcal{G}'_{n+1}, r)$ for all $n \in \mathbb{N}$ and $r \in [0, 1]$.

$\mathcal{E}(\mathcal{G}'_{n+1}, \frac{1}{2}) \geq \mathcal{E}(\mathcal{G}_{n+1}, \frac{1}{2}) = \mathcal{E}(\mathcal{G}'_{n+2}, \frac{1}{2}) \geq \mathcal{E}(\mathcal{G}_{n+2}, \frac{1}{2}) \cdots$

$\therefore \mathcal{E}(\mathcal{G}_n, \frac{1}{2}) \geq \mathcal{E}(\mathcal{G}'_m, \frac{1}{2})$ for all $m > n$.

OTOH $\mathcal{E}(\mathcal{G}'_m, \frac{1}{2}) \geq m$ for all $m \in \mathbb{N}$.

Therefore $\mathcal{E}(\mathcal{G}_n, \frac{1}{2}) = \infty$ for all $n \in \mathbb{N}$ and thus

$$\text{val}^*(\mathcal{G}_M) = \text{val}^*(\mathcal{G}_1) < \frac{1}{2}. $$
MIP* = RE from Compression: Part III

- Now suppose that $M$ does not halt.
- Then $\text{val}^*(G_n) = \text{val}^*(G_{n+1}')$ and $\mathcal{E}(G_n, r) = \mathcal{E}(G_{n+1}', r)$ for all $n \in \mathbb{N}$ and $r \in [0, 1]$.
- $\mathcal{E}(G_{n+1}', 1/2) \geq \mathcal{E}(G_{n+1}, 1/2) = \mathcal{E}(G_{n+2}', 1/2) \geq \mathcal{E}(G_{n+2}, 1/2) \cdots$
- $\therefore \mathcal{E}(G_n, 1/2) \geq \mathcal{E}(G_m', 1/2)$ for all $m > n$.
- OTOH $\mathcal{E}(G_m', 1/2) \geq m$ for all $m \in \mathbb{N}$.
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Isaac Goldbring (UCI)
Now suppose that $M$ does not halt.

Then $val^*(G_n) = val^*(G_{n+1})$ and $E(G_n, r) = E(G_{n+1}, r)$ for all $n \in \mathbb{N}$ and $r \in [0, 1]$.

$E(G_{n+1}, \frac{1}{2}) \geq E(G_{n+1}, \frac{1}{2}) = E(G_{n+2}, \frac{1}{2}) \geq E(G_{n+2}, \frac{1}{2}) \cdots$

$\therefore E(G_n, \frac{1}{2}) \geq E(G_m, \frac{1}{2})$ for all $m > n$.

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$$val^*(G_M) = val^*(G_1) < \frac{1}{2}.$$
Now suppose that $M$ does not halt.

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OTOH $\mathcal{E}(\mathcal{G}'_m, \frac{1}{2}) \geq m$ for all $m \in \mathbb{N}$.

Therefore $\mathcal{E}(\mathcal{G}_n, \frac{1}{2}) = \infty$ for all $n \in \mathbb{N}$ and thus

$$val^*(\mathcal{G}_M) = val^*(\mathcal{G}_1) < \frac{1}{2}. $$
Hand-waving about the proof of the Compression Theorem

- **Question reduction**
  - Get the players to sample questions for themselves.
  - Uses *rigidity of nonlocal games* and the *Heisenberg uncertainty principle*.
  - Brings the sampler complexity down from $\text{poly}(n)$ to $\text{poly}(\log n)$.

- **Answer reduction**
  - The players must now also compute the decision predicate $D_n(x, y, a, b)$ for themselves.
  - They must include a *succinct proof* that they computed $D_n$ correctly.
  - Uses *probabilistically checkable proofs* (PCP).
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