EXISTENTIALLY CLOSED MEASURE-PRESERVING ACTIONS OF APPROXIMATELY TREEABLE GROUPS

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Abstract. Given a countable group Γ, letting $\mathcal{K}_\Gamma$ denote the class of p.m.p. actions of Γ, we study the question of when the model companion of $\mathcal{K}_\Gamma$ exists. Berenstein, Henson, and Ibarlucía showed that the model companion of $\mathcal{K}_\Gamma$ exists when Γ is a nonabelian free group on a countable number of generators. We significantly generalize their result by showing that the model companion of $\mathcal{K}_\Gamma$ exists whenever Γ is an approximately treeable group. The class of approximately treeable groups contain the class of treeable groups as well as the class of universally free groups, that is, the class of groups with the same universal theory as nonabelian free groups. We prove this result using an open mapping characterization of when the model companion exists; moreover, this open mapping characterization provides concrete, ergodic-theoretic axioms for the model companion when it exists. We show how to simplify these axioms in the case of treeable groups, providing an alternate axiomatization for the model companion in the case of the free group, which was first axiomatized by Berenstein, Henson, and Ibarlucía using techniques from model-theoretic stability theory. Along the way, we prove a purely ergodic-theoretic result of independent interest, namely that finitely generated universally free groups (also known as limit groups) have Kechris' property MD. We also show that for groups with Kechris' EMD property, the profinite completion action is existentially closed, and for groups without property (T), the generic existentially closed action is weakly mixing, generalizing results of Berenstein, Henson, and Ibarlucía for the case of nonabelian free groups.

Contents

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1. Introduction

The work presented in this paper lie at the intersection of ergodic theory and model theory. Our main objects of study are "model-theoretically generic" probability measure-preserving (p.m.p.) actions of discrete countable groups on probability spaces. Here, the precise formulation of a model-theoretically generic p.m.p. action is that of an **existentially closed** action.

Given some axiomatizable class $\mathcal{K}$ of structures in an appropriate (classical or continuous) language L, for example, the class of fields, the class of groups, the class of graphs, the class of Banach spaces, the class of tracial von Neumann algebras, the class of C^*-algebras etc..., the Robinsonian philosophy for obtaining a good model-theoretic understanding of $\mathcal{K}$ is to understand the class of existentially closed members of $\mathcal{K}$, for these members of $\mathcal{K}$ represent "universal domains" for which one should work in when studying members of $\mathcal{K}$. Roughly speaking, e.c. members $\mathcal{M}$ of $\mathcal{K}$ contain all "solutions" to "equations" with "coefficients" from $\mathcal{M}$ that "should have" solutions, that is, which have solutions in some extension of $\mathcal{M}$ belonging to $\mathcal{K}$. For example, e.c. fields are precisely the algebraically closed fields, e.c. graphs are precisely the "random" graphs, and e.c. Banach spaces are precisely the Gurarij Banach spaces. In these examples, something fortuitous occurs: the class of e.c. members of $\mathcal{K}$, while a priori not expressible using first-order axioms, do actually themselves form an axiomatizable class. When this situation happens, we say that the **model companion** for $\mathcal{K}$ exists. An alternative formulation avoiding the first-order formalism states that the model companion for $\mathcal{K}$ exists precisely when the e.c. members of $\mathcal{K}$ are closed under ultraproducts. On the other hand, the classes of groups, tracial von Neumann algebras, and C^*-algebras do *not* admit model companions; from the model-theoretic point of view, these classes are "wild."

In this paper, we consider the class $\mathcal{K}_\Gamma$ of p.m.p. actions of a given discrete, countable group Γ (which is indeed an axiomatizable class in an appropriate continuous language L_Γ). Our motivating question is the following:

Question 1.1. For which groups Γ does the class $\mathcal{K}_\Gamma$ admit a model companion?

Right at the outset, we mention that there does not exist a single group Γ for which the answer to the above question is known to be negative. Our main contribution is to significantly enlarge the class of groups for which the answer to this question is positive.

The first example of a group Γ for which the previous question was shown to have a positive answer was the case $\Gamma = \mathbb{Z}$, as demonstrated by Berenstein and Henson in [4]. There, they show that the e.c. actions of $\mathbb{Z}$ are precisely the aperiodic ones and they use the classical Rohklin lemma to give a concrete axiomatization of this class. In unpublished work of Berenstein and Henson, they use the Ornstein-Weiss version of the Rohklin lemma to extend their result to the case of all amenable groups Γ, whose model companion is simply the axiomatizable class of free actions.

Besides the case of amenable groups, the only other instance of Question 1.1 known to have a positive answer is when $\Gamma = \mathbb{F}_k$, a finitely generated free group:

Fact 1.2 (Berenstein, Henson, and Ibarlucía [5]). *If $\Gamma = \mathbb{F}_k$ is a finitely generated free group, then $\mathcal{K}_\Gamma$ admits a model companion.*

In the introduction of [5], the authors point out that their result also holds for the free group $\mathbb{F}_\omega$ on a countably infinite set of generators.

By studying general preservation properties for when $\mathcal{K}_\Gamma$ admits a model companion, we significantly enlarge the class of groups for which Question 1.1 has a positive answer:

Theorem. *If Γ is a universally free group, then $\mathcal{K}_\Gamma$ admits a model companion.*

Here, a **universally free group** (sometimes called a **ω-residually free** or **fully residually free** group) is a group Γ which is a model of the universal theory of a free group, or, in logic-free terms, a group that embeds in an ultrapower of a free group. The finitely generated universally free groups are known as **limit groups** and are of great interest in geometric group theory.

With further effort, we can generalize the previous result even further. To explain this generalization, we recall that a group is called **treeable** if it admits a free p.m.p. action whose orbit equivalence relation can be equipped with a treeing (a precise definition is given in Subsection 2.5 below); an alternate description of treeability can be given using a theorem of Hjorth [26]: a group is treeable if it is measure equivalent to a free group. The class of treeable groups is quite rich and contains many interesting groups, such as surface groups and, more generally, groups admitting a planar Cayley graph. By a result from [12], the class of treeable groups does have some intersection with the aforementioned class of universally free groups: every **elementarily free group**, that is, every finitely generated group with the same first order theory as a nonabelian free group, is treeable.

One can generalize further by considering the class of **approximately treeable** groups (a precise definition is given in Subsection 2.5); approximately treeable groups were studied in [20] under the guise of groups which have approximate ergodic dimension at most 1. In Section 7, it is remarked that all universally free groups are approximately treeable. (It is a well-known open problem whether or not all limit groups are in fact treeable.) The class of approximately treeable groups is a proper extension of the class of all treeable groups as witnessed by the group $\mathbb{F}_2 \times \mathbb{Z}$ (which is not treeable by [19] and by [37]). In Subsection 5.5 below, we develop an "open mapping" characterization of when the model companion of $\mathcal{K}_\Gamma$ exists and use this characterization in Section 7 to prove the following substantial generalization of Fact 1.2:

Theorem. *If Γ is an approximately treeable group, then the model companion of $\mathcal{K}_\Gamma$ exists.*

While proving Fact 1.2, the authors give a concrete axiomatization of the class of e.c. members of $\mathcal{K}_\Gamma$. However, their proof adapts known model-theoretic techniques from the area of model theory known as **stability theory** and extracting an "ergodic-theoretic" axiomatization of this class from their axioms is not entirely straightforward.

The open mapping characterization of the existence of the model companion proved in Subsection 5.5 actually yields purely ergodic-theoretic axioms for the model companion whenever it exists. That being said, these axioms are not entirely illuminating and it seems desirable to find groups for which the axioms for the model companion of $\mathcal{K}_\Gamma$ can be made simpler. In Section 6 below, we prove the following result along these lines:

Theorem. *If Γ has the **extension-MD property** and the **definable cocycle property**, then the model companion of $\mathcal{K}_\Gamma$ exists. Moreover, one can list simple axioms for the model companion of an ergodic-theoretic nature.*

The first condition in the theorem is related to the **MD property**, first introduced by Kechris in [30]; roughly speaking, a residually finite group has the MD property if the set of profinite actions of the group is dense in the space of all actions. If the a priori stronger condition that the set of ergodic profinite actions is dense in the space of all actions, then group is said to have the **EMD property**. We say more about these properties in Section 4 below. The second condition in the theorem roughly states that every cochain that is close to satisfying the cocycle identity is near an actual cocycle.

In Subsections 6.1 and 6.2 respectively, we show, using fairly elementary means, that free groups satisfy both properties, whence one obtains a simple, purely ergodic-theoretic axiomatization of the model companions of $\mathcal{K}_{\mathbb{F}_k}$ and $\mathcal{K}_{\mathbb{F}_\omega}$. In Subsections 6.4 through 6.6, we generalize the result that free groups have the extension-MD property and the definable cocycle property to the wider class of **strongly treeable** groups:

Theorem. *Strongly treeable groups have the extension-MD property and the definable cocycle property.*

Here, a group is strongly treeable if *all* of its free p.m.p. actions admit a treeing. It is a well-known open problem whether there exists a group that is treeable but not strongly treeable. In any event, we now have that if Γ is a strongly treeable group, then not only does the model companion of $\mathcal{K}_\Gamma$ exist, but there is a very simple set of axioms for this model companion. We also show how to tweak these results to cover the case of a treeable (but not necessarily strongly treeable) group at the cost of slightly complicating the axioms. In the process, we obtain the following simple characterization of e.c. actions of treeable groups.

Theorem. *Let Γ be a treeable group. Then a p.m.p. action $\Gamma \curvearrowright^a (X, \mu)$ is e.c. if and only if all of the following hold:*

(1) $\Gamma \curvearrowright^a (X, \mu)$ *weakly contains a free treeable action of* Γ*;*
(2) *the trivial extension* $\Gamma \curvearrowright^{a \times \mathrm{id}} (X \times [0, 1], \mu \times \lambda)$ *where* Γ *acts on* $[0, 1]$ *by fixing every point, is an e.c. extension of* $\Gamma \curvearrowright^a (X, \mu)$*;*
(3) $B^1(a, \mathrm{Sym}(k))$ *is dense in* $Z^1(a, \mathrm{Sym}(k))$ *for every* $k \in \mathbb{N}$*.*

In [5], Berenstein, Henson, and Ibarlucía show that the profinite completion action of the free group is a concrete example of an e.c. action. In Section 4, we extend their result in an optimal way by proving the following:

Theorem. *The profinite completion of any group with the EMD property is an e.c. action.*

We note that being an EMD group is a necessary condition for the profinite completion to be an e.c. action, whence this result is indeed optimal. Along the way, we prove the following result, which is purely ergodic-theoretic and of independent interest:

Theorem. *Limit groups have property MD.*

Berenstein, Henson, and Ibarlucía also prove the existence of a weakly mixing e.c. action of the free group. Using a result of Kerr and Pichot, we extend their result to any group without property (T):

Theorem. *If Γ is a group without property (T), then the model-theoretically "generic" e.c. action of Γ is weakly mixing.*

A precise formulation of "generic" in the previous theorem will be given in Section 4. We note that, for the conclusion of the theorem to hold, the group cannot have property (T), whence the result is once again optimal. The main technical idea behind this proof is to use various descriptions of weakly mixing actions for nonamenable groups developed by Bergelson to show that they have a very particular infinitary first-order description.

Throughout this paper, we assume familiarity with basic continuous model theory as well as basic ergodic theory. We refer the reader to [3] for the former and [42] for the latter.

2. Preliminaries

In this section, we gather all of the necessary preliminaries that are used in the rest of the paper.

2.1. **Probability measure-preserving actions as metric structures.**

In this subsection, we explain how we view probability measure-preserving actions of a countable group Γ as structures in continuous logic. We follow closely the excellent presentation given in [27].

Throughout this paper, Γ denotes a countable, discrete group. Given a probability space $(X, \mathcal{B}, \mu)$ (sometimes abbreviated (X, μ) for the sake of brevity), a **probability measure-preserving (p.m.p.) action** of Γ on $(X, \mathcal{B}, \mu)$ is a homomorphism $a : \Gamma \to \mathrm{Aut}(X, \mathcal{B}, \mu)$. We often denote such actions by $\Gamma \curvearrowright^a (X, \mathcal{B}, \mu)$ or sometimes by the simpler notation $\Gamma \curvearrowright^a (X, \mu)$ or even $\Gamma \curvearrowright^a X$. We also write $\gamma^a \cdot x$, $\gamma^a x$, or even γx (when a can be inferred from the context) instead of $a(\gamma)(x)$ for $\gamma \in \Gamma$ and $x \in X$. Two actions $\Gamma \curvearrowright^a X$ and $\Gamma \curvearrowright^b X$ are **isomorphic** or **conjugate** if there is a measure space isomorphism $\Phi : X \to Y$ such that $\Phi(\gamma^a x) = \gamma^b \Phi(x)$ for all $\gamma \in \Gamma$ and a.e. $x \in X$.

Recall that one associates to a probability space $(X, \mathcal{B}, \mu)$ its **measure algebra** $\mathrm{MALG}(X, \mathcal{B}, \mu)$, or $\mathrm{MALG}(X)$ for the sake of brevity, which is defined to be the Boolean algebra of equivalence classes of elements of $\mathcal{B}$ with respect to the pseudometric $d_\mu(A, B) := \mu(A \triangle B)$. The pseudometric d_μ naturally induces a metric on $\mathrm{MALG}(X)$, still denoted by d_μ, and the set-theoretic operations of union, intersection, and complement on $\mathcal{B}$ induce operations on $\mathrm{MALG}(X)$ rendering it a measured Boolean algebra for which the Boolean algebra operations are uniformly continuous with respect to the metric d_μ. Moreover, the countable additivity of the measure μ implies that d_μ is a complete metric. Given $A \in \mathcal{B}$, we denote its equivalence class in $\mathrm{MALG}(X)$ by $[A]_\mu$, or sometimes by the corresponding lowercase letter a. The associated measure algebra is separable with respect to the metric d_μ if and only if X is a **standard** probability space.This process yields an equivalence of categories between the categories of probability spaces and the associated measure algebras. Moreover, if Γ is a countable group, then the p.m.p. actions of Γ on a measure space X correspond in this duality to isometric actions of Γ on $\mathrm{MALG}(X)$ that preserve the Boolean operations.

We let L_Γ be the language consisting of function symbols for the Boolean operations, a predicate symbol for the measure, and a unary function symbol u_γ for each $\gamma \in \Gamma$. The moduli of uniform continuity for these symbols will be assumed to be the natural ones. We let T_Γ denote the L_Γ-theory which axiomatizes isometric actions of Γ on measure algebras. More precisely, the axioms include the familiar axioms for measure algebras, axioms which state that each u_γ is an automorphism, and axioms stating that $u_{\gamma_1} \circ u_{\gamma_2} = u_{\gamma_1\gamma_2}$ for all $\gamma_1, \gamma_2 \in \Gamma$. These axioms are clearly universal. If $\Gamma \curvearrowright^a X$ is a p.m.p. action of Γ, we let $\mathcal{M}_a$ denote the corresponding model of T_Γ.

Given p.m.p. actions $\Gamma \curvearrowright^a (X, \mathcal{B}, \mu)$ and $\Gamma \curvearrowright^b (Y, \mathcal{C}, \nu)$ of Γ, recall that a **factor map** $\pi : (X, \mathcal{B}, \mu) \to (Y, \mathcal{C}, \nu)$ between these actions is a measurable map that commutes with the actions of Γ and for which $\pi_*\mu = \nu$; we then refer to b as a factor of a. Sometimes we abuse notation and simply write $\pi : X \to Y$ for a factor map. If Λ is a subgroup of Γ, then we call a measurable map $\pi : X \to Y$ a **Λ-equivariant factor map** if it a factor map between the restricted actions of Λ on X and Y.

Given such a factor map π, it is clear that $\pi^{-1}(\mathcal{C})$ is a Γ-invariant σ-subalgebra of $\mathcal{B}$. Conversely, every Γ-invariant σ-subalgebra of $\mathcal{B}$ arises from a factor map in this way. Consequently, there is a duality between substructures of models of T_Γ and factor maps. More precisely, if $\mathcal{M} \models T_\Gamma$ and $\mathcal{N}$ is a substructure of $\mathcal{M}$, then we can find p.m.p. actions $\Gamma \curvearrowright^a X$ and $\Gamma \curvearrowright^b Y$ for which $\mathcal{M} \cong \mathcal{M}_a$, $\mathcal{N} \cong \mathcal{M}_b$, and for which b is a factor of a.

In the sequel, it will behoove us to know that we can also axiomatize the **free** p.m.p. actions of Γ, that is, those actions $\Gamma \curvearrowright^a X$ for which, for every $\gamma \in \Gamma \setminus \{e\}$ and $x \in \mathrm{MALG}(X)$ with $\mu(x) > 0$ there is $y \subseteq x$ with $\mu(y) > 0$ and $\mu(y \cap \gamma y) = 0$ (when X is standard this is equivalent to $\mu(\{x \in X : \gamma^a x = x\}) = 0$). Indeed, as observed in [27, Def. 2.4], we can form the theory $T_{\Gamma,\mathrm{free}}$ consisting of the above axioms T_Γ together with axioms

$$\inf_x \max(|\mu(x) - 1/n|, \max_{i<j<n} \mu(\gamma^i x \cap \gamma^j x)) = 0$$

for each $(\gamma, n) \in \Gamma \times \mathbb{N}$ for which $n < |\langle\gamma\rangle| = \infty$ or $n = |\langle\gamma\rangle| < \infty$.

2.2. **Space of actions.** Let (X, μ) be a standard probability space; unless otherwise stated, we allow for the possibility that these probability spaces have atoms. We equip the automorphism group $\mathrm{Aut}(X, \mu)$ of (X, μ) with the weak topology, namely the weakest topology for which all maps

$$T \mapsto [T(A)]_\mu : \mathrm{Aut}(X, \mu) \to \mathrm{MALG}(X)$$

are continuous, as A ranges over measurable subsets of X. With this topology, $\mathrm{Aut}(X, \mu)$ is a Polish group. We also set $A(\Gamma, X, \mu)$ to be the set of p.m.p. actions of Γ on X. By viewing $A(\Gamma, X, \mu)$ as a closed subspace of $\mathrm{Aut}(X, \mu)^\Gamma$, it is also naturally a Polish space.

For our purposes, we will also need a relative version of this topology. Towards this end, consider another standard probability space (Y, ν) and let $F(Y, X)$ denote the space of all measure preserving maps from Y to X, where we identify two such maps if they agree almost everywhere. This is naturally a Polish space since it can be identified with the space of all isometric embeddings from the measure algebra of X into the measure algebra Y sending $[\emptyset]_\mu$ to $[\emptyset]_\nu$, equipped with the pointwise convergence topology.

Let $F(\Gamma, Y, X)$ denote the set of all triples (ϕ, b, a), where $b \in A(\Gamma, Y, \mu)$, $a \in A(\Gamma, X, \mu)$, and ϕ is a factor map from b to a. Then $F(\Gamma, Y, X)$ is a closed subspace of $F(Y, X) \times A(\Gamma, Y, \nu) \times A(\Gamma, X, \mu)$ hence is a Polish space. For a fixed action $a \in A(\Gamma, X, \mu)$, let $F_a(\Gamma, Y, X)$ denote the slice above a in $F(\Gamma, Y, X)$, which we identify with the set of all pairs $(\phi, b) \in F(Y, X) \times A(\Gamma, Y, \nu)$ where ϕ factors b onto a. Elements of $F_a(\Gamma, Y, X)$ are called **extensions of** a or a**-extensions**. Two a-extensions, $(\phi, b) \in F_a(Y, X)$ and $(\psi, c) \in F_a(X, Z)$ are **isomorphic** if there is a measure space isomorphism $S : Y \to Z$ such that $S \cdot (\phi, b) = (\psi, c)$, where $S \cdot (\phi, b) := (\phi S^{-1}, S \cdot b)$ and $\gamma^{S \cdot b} := S\gamma^b S^{-1}$.

Given finite collections $\mathcal{A}$ and $\mathcal{B}$ of measurable subsets of Y and X respectively, a finite subset F of Γ, and $\epsilon > 0$, we obtain a basic open neighborhood $U_a^{\mathcal{A}, \mathcal{B}, F, \epsilon}(\phi, b)$ of $(\phi, b) \in F_a(\Gamma, Y, X)$ consisting of all pairs $(\psi, c) \in F_a(\Gamma, Y, X)$ satisfying, for all $A \in \mathcal{A}$, $B \in \mathcal{B}$, and $\gamma \in F$:

$$\nu(\phi^{-1}A \triangle \psi^{-1}A) < \epsilon \text{ and } \nu(\gamma^b B \triangle \gamma^c B) < \epsilon.$$

2.3. **Ultraproducts of p.m.p. actions.** Suppose that $(\Gamma \curvearrowright^{a_i} (X_i, \mu_i))_{i \in I}$ is a family of p.m.p. actions of Γ and $\mathcal{U}$ is an ultrafilter on I. Let $\mathcal{M}_{a_i} \models T_\Gamma$ denote the L_Γ-structure associated to the action a_i. Then we can take the model-theoretic ultraproduct $\prod_\mathcal{U} \mathcal{M}_{a_i}$, which is necessarily an isometric action of Γ on the measure algebra associated to some p.m.p. action of Γ. This action is none other than the ultraproduct action $\Gamma \curvearrowright^{\prod_\mathcal{U} a_i} \prod_\mathcal{U}(X_i, \mu_i)$ (see, for example, [15]). More precisely, let $\prod_\mathcal{U} X_i$ denote the set-theoretic ultraproduct of the sets X_i and let μ_0 denote the finitely additive measure on the Boolean algebra $\mathcal{B}_0$ of subsets of $\prod_\mathcal{U} X_i$ of the form $\prod_\mathcal{U} A_i$, where each $A_i \subseteq X_i$ is measurable. Define $N \subseteq \prod_\mathcal{U} X_i$ to be μ_0-null if, for every $\epsilon > 0$, there is $A \in \mathcal{B}_0$ such that $N \subseteq A$ and $\mu_0(A) < \epsilon$. Then set $\mathcal{B}$ to be the σ-algebra on $\prod_\mathcal{U} X_i$ generated by $\mathcal{B}_0$ and the σ-ideal of null sets and let μ be the unique extension of μ_0 to a probability measure on $\mathcal{B}$. Then the diagonal action of Γ on $\prod_{i \in I} X_i$ induces a p.m.p. action $\Gamma \curvearrowright^{\prod_\mathcal{U} a_i} (\prod_\mathcal{U} X_i, \mu)$ and the corresponding action on measure algebras is precisely the model $\prod_\mathcal{U} \mathcal{M}_{a_i}$.

In regards to ultraproducts, we adopt the following notation. Given a sequence $x_i \in \prod_{i \in I} X_i$, we let $[x_i]_{\mathcal{U}}$ denote the corresponding element of $\prod_{\mathcal{U}} X_i$. Similarly, given a sequence of measurable sets $A_i \subseteq X_i$, we write $[A_i]_{\mathcal{U}}$ instead of $\prod_{\mathcal{U}} A_i$. Note that every element in the measure algebra of $\prod_{\mathcal{U}}(X_i, \mu_i)$ can be represented by a set of the form $[A_i]_{\mathcal{U}}$.

In general, we note that an ultraproduct action is almost never standard (unless the ultrafilter is somewhat trivial, for example countably complete), but one can always take a separable elementary substructure of the model-theoretic ultraproduct and then the corresponding action will be standard.

When each $\Gamma \curvearrowright^{a_i} (X_i, \mu_i) = \Gamma \curvearrowright^{a} (X, \mu)$ for some common action $\Gamma \curvearrowright^{a} (X, \mu)$, we speak of the **ultrapower action** $\Gamma \curvearrowright^{a_{\mathcal{U}}} (X, \mu)_{\mathcal{U}}$ (corresponding to the model-theoretic ultrapower $\mathcal{M}_a^{\mathcal{U}}$) and the diagonal embedding $\mathcal{M}_a \hookrightarrow \mathcal{M}_a^{\mathcal{U}}$ corresponds to the diagonal factor map $X_{\mathcal{U}} \to X$.

Recall that if $\Gamma \curvearrowright^{a} (X, \mu)$ and $\Gamma \curvearrowright^{b} (Y, \nu)$ are p.m.p. actions of Γ, then a is **weakly contained** in b, denoted $a \preceq b$, if for any finitely many measurable sets $A_1, \ldots, A_n \subseteq X$, finite $F \subseteq \Gamma$, and $\epsilon > 0$, there are measurable sets $B_1, \ldots, B_n \subseteq Y$ such that $|\mu(\gamma^a A_i \cap A_j) - \nu(\gamma^b B_i \cap B_j)| < \epsilon$ for all $1 \leq i, j \leq n$ and $\gamma \in F$.

Fact 2.1. *If (X, μ) and (Y, ν) are standard, non-atomic probability spaces, the following are equivalent:*

(1) $a \preceq b$.
(2) *There is a sequence of actions $(c_n)_{n \in \mathbb{N}} \in A(\Gamma, X, \mu)$ such that $c_n \cong b$ (conjugacy of actions) and $a = \lim_n c_n$*
(3) *a is contained in some ultrapower $b_{\mathcal{U}}$ of b, that is, if a is a factor of $b_{\mathcal{U}}$.*
(4) *$\mathcal{M}_a$ is embeddable in an ultrapower $\mathcal{M}_b^{\mathcal{U}}$ of $\mathcal{M}_b$.*

For a discussion of the preceding fact, see [13, Subection 2.2].

Convention 2.2. In the remainder of this article, unless explicitly stated, $\mathcal{U}$ always denotes a nonprincipal ultrafilter on a countable index set I. (Many of the facts to follow work for nonprincipal ultrafilters on arbitrary index sets, but some of them might require further mild hypotheses on the ultrafilter such as countable incompleteness.)

2.4. **Cocycles.** We will discuss cocycles using three different frameworks: ergodic theory, group cohomology, and continuous model theory. We start with the ergodic theory perspective.

Fix an action $\Gamma \curvearrowright^{a} X$ and a Polish group G. A **cochain** for a is a measurable map $\sigma : \Gamma \times X \to G$. A **cocycle** for a is a cochain $\sigma : \Gamma \times X \to G$ satisfying the cocycle identity $\sigma(\gamma_1\gamma_2, x) = \sigma(\gamma_1, \gamma_2 x)\sigma(\gamma_2, x)$ for all $\gamma_1, \gamma_2 \in \Gamma$ and almost all $x \in X$. We let $Z^1(a, G)$ denote the collection of cocycles for a with values in G.

We say that two cocycles $\sigma_1, \sigma_2 : \Gamma \times X \to G$ of the action a are **cohomologous** if there is a measurable map $s : X \to G$ for which $\sigma_2(\gamma, x) = s(\gamma x)^{-1}\sigma_1(\gamma, x)s(x)$ for all $\gamma \in \Gamma$ and for almost all $x \in X$. The set of cohomology classes of cocycles with values in G is called the **first cohomology space of** a **relative to** G, denoted $H^1(a, G)$.

The **trivial cocycle** for a sends each pair $(\gamma, x) \in \Gamma \times X$ to the identity $e_G \in G$. A cocycle for a cohomologous to the trivial cocycle is called a **coboundary** for a. The set of coboundaries for a with values in G is denoted by $B^1(a, G)$.

We consider $Z^1(a, G)$ as topologized so that $\sigma_n \to \sigma$ if and only if, for each $\gamma \in \Gamma$, we have that the functions $\sigma_n(\gamma, \cdot) : X \to G$ converge to $\sigma(\gamma, \cdot)$ in measure. We say that cocycles σ_1 and σ_2 are **approximately equivalent** if σ_1 belongs to the closure of the cohomology class of σ_2. In other words, σ_1 and σ_2 are approximately equivalent if and only if there are measurable functions $f_n : X \to G$ such that, for all $\gamma \in \Gamma$, we have that the functions $x \mapsto f_n(\gamma x)\sigma_1(\gamma, x)f_n(x)^{-1}$ converge in measure to $x \mapsto \sigma_2(\gamma, x)$.

In the case that (U, ρ) is a standard probability space and $G = \mathrm{Aut}(U, \rho)$, from a cocycle σ for a we can form the **skew-product extension** $X \times_\sigma (U, \rho)$, which is the p.m.p. action of Γ on $(X \times U, \mu \times \rho)$ given by $\gamma(x, u) := (\gamma x, \sigma(\gamma, x)u)$. (Our primary interest will be in the cases where (U, ρ) is either a finite set, equipped with its counting measure, in which case $\mathrm{Aut}(U, \rho)$ is simply S_n for some $n \in \mathbb{N}$, or where $U = [0, 1]$ and ρ is Lebesgue measure.) Note then that the projection map $X \times U \to X$ is a factor map. **Rohklin's skew-product theorem** asserts that every ergodic action that factors onto X is isomorphic to a skew-product extension of X (see, for example, [22]). We remark that when σ_1 and σ_2 are cohomologous, say with $s : X \to \mathrm{Aut}(U, \rho)$ measurable and satisfying $\sigma_2(\gamma, x) = s(\gamma x)^{-1}\sigma_1(\gamma, x)s(x)$ for all $\gamma \in \Gamma$ and for almost all $x \in X$, then the map

$$(x, u) \mapsto (x, s(x)^{-1}u) : X \times_{\sigma_1} U \to X \times_{\sigma_2} U$$

is an isomorphism of the associated skew-product extensions. In particular, when σ is a coboundary the corresponding skew-product extension is isomorphic to the product of a with the trivial action on U.

We remark that $\mathrm{Aut}([0, 1], \lambda)$, where λ is the Lebesgue measure, is a Polish group when equipped with the **weak topology**, which is the weakest topology for which all maps

$$T \mapsto [T(A)]_\mu : \mathrm{Aut}([0, 1], \lambda) \to \mathrm{MALG}([0, 1])$$

are continuous, as A ranges over Borel subsets of $[0, 1]$.

We now turn to the group cohomology perspective, which we use to access topological arguments that will streamline some of our proofs. Let G be any

Polish group. In addition to the usual left-shift action $\Gamma \curvearrowright^s G^\Gamma$ given by the rule

$$(\gamma^s \cdot y)(\delta) = y(\gamma^{-1}\delta),$$

we will also make use of the right-shift action $\Gamma \curvearrowright^r G^\Gamma$ given by the formula

$$(\gamma^r \cdot y)(\delta) = y(\delta\gamma).$$

For $y \in G^\Gamma$ and $\gamma \in \Gamma$ we will write y^γ in place of $\gamma^r \cdot y$ (this notation may require some care, since $(y^\gamma)^\delta = y^{\delta\gamma}$). Notice that G^Γ is a group with respect to coordinate-wise composition and that for every $\gamma \in \Gamma$ the map $y \in G^\Gamma \mapsto y^\gamma \in G^\Gamma$ is a group automorphism.

Define the space of 1-cochains $C^1(\Gamma, G^\Gamma)$ to be the set of all functions from Γ to G^Γ. Similarly, we define the space of 2-cochains $C^2(\Gamma, G^\Gamma)$ to be the set of all functions from $\Gamma \times \Gamma$ to G^Γ. We endow $C^1(\Gamma, G^\Gamma)$ and $C^2(\Gamma, G^\Gamma)$ with the product topologies obtained by identifying them with the sets $G^{\Gamma\times\Gamma}$ and $G^{\Gamma\times\Gamma\times\Gamma}$, respectively.

The coboundary map $\partial : C^1(\Gamma, G^\Gamma) \to C^2(\Gamma, G^\Gamma)$ is given by the formula

$$\partial c(\alpha, \beta) = c(\alpha\beta)^{-1} c(\beta)^\alpha c(\alpha).$$

Note that in the above equation $\partial c(\alpha, \beta)$ is an element of G^Γ and that the right-hand side is a product of elements of G^Γ. We call $c \in C^1(\Gamma, G^\Gamma)$ a cocycle if $\partial c(\alpha, \beta)(\gamma) = e_G$ for all $\alpha, \beta, \gamma \in \Gamma$, and we denote by $Z^1(\Gamma, G^\Gamma)$ the set of all cocycles.

We define actions of Γ on $C^1(\Gamma, G^\Gamma)$ and $C^2(\Gamma, G^\Gamma)$ via the formulas

$$(\alpha^t \cdot c_1)(\beta)(\gamma) = c_1(\beta)(\alpha^{-1}\gamma)$$
$$(\alpha^t \cdot c_2)(\beta, \gamma)(\delta) = c_2(\beta, \gamma)(\alpha^{-1}\delta)$$

for $c_1 \in C^1(\Gamma, G^\Gamma)$, $c_2 \in C^2(\Gamma, G^\Gamma)$ and $\alpha, \beta, \gamma, \delta \in \Gamma$. Equivalently, these actions are described in terms of the left-shift action $\Gamma \curvearrowright^s G^\Gamma$ via the formulas

$$(\alpha^t \cdot c_1)(\beta) = \alpha^s \cdot (c_1(\beta))$$
$$(\alpha^t \cdot c_2)(\beta, \gamma) = \alpha^s \cdot (c_2(\beta, \gamma)).$$

Using the fact that the left and right shift actions of Γ on G^Γ commute we have that for $c \in C^1(\Gamma, G^\Gamma)$ and $\alpha, \beta, \gamma \in \Gamma$

$$(\gamma^t \cdot c)(\beta)^\alpha = (\gamma^s \cdot (c(\beta)))^\alpha = \gamma^s \cdot (c(\beta)^\alpha).$$

Therefore

$$\begin{aligned}\partial(\gamma^t \cdot c)(\alpha, \beta) &= (\gamma^t \cdot c)(\alpha\beta)^{-1} \cdot (\gamma^t \cdot c)(\beta)^\alpha \cdot (\gamma^t \cdot c)(\alpha) \\ &= (\gamma^s \cdot (c(\alpha\beta)))^{-1} \cdot (\gamma^s \cdot (c(\beta)^\alpha)) \cdot (\gamma^s \cdot (c(\alpha))) \\ &= \gamma^s \cdot (c(\alpha\beta)^{-1} c(\beta)^\alpha c(\alpha)) \\ &= \gamma^s \cdot (\partial c(\alpha, \beta))\end{aligned}$$

$$= (\gamma^t \cdot \partial c)(\alpha, \beta).$$

As a result, we have that the coboundary map ∂ is Γ-equivariant. This additionally implies that $Z^1(\Gamma, G^\Gamma)$ is Γ-invariant.

Lemma 2.3. *Let $\Gamma \curvearrowright^a (X, \mu)$ be a p.m.p. action and let G be a Polish group. Then there is a one-to-one correspondence between measurable cochains $\theta : \Gamma \times X \to G$ and measurable equivariant maps $c : X \to C^1(\Gamma, G^\Gamma)$ given by the relations*

$$c(x)(\beta)(\alpha) = \theta(\beta^{-1}, (\alpha^{-1})^a \cdot x) \quad \textit{and} \quad \theta(\gamma, x) = c(x)(\gamma^{-1})(e). \tag{1}$$

Moreover, this correspondence produces a bijection between the measurable cocycles $\sigma : \Gamma \times X \to G$ and measurable equivariant maps $z : X \to Z^1(\Gamma, G^\Gamma)$.

Proof. It is immediately apparent that if $\theta : \Gamma \times X \to G$ is any measurable map (that is, a cochain) then the function $c : X \to C^1(\Gamma, G^\Gamma)$ given by the left-hand side of (1) is measurable and that θ is recovered from c by the formula in the right-hand side of (1). Additionally, when c is obtained from θ using the left side of (1), c is automatically equivariant since

$$c(\gamma^a \cdot x)(\beta)(\alpha) = \theta(\beta^{-1}, (\alpha^{-1}\gamma)^a \cdot x) = c(x)(\beta)(\gamma^{-1}\alpha) = (\gamma^t \cdot c(x))(\beta)(\alpha).$$

Conversely, given any measurable equivariant map $c : X \to C^1(\Gamma, G^\Gamma)$, if θ is defined using the right side of (1) then the equivariance of c implies that

$$\theta(\beta^{-1}, (\alpha^{-1})^a \cdot x) = c((\alpha^{-1})^a \cdot x)(\beta)(e) = ((\alpha^{-1})^t \cdot c(x))(\beta)(e) = c(x)(\beta)(\alpha),$$

and thus c is recovered from θ using the left side of (1). Finally, for any $x \in X$ we have that $c(x) \in Z^1(\Gamma, G^\Gamma)$ if and only if $c(x)(\alpha\beta)(\gamma)$ is equal to

$$c(x)(\beta)^\alpha(\gamma)c(x)(\alpha)(\gamma) = c(x)(\beta)(\gamma\alpha)c(x)(\alpha)(\gamma)$$

for all $\alpha, \beta, \gamma \in \Gamma$. When c and θ are related according to the formulas in (1), this is equivalent to the requirement that $\theta(\beta^{-1}\alpha^{-1}, (\gamma^{-1})^a \cdot x)$ is equal to $\theta(\beta^{-1}, (\alpha^{-1}\gamma^{-1})^a \cdot x)\theta(\alpha^{-1}, (\gamma^{-1})^a \cdot x)$ for all $\alpha, \beta, \gamma \in \Gamma$. Thus $c(x) \in Z^1(\Gamma, G^\Gamma)$ for μ-almost-every x if and only if θ is a cocycle. ☐

Lastly, we discuss the model-theoretic perspective on cocycles, exclusively looking at the case where $G = K$ is a finite group. In what follows, we use the terminology around definability presented in the first author's article [24]. For any finite group K, we may assume that we have a sort S_K in our langugage, whose intended interpretation in $\mathcal{M}_a$ is simply the Cartesian product $\mathcal{M}_a^{\Gamma \times K}$ (equipped with some fixed compatible complete metric), and that we have function symbols $\pi_{\gamma,k}$ for each $\gamma \in \Gamma$ and $k \in K$ whose intended interpretations are the projections maps $\pi_{\gamma,k}^{\mathcal{M}_a} : \mathcal{M}_a^{\Gamma \times K} \to \mathcal{M}_a$. We identify an element $B \in \mathcal{M}_a^{\Gamma \times K}$ with the cochain $\sigma_B : \Gamma \times X \to K$ for which $\pi_{\gamma,k}(B) = \{x \in X : \sigma_B(\gamma, x) = k\}$. Conversely, given a cochain $\sigma : \Gamma \times X \to K$, we let $B_\sigma \in \mathcal{M}_a^{\Gamma \times K}$ denote the corresponding tuple.

Fix bijections $e : \Gamma \to \mathbb{N}$ and $f : \Gamma \times \Gamma \times K \to \mathbb{N}$. We define the T_Γ-formula Cocy_K whose interpretations are given by $\mathrm{Cocy}_K^{\mathcal{M}_a} : \mathcal{M}_a^{\Gamma \times K} \to [0, 1]$, where

$$\mathrm{Cocy}_K^{\mathcal{M}_a}(B) = \max(\Phi_1(B), \Phi_2(B)),$$

with $\Phi_1(B)$ the T_Γ-formula

$$\sum_{\gamma \in \Gamma} 2^{-e(\gamma)} d((B(\gamma, k))_{k \in K}, \mathrm{Part}_K)$$

and $\Phi_2(B)$ the T_Γ-formula

$$\sum_{(\gamma,\delta,k) \in \Gamma \times \Gamma \times K} 2^{-f(\gamma,\delta,k)} d\left(\bigcup_{h \in K} (\delta^{-1} B(\gamma, h) \cap B(\delta, h^{-1}k)), B(\gamma\delta, k)\right).$$

In the definition of Φ_1, Part_K denotes the T_Γ-definable set given by partitions of the measure algebra indexed by K. We note that $\mathrm{Cocy}_K^{\mathcal{M}_a}(B) = 0$ if and only if σ_B is a cocycle of the action a. In other words, the T_Γ-functor corresponding to K-valued cocycles is a T_Γ-zeroset that we denote $Z(\mathrm{Cocy}_K^{\mathcal{M}_a})$.

2.5. **Treeability notions for groups.** Consider a countable group Γ and a p.m.p. action $\Gamma \curvearrowright^a (X, \mu)$. Let $\mathcal{R}_a$ be the equivalence relation on X given by the orbits. We call a set $\mathcal{G} \subseteq \mathcal{R}_a$ a **directed measurable graph** if $\mathcal{G}$ is a measurable, antisymmetric, irreflexive subset of $\mathcal{R}_a$. We view such a $\mathcal{G}$ as a collection of edges on the vertices X, and we write $\mathcal{R}_\mathcal{G}$ for the equivalence relation given by the connected components of $\mathcal{G}$ (ignoring edge direction). A **graphing** of $\mathcal{R}_a$ is a measurable directed graph $\mathcal{G} \subseteq \mathcal{R}_a$ having the property that $\mathcal{R}_\mathcal{G} = \mathcal{R}_a$. A **treeing** of $\mathcal{R}_a$ is a graphing of R_a having no cycles. The action $\Gamma \curvearrowright^a (X, \mu)$ is called **treeable** if $\mathcal{R}_a$ admits some treeing. Similarly, if for every finite set $H \subseteq \Gamma$ and every $\epsilon > 0$ there is a measurable directed graph $\mathcal{G} \subseteq \mathcal{R}_a$ that has no cycles and satisfies

$$\mu(\{x \in X : (h_1^a \cdot x, h_2^a \cdot x) \in \mathcal{R}_\mathcal{G} \text{ for all } h_1, h_2 \in H\}) > 1 - \epsilon,$$

then the action $\Gamma \curvearrowright^a (X, \mu)$ is called **approximately treeable**. Note that treeable actions are approximately treeable.

The group Γ is called **strongly treeable** if every free p.m.p. action of Γ is treeable, and it is called **treeable** if at least one free p.m.p. action of Γ is treeable. More generally, Γ is called **approximately treeable** if at least one free p.m.p. action of Γ is approximately treeable.

Amenable groups, free groups, finitely generated groups admitting planar Cayley graphs, elementarily free groups (that is, groups with the same first-order theory as a nonabelian free group), and the group of isometries of the hyperbolic plane and all its closed subgroups are examples of strongly treeable groups

[14, 21]. It is a prominent open question whether there is a group that is treeable but not strongly treeable. As we will see in Section 7, the class of approximately treeable groups includes all treeable groups and all universally free groups and is closed under increasing unions and extensions by amenable groups.

While the above are the conventional definitions for these properties, we find it more convenient to work with alternative characterizations that we now introduce.

Let d be the diagonal translation action of Γ on $\Gamma \times \Gamma$, specifically $\gamma^d \cdot (\alpha, \beta) = (\gamma\alpha, \gamma\beta)$. We similarly denote by d the action of Γ on the space $\mathrm{Pow}(\Gamma \times \Gamma)$ of all subsets of $\Gamma \times \Gamma$, so that $(\alpha, \beta) \in A \Leftrightarrow (\gamma\alpha, \gamma\beta) \in \gamma^d \cdot A$. We endow $\mathrm{Pow}(\Gamma \times \Gamma)$ with the topology given by pointwise convergence of indicator functions. For $S \in \mathrm{Pow}(\Gamma \times \Gamma)$ write $\bar{S}$ for the symmetric reflection of S, that is,

$$\bar{S} = \{(\beta, \alpha) \in \Gamma \times \Gamma : (\alpha, \beta) \in S\}.$$

Let $\mathcal{F}(\Gamma) \subseteq \mathrm{Pow}(\Gamma \times \Gamma)$ denote the space of all directed forests on Γ, specifically, $F \subseteq \Gamma \times \Gamma$ belongs to $\mathcal{F}(\Gamma)$ precisely when $F \cap \bar{F} = \varnothing$ and $F \cup \bar{F}$ is an acyclic graph on Γ. Notice in this case that $\bar{F}$ is the same graph but with the direction of all edges reversed. We also let $\mathcal{T}(\Gamma) \subseteq \mathcal{F}(\Gamma)$ denote the space of all directed trees with vertex set Γ.

Let $\mathcal{E}(\Gamma) \subseteq \mathrm{Pow}(\Gamma \times \Gamma)$ be the space of all equivalence relations on Γ. For $F \in \mathcal{F}(\Gamma)$ we write $E_F \in \mathcal{E}(\Gamma)$ for the equivalence relation given by the connected components of F (ignoring the direction of the edges), and for $E \in \mathcal{E}(\Gamma)$ we write Γ/E for the set of E-equivalence classes. Notice that d provides an action of Γ on both $\mathcal{F}(\Gamma)$ and $\mathcal{E}(\Gamma)$, and that $\gamma^d \cdot E_F = E_{\gamma^d \cdot F}$ for $\gamma \in \Gamma$ and $F \in \mathcal{F}(\Gamma)$.

Given a p.m.p. action $\Gamma \curvearrowright^a (X, \mu)$, there is a one-to-one correspondence between measurable sets $\mathcal{Y} \subseteq \mathcal{R}_a$ and equivariant measurable maps $\phi : X \to \mathrm{Pow}(\Gamma \times \Gamma)$ given by the rule

$$(\alpha, \beta) \in \phi(x) \Leftrightarrow ((\alpha^{-1})^a \cdot x, (\beta^{-1})^a \cdot x) \in \mathcal{Y}.$$

This correspondence identifies measurable equivalence subrelations $\mathcal{R}$ of $\mathcal{R}_a$ with equivariant measurable maps to $\mathcal{E}(\Gamma)$, identifies treeings $\mathcal{G} \subseteq \mathcal{R}_a$ with measurable equivariant maps to $\mathcal{T}(\Gamma)$, and identifies directed measurable graphs $\mathcal{G} \subseteq \mathcal{R}_a$ having no cycles with equivariant measurable maps to $\mathcal{F}(\Gamma)$. Additionally, when $\mathcal{G} \subseteq \mathcal{R}_a$ is a measurable directed graph having no cycles, if we let $\phi_{\mathcal{G}}$ and $\phi_{\mathcal{R}_{\mathcal{G}}}$ denote the maps associated with $\mathcal{G}$ and $\mathcal{R}_{\mathcal{G}}$, respectively, we have that $E_{\phi_{\mathcal{G}}(x)} = \phi_{\mathcal{R}_{\mathcal{G}}}(x)$ for all $x \in X$.

From these observations we obtain the following characterizations:

Proposition 2.4. *A group Γ is:*

(1) *treeable if and only if there exists a* Γ*-invariant Borel probability measure on* $\mathcal{T}(\Gamma)$*;*
(2) *strongly treeable if and only if: for every free p.m.p. action* $\Gamma \curvearrowright^a (X, \mu)$*, there is a* Γ*-invariant Borel probability measure* ν *on* $\mathcal{T}(\Gamma)$ *such that* $\Gamma \curvearrowright^a (X, \mu)$ *factors onto* $\Gamma \curvearrowright^d (\mathcal{T}(\Gamma), \nu)$*;*
(3) *approximately treeable if and only if: for every finite set* $H \subseteq \Gamma$ *and every* $\epsilon > 0$*, there is a* Γ*-invariant measure* μ *on* $\mathcal{F}(\Gamma)$ *satisfying*

$$\mu(\{F \in \mathcal{F}(\Gamma) : H \times H \not\subseteq E_F\}) < \epsilon.$$

Proof. The only implication not obvious from the above discussion is the "if" direction of (1); to see this, take the direct product of $\Gamma \curvearrowright^d (\mathcal{T}(\Gamma), \nu)$ with any free p.m.p. action of Γ, let ϕ be the projection map to $\mathcal{T}(\Gamma)$, and apply the above correspondence. □

We write $\mathrm{Prob}(\mathrm{Pow}(\Gamma \times \Gamma))$ for the space of all Borel probability measures on $\mathrm{Pow}(\Gamma\times\Gamma)$, equipped with the weak* topology, and write $\mathrm{Prob}(\mathcal{F}(\Gamma))$ and $\mathrm{Prob}(\mathcal{T}(\Gamma))$ for the subspaces of probability measures on $\mathcal{F}(\Gamma)$ and $\mathcal{T}(\Gamma)$, respectively (equipped with the subspace topoloogy).

Corollary 2.5. *A group* Γ *is approximately treeable if and only if: for every weak* open neighborhood* U *of the point-mass* $\delta_{\Gamma\times\Gamma} \in \mathrm{Prob}(\mathcal{E}(\Gamma))$*, there is an invariant Borel probability measure* μ *on* $\mathcal{F}(\Gamma)$ *so that the pushforward of* μ *under the map* $F \in \mathcal{F}(\Gamma) \mapsto E_F \in \mathcal{E}(\Gamma)$ *belongs to* U*.*

3. Existentially closed actions

In this section, we gather a number of basic facts about e.c. p.m.p.actions of Γ. We also consider study the Rokhlin entropy of e.c. actions and cocycles on e.c. actions.

3.1. **Definitions, first properties, and a useful reformulation.**

Definition 3.1. Given actions $\Gamma \curvearrowright^a X$ and $\Gamma \curvearrowright^b Y$ with $\mathcal{M}_a \subseteq \mathcal{M}_b$, we say that $\mathcal{M}_a$ is **existentially closed** (or **e.c.** for short) **in** $\mathcal{M}_b$ if: for any quantifier-free L_Γ-formula $\varphi(x, y)$, where x and y are finite tuples of variables, and any $c \in \mathcal{M}_a$ of the same length as x, we have

$$\left(\inf_y \varphi(c, y)\right)^{\mathcal{M}_a} = \left(\inf_y \varphi(c, y)\right)^{\mathcal{M}_b}.$$

We say that $\mathcal{M}_a$ is **existentially closed** (or e.c.) if it is existentially closed in $\mathcal{M}_b$ whenever $\mathcal{M}_a \subseteq \mathcal{M}_b$.

In the context of the above definition, considering the dual situation of a factor map $Y \to X$, we may also say that the factor map $Y \to X$ is existentially closed if the corresponding inclusion of models of T_Γ is existentially closed.

The following is a useful, well-known "logic-free" characterization of e.c. inclusions:

Fact 3.2. *The inclusion $\mathcal{M}_a \subseteq \mathcal{M}_b$ is e.c. if and only if there is an ultrafilter $\mathcal{U}$ and an embedding $\iota : \mathcal{M}_b \to \mathcal{M}_a^{\mathcal{U}}$ such that the restriction $\iota|\mathcal{M}_a$ of ι to $\mathcal{M}_a$ is the usual diagonal embedding $\mathcal{M}_a \hookrightarrow \mathcal{M}_a^{\mathcal{U}}$. Dually, the factor map $Y \to X$ is e.c. if and only if there is a factor map $X_{\mathcal{U}} \to Y$ such that the composite factor map $X_{\mathcal{U}} \to Y \to X$ is the usual diagonal factor map.*

The following is a well-known fact about the existence of e.c. actions, relativized to our current setting; see, for example, [41, Fact 2.8].

Fact 3.3. *E.c. actions exist. In fact, given any action $\Gamma \curvearrowright^a X$, there is an e.c. action $\Gamma \curvearrowright^b Y$ with $\mathcal{M}_a \subseteq \mathcal{M}_b$. Moreover, if X is standard, then Y can also be taken to be standard.*

We record the following immediate consequence of the definition of e.c. actions:

Lemma 3.4. *Suppose that $\Gamma \curvearrowright^a X$ is an e.c. action. Then X is an atomless probability space and a is a free action.*

Proof. Consider the factor map $X \times [0,1] \to X$, where Γ acts on $[0,1]$ in the trivial manner. Since this latter action is on an atomless space and being atomless can be axiomatized using existential axioms, the first statement follows from the definition of e.c. actions. (Alternatively, one can use the characterization in Fact 3.2 to find a factor map $X_{\mathcal{U}} \to X \times [0,1]$ such that the composite map $X_{\mathcal{U}} \to X \times [0,1] \to X$ is the diagonal map; this immediately implies that X is atomless.) Since there is also a factor map $Y \to X$ with Y a free action, the same reasoning, together with the fact that being a free action is axiomatizable by existential conditions (see Subsection 2.1), implies that a is a free action. □

We end this subsection with a useful, ergodic-theoretic reformulation for a factor map to be e.c. that will be used throughout the paper. First, we introduce the following notation. Throughout this paper, we view natural numbers as ordinals, that is, $p = \{0, 1, \ldots, p-1\}$ for every natural number p.

Notation 3.5. If Γ acts on a set X, $\alpha : X \to p$ is a function, and $F \subseteq \Gamma$ is a subset, then we define the function $\alpha_F : X \to p^F$ by $\alpha_F(x)(f) = \alpha(f^{-1} \cdot x)$.

Proposition 3.6. *The factor map $\phi : Y \to X$ is e.c. if and only if: for all $p, q \in \mathbb{N}$, measurable maps $\alpha : Y \to p$, $\beta : X \to q$, finite $S \subseteq \Gamma$, and $\epsilon > 0$, there is a measurable* *map* $\tilde{\alpha} : X \to p$ *such that*

$$\left|\nu(\alpha_S^{-1}(\pi) \cap \phi^{-1}(\beta^{-1}(j)) - \mu(\tilde{\alpha}_S^{-1}(\pi) \cap \beta^{-1}(j))\right| < \epsilon$$

for all $\pi \in p^S$ *and* $j \in q$*. Furthermore, if* $\mathcal{A}_n$ *is an increasing sequence of algebras whose union is dense in* $\mathcal{B}_X$*, then it suffices to verify for each* n *that the above condition holds for every* $\mathcal{A}_n$*-measurable map* $\beta : X \to q$*.*

Proof. This is simply translating definitions from one framework to another. Specifically, in place of discussing tuples of sets as done in the formal definition above, one can instead discuss the finitely many atoms of the Boolean algebra that those sets generate. For the final statement, each measurable map β can be approximated by maps β_n where β_n is $\mathcal{A}_n$-measurable. We leave the details to the reader. □

3.2. **E.c. actions are maximal with respect to weak containment.** Recall from Subsection 2.3 above that the action $\Gamma \curvearrowright^a X$ is weakly contained in the action $\Gamma \curvearrowright^b Y$ if $\mathcal{M}_a$ embeds into an ultrapower $\mathcal{M}_b^\mathcal{U}$ of $\mathcal{M}_b^\mathcal{U}$. Recalling the statement of Fact 3.2, we immediately see:

Lemma 3.7. *If* $\mathcal{M}_a$ *is e.c. in* $\mathcal{M}_b$*, then* b *is weakly contained in* a*.*

In the study of weak containment of p.m.p. actions of countable groups, special attention has been paid to actions a that are **maximal for weak containment**, that is, every other p.m.p. action is weakly contained in a; for example, see the wonderful survey article [13, Section 5]. Following nomenclature recently used in the model theory of operator algebras (see [17]), we might also sometimes refer to such actions as **locally universal**. As mentioned in [13], locally universal actions always exist. (This is a special case of a much more general fact, as argued in [17].) The proof of Lemma 3.4 shows also that locally universal actions are free. We note the following well-known result, relativized to our current setting:

Lemma 3.8. *If* $\mathcal{M}_a$ *is an e.c. model of* T_Γ*, then* a *is a locally universal action.*

Proof. Let $\Gamma \curvearrowright^b Y$ be any action of Γ. Then since $\mathcal{M}_a \subseteq \mathcal{M}_{a\times b}$, by Lemma 3.7, we see that $a \times b$ is weakly contained in a, whence b is also weakly contained in a. It follows that a is a locally universal action. □

For certain groups, "concrete" examples of locally universal actions are known. In particular, for so-called **EMD** groups (see Subsection 4.1 for the definition), the profinite completion action is locally universal. In Subsection 4.1 below, we will generalize this fact by showing that the profinite completion action is an e.c. action when the group has property EMD.

Lemma 3.9. *For each group Γ, there is a theory $T_{\Gamma,\max}$ whose models are precisely the locally universal actions of Γ.*

Proof. Fix a locally universal action $\Gamma \curvearrowright^a (X,\mu)$ of Γ. Let T_Γ denote the set of existential sentences true in $\mathcal{M}_a$. Since any two locally universal actions satisfy the same existential sentences, we have that all locally universal actions model T_Γ. Conversely, if $\mathcal{M}_b$ is a model of $T_{\Gamma,\max}$, then $\mathcal{M}_a$ is a model of the universal theory of $\mathcal{M}_b$, whence $\mathcal{M}_a$ embeds in an ultrapower of $\mathcal{M}_b$; since $\mathcal{M}_a$ is locally universal, so is $\mathcal{M}_b$. □

3.3. **Rokhlin entropy of e.c. actions.** In this subsection, we prove an interesting result about the Rohklin entropy of e.c. actions of non-amenable groups on standard spaces. This result will not be used in the remainder of the paper.

Suppose that $\Gamma \curvearrowright (X,\mu)$ is an aperiodic p.m.p. action on a standard Borel probability space. For a countable Borel partition α, we denote by $H(\alpha)$ the **Shannon entropy of** α:

$$H(\alpha) = \sum_{A\in\alpha} -\mu(A)\log\mu(A),$$

where we use the convention that $0\log 0 = 0$. Similarly, when β is a countable Borel partition of X satisfying $H(\beta) < \infty$ and $\mathcal{F}$ is a σ-algebra of Borel sets, the **relative Shannon entropies** are defined as

$$H(\alpha\,|\,\beta) = H(\alpha\vee\beta) - H(\beta)$$
$$H(\alpha\,|\,\mathcal{F}) = \inf_{\beta\subseteq\mathcal{F}} H(\alpha\,|\,\beta),$$

where the infimum in the second line is over all finite partitions $\beta \subseteq \mathcal{F}$.

Write $\sigma\text{-alg}_\Gamma(\alpha)$ for the smallest Γ-invariant σ-algebra containing α. For any collection $\mathcal{C}$ of Borel susets of X and any Γ-invariant σ-algebra $\mathcal{F}$ of Borel sets, the **Rokhlin entropy** $h_\Gamma(\mathcal{C}\mid\mathcal{F})$ **of** $\mathcal{C}$ **relative to** $\mathcal{F}$ is defined to be the infimum of $H(\alpha|\mathcal{F})$, as α ranges over all countable Borel partitions of X satisfying $\sigma\text{-alg}_\Gamma(\alpha)\vee\mathcal{F}\supseteq\mathcal{C}$. When $\mathcal{F} = \{\emptyset, X\}$ is trivial, we write $h_\Gamma(\mathcal{C})$ in place of $h_\Gamma(\mathcal{C}\mid\mathcal{F})$. The **Rokhlin entropy of** $\Gamma \curvearrowright (X,\mu)$, denoted $h_\Gamma(X,\mu)$, is defined to be $h_\Gamma(\mathcal{B}(X))$, where $\mathcal{B}(X)$ is the Borel σ-algebra of X.

Rokhlin entropy was introduced by the second author in 2019 [39] and is one of two extensions of the classical Kolmogorov–Sinai entropy theory for actions of countable amenable groups. Specifically, in the case that Γ is amenable and the action is free, Rokhlin entropy coincides with Kolmogorov–Sinai entropy [2, Cor. 1.9]. The other extension of Kolmogorov–Sinai entropy is sofic entropy, which was introduced by Lewis Bowen in 2010 [8]. Although sofic entropy is more practical to compute and has been studied in greater depth, we work with Rokhlin entropy here because it has the advantage of being defined for actions of all countable groups (sofic entropy is defined only for actions of sofic groups). Additionally, Rokhlin entropy is an upper bound to sofic entropy whenever the latter is defined [2, Prop. 1.10] (see also [8, Prop. 5.3]), which means that the proposition below automatically provides the optimal result for sofic entropy as well.

Proposition 3.10. *Let $\Gamma \curvearrowright (X, \mu)$ be a p.m.p. action on a standard Borel probability space. If the action is e.c. and Γ is non-amenable, then $h_\Gamma(X, \mu) = 0$.*

Proof. Let $(\alpha_n)_{n\in\mathbb{N}}$ be an increasing sequence of finite Borel partitions whose union generates the entire Borel σ-algebra on X. Since Rokhlin entropy is countably subadditive [2, Corollary 1.5], we have

$$h_\Gamma(X, \mu) \leq \sum_{n\in\mathbb{N}} h_\Gamma(\alpha_n).$$

Thus it suffices to show that $h_\Gamma(\alpha) = 0$ for every finite Borel partition α of X.

Fix a finite Borel partition α of X. Since Γ is non-amenable, by a result of Bowen [9] there exists a free p.m.p. action $\Gamma \curvearrowright (Y, \nu)$ with $h_\Gamma(Y, \nu) = 0$ for which there is a factor map $\phi : Y \to X$. Fix $\epsilon > 0$ and pick a countable Borel partition β of Y satisfying $H(\beta) < \epsilon$ and $\sigma\text{-alg}_\Gamma(\beta) = \mathcal{B}(Y)$. Since $\phi^{-1}(\alpha) \subseteq \sigma\text{-alg}_\Gamma(\beta)$ we have

$$0 = H(\phi^{-1}(\alpha) \mid \sigma\text{-alg}_\Gamma(\beta)) = \inf_{\substack{F\subseteq\Gamma \\ F \text{ finite}}} H\left(\phi^{-1}(\alpha) \,\middle|\, \bigvee_{g\in F} g\cdot\beta\right),$$

where the second equality is a basic property of Shannon entropy (see [16, Lemma 1.7.11]). So there is a finite set $F \subseteq \Gamma$ with $H(\phi^{-1}(\alpha) \mid \bigvee_{g\in F} g\cdot\beta) < \epsilon$. Since $\Gamma \curvearrowright (X, \mu)$ is existentially closed, there must exist a finite Borel partition β' of X satisfying $H(\beta') < \epsilon$ and $H(\alpha \mid \bigvee_{g\in F} g\cdot\beta') < \epsilon$. Consequently, by subadditivity of Rokhlin entropy,

$$h_\Gamma(\alpha) \leq h_\Gamma(\beta') + h_\Gamma(\alpha \mid \sigma\text{-alg}_\Gamma(\beta')) \leq H_\mu(\beta') + H_\mu\left(\alpha \,\middle|\, \bigvee_{g\in F} g\cdot\beta'\right) < 2\epsilon.$$

As ϵ was arbitrary, we conclude that $h_\Gamma(\alpha) = 0$. □

If Γ is an amenable group and $r \in (0, +\infty]$, then there exists a free p.m.p. action of Γ on a standard Borel probability space having Kolmogorov–Sinai entropy equal to r (for instance, any Bernoulli shift over Γ whose base space has Shannon entropy equal to r). Such an action would be e.c. since all free actions of amenable groups are e.c. and it would have Rokhlin entropy r since Rokhlin entropy and Kolmogorov–Sinai entropy coincide for free actions of amenable groups [2, Corollary 1.9]. The assumption in the above proposition that Γ be non-amenable is therefore necessary.

3.4. **Cocycles on e.c. actions.**

Lemma 3.11. *Suppose that* $\Gamma \curvearrowright^a (X, \mu)$ *is e.c. Then, for any finite group* K, $B^1(a, K)$ *is dense in* $Z^1(a, K)$.

Proof. Let Y be the skew-product extension $X \times_\sigma K$. If $p_0 : Y \to K$ is the projection map, then we have $p_0(\gamma \cdot (x, k))p_0(x, k)^{-1} = \sigma(\gamma, x)$ for all γ, x, and k. Thus, since $\Gamma \curvearrowright^a (X, \mu)$ is e.c. for any finite $F \subseteq \Gamma$ and $\epsilon > 0$, there is a map $p : X \to K$ such that

$$\mu(\{x \in X \ : \ p(\gamma x)p(x)^{-1} = \sigma(\gamma, x) \text{ for all } \gamma \in F\}) > 1 - \epsilon,$$

proving the lemma. □

The above fact can be strengthened in the case where Γ has property (T).

Corollary 3.12. *If* Γ *has property (T), then for any e.c. action* $\Gamma \curvearrowright^a (X, \mu)$ *and any finite group* K, *we have* $H^1(a, K) = 0$.

Proof. For ergodic actions of property (T) groups, its known that $B^1(a, K)$ is clopen in $Z^1(a, K)$ [38, Lemma 4.2] (alternatively see [18, Theorem 4.2]). If our e.c. action were ergodic we could combine this with Lemma 3.11 and be done. However, e.c. actions of property (T) groups are never ergodic, so we instead adapt the proof of [18, Theorem 4.2] to the non-ergodic setting.

Since Γ has property (T), there is a finite set $F \subseteq \Gamma$ and $\epsilon > 0$ such that for every unitary representation $\pi : \Gamma \to \mathcal{U}(\mathcal{H})$ and every unit vector ξ_0, if $|\langle \pi(\gamma)(\xi_0), \xi_0 \rangle| \geq 1 - \epsilon$ for every $\gamma \in F$ then there is a Γ-invariant unit vector $\xi \in \mathcal{H}$ with $\|\xi - \xi_0\| < 1/16$.

We claim that for every measurable Γ-invariant set $Y \subseteq X$ of positive measure and cocycles $\alpha, \beta : \Gamma \times X \to K$ satisfying, for every $\gamma \in F$:

$$\mu(\{x \in X : \alpha(\gamma^{-1}, x) \neq \beta(\gamma^{-1}, x)\}) \leq \epsilon\mu(Y)$$

there is a measurable Γ-invariant set $Z \subseteq Y$ and a measurable function $f : Z \to K$ satisfying $\mu(Z) \geq \mu(Y)/2$ and $\alpha(\gamma, x) = f(\gamma x)\beta(\gamma, x)f(x)^{-1}$ for a.e. $x \in Z$ and every $\gamma \in \Gamma$.

Assume for now that the claim holds. Set $X_0 = \emptyset$ and inductively assume that X_m has been defined for all $m \leq n$. If $\mu(\bigcup_{m \leq n} X_m) = 1$ then the induction can stop, but otherwise it proceeds as follows. By Lemma 3.11 $B^1(a, K)$ is dense in $Z^1(a, K)$, so we can pick a cocycle $\sigma_{n+1} \in B^1(a, K)$ satisfying, for every $\gamma \in F$:

$$\mu(\{x \in X : \sigma(\gamma^{-1}, x) \neq \sigma_{n+1}(\gamma^{-1}, x)\}) \leq \epsilon \cdot \mu\left(X \setminus \bigcup_{m \leq n} X_m\right).$$

Since σ_{n+1} is a coboundary, we can pick a measurable function $h_{n+1} : X \to K$ satisfying $\sigma_{n+1}(\gamma, x) = h_{n+1}(\gamma x)h_{n+1}(x)^{-1}$ for a.e. x and every γ. Next apply the claim of the previous paragraph to $Y = X \setminus \bigcup_{m \leq n} X_m$, $\alpha = \sigma$, and $\beta = \sigma_{n+1}$ to obtain a Γ-invariant measurable set $X_{n+1} \subseteq X \setminus \bigcup_{m \leq n} X_m$ with

$$\mu(X_{n+1}) \geq \frac{1}{2}\mu\left(X \setminus \bigcup_{m \leq n} X_m\right)$$

and a measurable function $f_{n+1} : X_{n+1} \to K$ such that

$$\sigma(\gamma, x) = f_{n+1}(\gamma x)\sigma_{n+1}(\gamma, x)f_{n+1}(x)^{-1}$$

for a.e. $x \in X_{n+1}$ and every $\gamma \in \Gamma$. Then the sets X_n will be pairwise disjoint and their union will be conull and the function $f : \bigcup_n X_n \to K$ defined by $f(x) = f_n(x)h_n(x)$ for $x \in X_n$ will satisfy $\sigma(\gamma, x) = f(\gamma x)f(x)^{-1}$ for a.e. x and every γ, implying that σ is a coboundary as desired.

We now prove the claim. Let Y, α and β be as described. Define an action of Γ on $Y \times K$ by $\gamma \cdot (x, k) = (\gamma x, \alpha(\gamma, x)k\beta(\gamma, x)^{-1})$. Set $\mathcal{H} = L^2(Y \times K, \frac{1}{\mu(Y)}\mu \times c)$ where c is the counting measure on K, and let $\pi : \Gamma \to \mathcal{U}(\mathcal{H})$ be the unitary representation $\pi(\gamma)(\eta)(x, k) = \eta(\gamma^{-1} \cdot (x, k))$. Let ξ_0 be the unit vector $1_{Y \times \{e_K\}}$ and observe that for $\gamma \in F$

$$|\langle \pi(\gamma)(\xi_0), \xi_0 \rangle| = \frac{1}{\mu(Y)}\mu(\{x \in Y : \alpha(\gamma^{-1}, x) = \beta(\gamma^{-1}, x)\}) \geq 1 - \epsilon.$$

It follows from our choice of F and ϵ that there is a Γ-invariant unit vector ξ satisfying $\|\xi - \xi_0\| < 1/16$.

Define Z to be the set of $x \in Y$ for which there is a unique $k \in K$ maximizing the value of $|\xi(x, k)|$ and, in this case, define $f(x)$ to be that unique element of K. Note that since $f^{-1}(k_0) = \bigcap_{k \in K \setminus \{k_0\}} \{x \in Y : |\xi(x, k)| < |\xi(x, k_0)|\}$ and $Z = \bigcup_{k_0 \in K} f^{-1}(k_0)$, both f and Z are measurable. The invariance of ξ tells us that

$$\xi(\gamma x, k) = \pi(\gamma^{-1})(\xi)(x, \alpha(\gamma, x)^{-1}k\beta(\gamma, x)) = \xi(x, \alpha(\gamma, x)^{-1}k\beta(\gamma, x)),$$

and since the map $k \in K \mapsto \alpha(\gamma, x)^{-1}k\beta(\gamma, x)$ is a permutation of K, we see that Z is Γ invariant and that $\alpha(\gamma, x)^{-1}f(\gamma x)\beta(\gamma, x) = f(x)$ for all $x \in Z$.

Finally, it only remains to check that $\mu(Z) \geq \mu(Y)/2$. Consider the sets

$$D_1 = \left\{x \in Y : |1 - \xi(x, e_K)|^2 \geq \frac{1}{4}\right\}$$

$$D_2 = \left\{x \in Y : \sum_{k \in K \setminus \{e_K\}} |\xi(x, k)|^2 \geq \frac{1}{4}\right\}.$$

If $x \in Y \setminus (D_1 \cup D_2)$ then $|\xi(x, e_K)|^2 \geq \frac{1}{4}$ while $\sum_{k \in K \setminus \{e_K\}} |\xi(x,k)|^2 < \frac{1}{4}$. So $Y \setminus (D_1 \cup D_2) \subseteq f^{-1}(e_K) \subseteq Z$. Since

$$\frac{1}{\mu(Y)} \int_Y \left(|1 - \xi(x, e_K)|^2 + \sum_{k \in K \setminus \{e_K\}} |\xi(x,k)|^2 \right) \, d\mu = \|\xi - \xi_0\|^2 < \frac{1}{16},$$

we have that $\mu(D_1) < \mu(Y)/4$ and $\mu(D_2) < \mu(Y)/4$ and therefore

$$\mu(Z) \geq \mu(Y \setminus (D_1 \cup D_2)) \geq \mu(Y)/2. \qquad \square$$

The previous corollary can be extended to all groups at the expense of restricting to e.c. actions which are ultraproducts.

Corollary 3.13. *If* $\Gamma \curvearrowright^{a_i} (X_i, \mu)$ *is a family of actions and the nonprincipal ultraproduct action* $\Gamma \curvearrowright^{a} (X, \mu)$ *is e.c. then* $H^1(a, K) = 0$ *for every finite group* K.

This immediately follows from Lemma 3.11 and the following general observation below.

Lemma 3.14. *Suppose that* $\Gamma \curvearrowright^{a} (X, \mu)$ *is a nonprincipal ultraproduct action. Then* $B^1(a, K)$ *is closed in* $Z^1(a, K)$ *for any finite group* K.

Proof. Suppose that $\mathcal{U}$ is a nonprincipal ultrafilter on a set I, $\Gamma \curvearrowright^{a_i} (X_i, \mu_i)$ is a p.m.p. action for every $i \in I$, $(X, \mu) = \prod_{\mathcal{U}} (X_i, \mu_i)$ and $a = \prod_{\mathcal{U}} a_i$. Let $\sigma : \Gamma \times X \to K$ belong to the closure of $B^1(a, K)$ in $Z^1(a, K)$. Choose an increasing sequence of finite sets $W_n \subseteq \Gamma$ with $\bigcup_{n \in \mathbb{N}} W_n = \Gamma$ and for each $n \in \mathbb{N}$ pick a cocycle $\sigma_n \in B^1(a, K)$ satisfying

$$\mu(S_{\gamma,k} \triangle S_{n,\gamma,k}\}) < 2^{-n} \text{ for all } \gamma \in W_n \text{ and } k \in K,$$

where $S_{\gamma,k} = \{x \in X : \sigma(\gamma, x) = k\}$ and $S_{n,\gamma,k} = \{x \in X : \sigma_n(\gamma, x) = k\}$.

For each $n \in \mathbb{N}$, pick a measurable function $f_n : X \to K$ satisfying $f_n(\gamma x) f_n(x)^{-1} = \sigma_n(\gamma, x)$ for a.e. $x \in X$ and every $\gamma \in \Gamma$. Choose measurable functions $f_n^i : X_i \to K$ with $f_n^{-1}(k) = [(f_n^i)^{-1}(k)]_{\mathcal{U}}$ for every $k \in K$ and define the cocycle $\sigma_n^i(\gamma, x) = f_n^i(\gamma x) f_n^i(x)^{-1}$. Set $S^i_{n,\gamma,k} = \{x \in X_i : \sigma_n^i(\gamma, x) = k\}$.

Choose measurable sets $S^i_{\gamma,k} \subseteq X_i$ satisfying $S_{\gamma,k} = [S^i_{\gamma,k}]_{\mathcal{U}}$ for all $\gamma \in \Gamma$ and $k \in K$. Since $\mathcal{U}$ is nonprincipal, we can fix a function $M : I \to \mathbb{N}$ satisfying $\lim_{\mathcal{U}} M(i) = \infty$. For each $i \in I$, define

$$m(i) = \max\{n \leq M(i) : \mu_i(S^i_{n,\gamma,k} \triangle S^i_{\gamma,k}) \leq 2^{-n+1} \text{ for all } \gamma \in W_n \text{ and } k \in K\}.$$

Since for every $n \in \mathbb{N}$ we have $\{i : M(i) \geq n\} \in \mathcal{U}$ and

$$\lim_{\mathcal{U}} \mu_i(S^i_{n,\gamma,k} \triangle S^i_{\gamma,k}) = \mu(S_{n,\gamma,k} \triangle S_{\gamma,k}) < 2^{-n} \text{ for all } \gamma \in W_n \text{ and } k \in K,$$

it is immediately seen that $\{i : m(i) \geq n\} \in \mathcal{U}$ and thus $\lim_{\mathcal{U}} m(i) = \infty$.

Since $\lim_{\mathcal{U}} m(i) = \infty$ and $\bigcup_n W_n = \Gamma$, we have that $\lim_{\mathcal{U}} \mu_i(S^i_{m(i),\gamma,k} \triangle S^i_{\gamma,k}) \leq \lim_{\mathcal{U}} 2^{-m(i)+1} = 0$ for every $\gamma \in \Gamma$ and $k \in K$. Therefore $S_{\gamma,k} = [S^i_{\gamma,k}]_{\mathcal{U}} = [S^i_{m(i),\gamma,k}]_{\mathcal{U}}$ for all $\gamma \in \Gamma$ and $k \in K$. Consequently,

$$\sigma(\gamma, [x_i]_{\mathcal{U}}) = \lim_{\mathcal{U}} \sigma^i_{m(i)}(\gamma, x_i) = \lim_{\mathcal{U}} f^i_{m(i)}(\gamma x_i) f^i_{m(i)}(x_i)^{-1},$$

and defining $h([x_i]_{\mathcal{U}}) = \lim_{\mathcal{U}} f^i_{m(i)}(x_i)$ we have $h : X \to K$ is measurable and $\sigma(\gamma, x) = h(\gamma x) h(x)^{-1}$ for a.e. x and all $\gamma \in \Gamma$. We conclude that $\sigma \in B^1(a, K)$. □

4. Special e.c. actions

In this section, we study when the profinite completion action is e.c. and when there is a weakly mixing e.c. action. In the process, we prove a fact of independent interest, namely that limit groups have Kechris' property MD.

4.1. **Profinite completions.**

In [5], the authors show that the natural action of a finitely generated free group $\mathbb{F}$ on its profinite completion $\widehat{\mathbb{F}}$ is an e.c. action. In this section, we generalize this result to the largest class of groups for which it could possibly hold.

First, recall that the profinite completion $\hat{\Gamma}$ of a countable residually finite group Γ is the inverse limit of the finite groups Γ/Λ as Λ varies over the normal finite-index subgroups of Γ. The profinite completion $\hat{\Gamma}$ is a compact group and thus admits a unique Haar probability measure $\mu_{\hat{\Gamma}}$. Γ naturally embeds into $\hat{\Gamma}$ and thus acting by left-translation yields an ergocdic p.m.p. action of Γ on $(\hat{\Gamma}, \mu_{\hat{\Gamma}})$. For each normal finite-index subgroup $\Lambda \lhd \Gamma$, the closure $\bar{\Lambda}$ of Λ in $\hat{\Gamma}$ is a finite-index clopen subgroup of $\hat{\Gamma}$, and the partition $\mathcal{C}_{\bar{\Lambda}}$ of $\hat{\Gamma}$ into its left $\bar{\Lambda}$-cosets is Γ-invariant (that is, Γ permutes the cosets). Moreover, there is a decreasing sequence Λ_n of finite-index normal subgroups of Γ such that the sequence $\mathcal{C}_{\bar{\Lambda}_n}$ separates points.

An action $\Gamma \curvearrowright^a (X, \mu)$ is called **profinite** if there is a decreasing sequence of finite Γ-invariant measurable partitions of X which separate points. Following Kechris [30], a residually finite group Γ is said to be **MD** if the set of profinite actions of Γ on (X, μ) is dense in the space $A(\Gamma, X, \mu)$ of all actions of Γ. The group Γ is said to have the a priori stronger property **EMD** if the set of ergodic profinite actions of Γ on (X, μ) is dense in $A(\Gamma, X, \mu)$. It is an open question whether or not the two notions coincide for all groups, but by work of the third author ([40, Corollary 4.7 and Theorem 4.10]), we have that they coincide for all groups without property (T) and that they coincide for all groups if and only if property

MD implies the negation of property (T). For examples and closure properties of these (somewhat mysterious) classes of groups, see [13, Section 5].

Kechris [30, Propositions 4.2, 4.5, and 4.8] showed that a group Γ has property EMD precisely when its action on its profinite completion $\hat{\Gamma}$ is locally universal while it has property MD precisely when its action on $\hat{\Gamma} \times [0, 1]$ is locally universal (where the action on the second coordinate is trivial). In this section, we show that for these classes of groups, the associated actions are in fact existentially closed. (Since being existentially closed implies being locally universal, these results are optimal.)

Lemma 4.1. *Let Γ be a countable group and let $\Lambda \leq \Gamma$ be a subgroup.*

(1) *If Γ has property MD, then Λ has MD as well.*
(2) *If Γ has property EMD and Λ has finite index in Γ, then Λ has EMD as well.*

Proof. (1) is observed in [30, Section 4]. Finally, (2) follows from (1) together with the following facts: no group with property (T) can have property EMD [30, Proposition 6]; for groups without property (T), EMD and MD are equivalent [40, Corollary 4.7]; and, since Λ is a finite index subgroup of Γ, Λ has property (T) if and only if Γ does. □

The following is the main result of this section:

Theorem 4.2. *Let Γ be a countable residually finite group and let $(\hat{\Gamma}, \mu_{\hat{\Gamma}})$ denote the profinite completion of Γ equipped with its normalized Haar probability measure. Finally, let λ denote Lebesgue measure on $[0, 1]$.*

(1) *If Γ has property EMD, then the action $\Gamma \curvearrowright (\hat{\Gamma}, \mu_{\hat{\Gamma}})$ is existentially closed.*
(2) *If Γ has property MD, then the action $\Gamma \curvearrowright (\hat{\Gamma} \times [0, 1], \mu_{\hat{\Gamma}} \times \lambda)$ is existentially closed.*

Proof. In case (1), set $(X, \mu) = (\hat{\Gamma}, \mu_{\hat{\Gamma}})$ and in case (2), set $(X, \mu) = (\hat{\Gamma} \times [0, 1], \mu_{\hat{\Gamma}} \times \lambda)$.

Let $\Gamma \curvearrowright (Y, \nu)$ be a p.m.p. action and $\phi : Y \to X$ a Γ-equivariant factor map. Also let $p \in \mathbb{N}$ and $\alpha : Y \to p$ be measurable. Let $S \subseteq \Gamma$ be finite with $e \in S$ and let $\epsilon > 0$.

Let H_n be the intersection of all subgroups of Γ having index at most n. Write $\bar{H}_n$ for the closure of H_n in $\hat{\Gamma}$. Let $\mathcal{B}_{[0,1]}$ be the Borel σ-algebra on $[0, 1]$. In case (1) set $\mathcal{A}_n = \{\gamma \bar{H}_n : \gamma \in \Gamma\}$ and in case (2) set $\mathcal{A}_n = \{\gamma \bar{H}_n \times B : \gamma \in \Gamma, \ B \in \mathcal{B}\}$. Then the $\mathcal{A}_n$'s are an increasing sequence of algebras whose union is dense in X. So the stronger form of Proposition 3.6 says that its enough to consider partitions of X which are $\mathcal{A}_n$-measurable for some n.

Fix n, set $H = H_n$ and $\bar{H} = \bar{H}_n$. Pick a choice $r : \Gamma/H \to \Gamma$ of representatives for the cosets of H in Γ with $r(H) = e$, and let $\rho : (\Gamma/H) \times \Gamma \to H$ be the cocycle $\rho(aH, \gamma) = r(aH)\gamma r(a\gamma H)^{-1}$. Define the finite set $T = \{r(aH) : a \in \Gamma\}$. In case (1) we let $q = 1 = \{0\}$ and let $\beta : X \to q$ be the constant function, and in case (2) we let $q \in \mathbb{N}$ and let $\beta : X \to q$ be a $\{\hat{\Gamma} \times B : B \in \mathcal{B}\}$-measurable map satisfying $\mu(\beta^{-1}(j)) > 0$ for every $j \in q$. Note that β is Γ-invariant.

For $t \in T$ and $j \in q$, set $X_t^j = \beta^{-1}(j) \cap t^{-1}\bar{H} \in \mathcal{A}_n$, and notice that these sets partition X. Also set $X_t = t^{-1}\bar{H} = \bigcup_{j \in q} X_t^j$. By Proposition 3.6 we will be done if we can find a measurable function $\tilde{\alpha} : X \to p$ with the property that for every $\pi \in p^S$, $j \in q$, and $t \in T$

$$\left| \mu(\tilde{\alpha}_S^{-1}(\pi) \cap X_t^j) - \nu(\alpha_S^{-1}(\pi) \cap \phi^{-1}(X_t^j) \right| < \epsilon.$$

Consider the functions α_{tS}, $t \in T$. We always have $t \cdot \alpha_S(y) = \alpha_{tS}(t \cdot y)$ since for $s \in S$

$$(t \cdot \alpha_S(y))(ts) = \alpha_S(y)(s) = \alpha(s^{-1} \cdot y) = \alpha(s^{-1}t^{-1}t \cdot y) = \alpha_{tS}(t \cdot y)(ts).$$

Therefore for all $t \in T$ and $\pi \in p^S$ we have $t \cdot \alpha_S^{-1}(\pi) = \alpha_{tS}^{-1}(t \cdot \pi)$. In particular, for every $j \in q$, we have

$$t \cdot \left(\alpha_S^{-1}(\pi) \cap \phi^{-1}(X_t^j) \right) = \alpha_{tS}^{-1}(t \cdot \pi) \cap \phi^{-1}(X_e^j). \tag{2}$$

Similarly, since $e \in S$, for $t \in T$, $s \in S$, and $y \in Y$, we have

$$\begin{aligned} \alpha_{tS}(y)(ts) = \alpha(s^{-1}t^{-1} \cdot y) &= \alpha(r(tsH)^{-1}\rho(tH, s)^{-1} \cdot y) \\ &= \alpha_{r(tsH)S}(\rho(tH, s)^{-1} \cdot y)(r(tsH)). \end{aligned}$$

So for every $t \in T$ and $j \in q$, we have

$$\bigcup_{s \in S} \{y \in \phi^{-1}(X_e^j) : \alpha_{r(tsH)S}(\rho(tH, s)^{-1} \cdot y)(r(tsH)) \neq \alpha_{tS}(y)(ts)\} = \emptyset. \tag{3}$$

For $j \in q$, let μ_j denote the normalized restriction of μ to X_e^j, and similarly define ν_j to be the normalized restriction of ν to $\phi^{-1}(X_e^j)$. The profinite completion of H is isomorphic to $\bar{H}$ and its normalized Haar probability measure $\mu_{\bar{H}}$ coincides with the normalized restriction of $\mu_{\hat{\Gamma}}$ to $\bar{H}$. Notice that in case (1) $j \in q$ can only have value 0 and $H \curvearrowright (X_e^0, \mu_0)$ is isomorphic to $H \curvearrowright (\bar{H}, \mu_{\bar{H}})$, and in case (2) $H \curvearrowright (X_e^j, \mu_j)$ is isomorphic to $H \curvearrowright (\bar{H} \times [0, 1], \mu_{\bar{H}} \times \lambda)$ for every $j \in q$. It follows from the assumptions of cases (1) and (2) and Lemma 4.1 that the action $H \curvearrowright (X_e^j, \mu_j)$ weakly contains all H-actions for every $j \in q$. Consequently, we can find measurable functions $\gamma_t : X_e \to p^{tS}$ for $t \in T$ satisfying the following two conditions. First, relative to each of the sets X_e^j, the γ_t's will have distribution in measure close to the α_{tS}'s, meaning that for all $t \in T$, $j \in q$, and $\pi \in p^S$, we have

$$\left| \mu_j\left(\gamma_t^{-1}(t \cdot \pi) \cap X_e^j\right) - \nu_j\left(\alpha_{tS}^{-1}(t \cdot \pi) \cap \phi^{-1}(X_e^j)\right) \right| < |\Gamma : H|\epsilon/2. \tag{4}$$

Second, we control how the functions γ_t relate to the action of H and demand, in view of (3), that $\mu_j(D_t^j) < |\Gamma : H|\epsilon/2$ for all $t \in T$ and $j \in q$, where

$$D_t^j = \bigcup_{s \in S} \{x \in X_e^j : \gamma_{r(tsH)}(\rho(tH, s)^{-1} \cdot x)(r(tsH)) \neq \gamma_t(x)(ts)\}.$$

Define $\tilde{\alpha} : X \to p$ by setting $\tilde{\alpha}(x) = \gamma_t(t \cdot x)(t)$ when $t \in T$ and $x \in X_t$. Notice that when $x \in X_t^j \setminus t^{-1} \cdot D_t^j$ we have $t \cdot \tilde{\alpha}_S(x) = \gamma_t(t \cdot x)$, since for any $s \in S$ we have $s^{-1} \cdot x \in X_{r(tsH)}$ and

$$\begin{aligned}(t \cdot \tilde{\alpha}_S(x))(ts) = \tilde{\alpha}_S(x)(s) = \tilde{\alpha}(s^{-1} \cdot x) &= \gamma_{r(tsH)}(r(tsH)s^{-1} \cdot x)(r(tsH)) \\ &= \gamma_{r(tsH)}(\rho(tH, s)^{-1} t \cdot x)(r(tsH))\end{aligned}$$

and the final term above is equal to $\gamma_t(t \cdot x)(ts)$ since $t \cdot x \notin D_t^j$. It follows that

$$\left(t \cdot (\tilde{\alpha}_S^{-1}(\pi) \cap X_t^j)\right) \triangle \left(\gamma_t^{-1}(t \cdot \pi) \cap X_e^j\right) \subseteq D_t^j. \tag{5}$$

For $\pi \in p^S$ and $j \in q$, equation (2) implies that

$$\nu(\alpha_S^{-1}(\pi) \cap \phi^{-1}(X_t^j)) = \nu(\alpha_{tS}^{-1}(t \cdot \pi) \cap \phi^{-1}(X_e^j))$$

and equation (5) implies

$$\left| \mu(\tilde{\alpha}_S^{-1}(\pi) \cap X_t^j) - \mu(\gamma_t^{-1}(t \cdot \pi) \cap X_e^j) \right| \leq \mu(D_t^j) = |\Gamma : H|^{-1} \mu_j(D_t^j) < \epsilon.$$

Since μ_j and ν_j are the normalized restrictions of μ and ν to X_e^j and $\phi^{-1}(X_e^j)$, respectively, and since $\mu(X_e^j) = \nu(\phi^{-1}(X_e^j))$, it follows from (4) and the above two equations that

$$\left| \nu(\alpha_S^{-1}(\pi) \cap \phi^{-1}(X_t^j)) - \mu(\tilde{\alpha}_S^{-1}(\pi) \cap X_t^j) \right| < \epsilon.$$

We conclude that the action $\Gamma \curvearrowright (X, \mu)$ is existentially closed. □

Since the finitely generated free group $\mathbb{F}$ has property EMD (see [30, Theorem 1] and, using different terminology, Bowen [7]), the previous theorem generalizes [5, Theorems 6.7 and 6.18].

The following is a nice application of Theorem 4.2:

Proposition 4.3. *Suppose that Γ has a coamenable normal subgroup Λ such that Γ/Λ is residually finite. Further suppose that Λ can be written as the increasing union of a sequence $(\Lambda_n)_{n \in \mathbb{N}}$ of subgroups such that:*

- *Each* Λ_n *has property MD;*
- *Each finite index subgroup of each* Λ_n *is closed in the profinite topology on* Γ*.*

Then Γ *has property MD.*

Proof. Since each finite index subgroup of each Λ_n is closed in the profinite topology of Γ, the action of Λ_n on $\hat{\Gamma}$ is isomorphic to the product of the action of Λ_n on $\widehat{\Lambda_n}$ with an identity action of Λ_n, so this action of Λ_n is existentially closed by Theorem 4.2. Thus, the action of Λ on $\hat{\Gamma}$ is existentially closed as well. The action of Γ on $\hat{\Gamma}$ factors onto $\hat{\Gamma}/\bar{\Lambda}$, where $\bar{\Lambda}$ denotes the closure of Λ in $\hat{\Gamma}$. Since Γ/Λ is residually finite, the action of Γ/Λ on $\hat{\Gamma}/\bar{\Lambda}$ is free. The action of Γ on $\hat{\Gamma}$ therefore satisfies the hypotheses of Theorem 5.18, hence it is existentially closed. In particular, Γ has property MD. □

A consequence of the previous proposition is the following, expanding the collection of examples of groups known to have property MD:

Theorem 4.4. *Limit groups have property MD.*

Proof. Let Γ be a limit group. A result of Kochloukova[34, Corollary B] is that Γ has a free normal subgroup Λ such that Γ/Λ is torsion-free nilpotent (and, in particuar, residually finite). Write Λ as a union of an increasing sequence of finitely generated free subgroups Λ_n, whence each Λ_n has property MD. By a result of Wilton [43], limit groups are subgroup separable, meaning that each finitely generated subgroup of Γ is closed in the profinite topology of Γ. In particular, each finite index subgroup of each Λ_n is closed in the profinite topology of Γ. Hence Γ has property MD by Proposition 4.3. □

Remark 4.5. Theorem 4.4 implies that limit groups have property FD, the representation theoretic analogue of MD introduced by Lubotzky and Shalom in [35] (note that FD was introduced prior to MD). Property MD implies property FD by [30], although the converse is open. Property FD for limit groups could also be deduced from [34] and [43] by appealing to [35, Corollary 2.5]. While our Proposition 4.3 is an analogue of [35, Corollary 2.5], its proof is conceptually a bit different, since it makes critical use of existentially closed actions. One may also give a somewhat ad hoc proof of Proposition 4.3, avoiding the use of existentially closed actions, that more closely parallels the proof of [35, Corollary 2.5], by using an approach similar to [11].

4.2. **Weakly mixing e.c. actions.** Recall that the p.m.p. action $\Gamma \curvearrowright^{a} X$ is **weakly mixing** if the product action $\Gamma \curvearrowright^{a\times a} X \times X$ is ergodic. It follows from [5, Theorems 4.3 and 6.6] and [33] that there is a weakly mixing e.c. action of the free group. In this subsection, we generalize this result to the case of any group without property (T) and in fact show that the "model-theoretically generic" e.c. action (in a sense we make precise below) is weakly mixing. First, we need the following result, which is somewhat implicit in Bergelson's [6]:

Proposition 4.6. *The action $\Gamma \curvearrowright X$ is weakly mixing if and only if: for any measurable sets $A_1, \ldots, A_n, B_1, \ldots, B_n \subseteq X$ and any $\epsilon > 0$, we have*

$$\bigcap_{i=1}^{n} \{\gamma \in \Gamma \ : \ |\mu(A_i \cap \gamma B_i) - \mu(A_i)\mu(B_i)| < \epsilon\} \neq \emptyset.$$

Proof. First suppose that the action is weakly mixing. By [6, Theorem 4.7], each set appearing in the above intersection is a central* subset of Γ. Since the family of central* subsets of Γ has the finite intersection property, we see that the above intersection is nonempty.

Now suppose that the above condition holds. In order to show that the action is weakly mixing, by [6, Exercise 21], it suffices to show: for any $f_1, \ldots, f_n \in L^2_0(X)$ and any $\epsilon > 0$, we have

$$\bigcap_{i=1}^{n} \{\gamma \in \Gamma \ : |\langle U_\gamma f_i, f_i \rangle| < \epsilon\} \neq \emptyset.$$

Here, U_γ is the Koopman representation associated to the action and $L^2_0(X)$ is the orthogonal complement of the subspace of $L^2(X)$ consisting of vectors of integral 0. For each $i = 1, \ldots, n$, take simple functions $h_1, \ldots, h_n \in L^2(X)$ such that $\|f_i - h_i\|_2 < \delta$ for some sufficiently small $\delta < \sqrt{\frac{\epsilon}{3}}$ so that

$$|\langle U_\gamma f_i, f_i \rangle| - |\langle U_\gamma h_i, h_i \rangle| < \frac{\epsilon}{3}$$

for all $i = 1, \ldots, n$ and all $\gamma \in \Gamma$. Write $h_i = \sum_j c_{ij} 1_{A_{ij}}$. Then

$$\langle U_\gamma h_i, h_i \rangle = \sum_{j,k} c_{ij}\overline{c_{ik}}\mu(\gamma A_{ij} \cap A_{ik}).$$

Fix $\eta > 0$ so that $\eta \sum_{j,k} |c_{ij}c_{ik}| < \frac{\epsilon}{3}$ for all $i = 1, \ldots, n$. By assumption, there is $\gamma \in \Gamma$ such that $|\mu(\gamma A_{ij} \cap A_{ik}) - \mu(A_{ij})\mu(A_{ik})| < \eta$ for each i, j, k. It follows that $\left| |\langle U_\gamma h_i, h_i \rangle| - |\sum_{jk} c_{ij}\overline{c_{ik}}\mu(A_{ij})\mu(A_{ik})| \right| \leq \sum_{jk} |c_{ij}\overline{c_{ik}}|\eta < \frac{\epsilon}{3}$. But

$$\left| \sum_{jk} c_{ij}\overline{c_{ik}}\mu(A_{ij})\mu(A_{ik}) \right| = \left| \int h_i \right|^2 = \left| \int (f_i - h_i) \right|^2 \leq \|f_i - h_i\|_2^2 < \delta^2 < \frac{\epsilon}{3}.$$

Consequently, for this $\gamma \in \Gamma$ and all $i = 1, \ldots, n$, we have $|\langle U_\gamma f_i, f_i \rangle| < \epsilon$, as desired. □

The previous proposition shows that weak mixing can be written in an infinitary first-order way of a particularly simple form. Recall that a $\forall\bigvee\exists$**-sentence** is one of the form $\sup_x \bigvee_{n\in\mathbb{N}} \varphi_n(\vec{x})$, where each $\varphi(x)$ is an existential formula. We call a class $\mathcal{C}$ of models of T_Γ $\forall\bigvee\exists$**-axiomatizable** if there is a collection σ_n of $\forall\bigvee\exists$-sentences such that, for all $\mathcal{M} \models T_\Gamma$, we have $\mathcal{M} \in \mathcal{C}$ if and only if $\mathcal{M} \models \sigma_n$ for each $n \in \mathbb{N}$.

Corollary 4.7. *The class of weakly mixing models of* T_Γ *is* $\forall\bigvee\exists$*-axiomatizable.*

Proof. For each n, let σ_n be the sentence

$$\sup_{x_1,\ldots,x_n,y_1,\ldots,y_n} \bigvee_{\gamma\in\Gamma} \max_{i=1,\ldots,n} |\mu(x_i \cap \gamma y_i) - \mu(x_i)\mu(y_i)|.$$

Then an action a is weakly mixing if and only if $\sigma_n^{\mathcal{M}_a} = 0$ for all $n \in \mathbb{N}$. □

The following corollary uses the notion of an **enforceable property**, which is a model-theoretic notion of genericity; a precise definition can be found in the first author's article [25].

Corollary 4.8. *Suppose that* Γ *is a group without property* (*T*)*. Then being a weakly mixing action is an enforceable property of actions of* Γ*.*

Proof. By a result of Kerr and Pichot [33], since Γ does not have property (T), there is a locally universal weakly mixing action of Γ. Using this fact, Corollary 4.7, and [25, Proposition 2.6], the result follows. □

Since being e.c. is also an enforceable property ([25, Proposition 2.10]) and since the conjunction of two enforceable properties is also enforceable, we get:

Corollary 4.9. *Suppose that* Γ *is a group without property* (*T*)*. Then there is an e.c. weakly mixing action of* Γ*.*

Note that if a group has property (T), then no locally universal action of it can be weakly mixing (or even ergodic), whence the previous corollary is optimal.

5. Generalities on model companions of meaure-preserving actions

In this section, we establish a number of interesting general results on the existence of the model companion for T_Γ. A consequence of these results is that T_Γ exists whenever Γ is a universally free group, a fact we generalize later in Section 7. We conclude this section with an open mapping criterion for the existence of the model companion for T_Γ.

5.1. Definition of model companions.

Definition 5.1. We say that **the model companion of T_Γ exists** if there is a set T of L_Γ-sentences such that, for all $\mathcal{M}_a \models T_\Gamma$, we have $\mathcal{M}_a \models T$ if and only if a is an e.c. action. In this case, there is a unique such theory T (up to logical equivalence), which we denote by T_Γ^*.

Remark 5.2. The model theorist will recognize that this is not the official definition of the model companion but is rather an equivalent reformulation (which holds in our context since the theory T_Γ is $\forall\exists$-axiomatizable).

There is a useful test for when the model companion exists:

Fact 5.3. *T_Γ^* exists if and only if: whenever $(\mathcal{M}_{a_i})_{i\in I}$ is a family of e.c. actions of Γ and $\mathcal{U}$ is an ultrafilter on I, we have that $\prod_{\mathcal{U}} \mathcal{M}_{a_i}$ is also an e.c. action of Γ.*

The proof of the previous fact hinges on an abstract characterization of when a class of structures in some language is the set of models of some theory, namely when the class is closed under isomorphism, ultraproduct, and ultraroot (see, for example, [3, Proposition 5.14]). In general, the class of e.c. structures is closed under elementary substructures, whence the nontrivial closure condition in the previous fact is that of being closed under ultraproducts.

As mentioned in the introduction, when Γ is amenable, T_Γ^* exists and $T_\Gamma^* = T_{\Gamma,\mathrm{free}}$. By a recent result of Berenstein, Ibarlucia, and Henson [5], for any finitely generated free group $\mathbb{F}$, we have that $T_{\mathbb{F}}^*$ exists. We will generalize this fact in a number of ways throughout this paper.

5.2. Model-theoretic shenanigans.

In this subsection, we establish some purely model-theoretic results; these results will be applied in the next subsection to the case of actions.

Definition 5.4. Suppose that $L_1 \subseteq L_2$ are languages, T_2 is an L_2-theory, and $T_1 := T_2|L_1$, that is, the set of sentences in T_2 that are actually L_1-sentences. We say that the pair (T_1, T_2) has the:

- **expansion property** if, given any $\mathcal{M} \models T_1$, there is $F(\mathcal{M}) \models T_2$ such that $\mathcal{M} \subseteq F(\mathcal{M})|L_1$;
- **relative expansion property** if, given any $\mathcal{N} \models T_2$ and $\mathcal{M} \models T_1$ with $\mathcal{N}|L_1 \subseteq \mathcal{M}$, then there is $G(\mathcal{M},\mathcal{N}) \models T_2$ with $\mathcal{M} \subseteq G(\mathcal{M},\mathcal{N})|L_1$ and $\mathcal{N} \subseteq G(\mathcal{M},\mathcal{N})$.

Remark 5.5. In the notation of the previous definition, if L_2 is countable, then a simple compactness argument implies that, to show that the pair (T_1, T_2) has the (relative) expansion property, it suffices to consider only countable models.

Remark 5.6. In the next section, we will suppose that Λ is a subgroup of Γ and work in the setting of the previous definition with $L_1 := L_\Lambda$, $L_2 := L_\Gamma$, and $T_2 := T_\Gamma$, so that $T_1 = T_\Lambda$. We will then show that (T_Λ, T_Γ) has both the expansion property and the relative expansion property.

Lemma 5.7. *Suppose that the pair* (T_1, T_2) *has both the expansion property and the relative expansion property. Further suppose that both* T_1 *and* T_2 *have the amalgamation property and that* T_2 *is* $\forall\exists$*-axiomatizable. Then if* T_2 *has a model companion, then so does* T_1.

Remark 5.8. For the sake of simplicity, we carry out the proof of this lemma as well as the two results that follow in the setting of classical (discrete) logic. The case of continuous logic is no more difficult, just slightly more annoying to write down.

Proof of Lemma 5.7. Fix a family $(\mathcal{M}_i)_{i\in I}$ of e.c. models of T_1 and an ultrafilter $\mathcal{U}$ on I. Set $\mathcal{M} := \prod_{\mathcal{U}} \mathcal{M}_i$. We wish to show that $\mathcal{M}$ is also an e.c. model of T_1. Fix $\mathcal{M}' \models T_1$ with $\mathcal{M} \subseteq \mathcal{M}'$, a quantifier-free L_1-formula $\varphi(x, y)$, and elements $a_i \in \mathcal{M}_i$ such that $\mathcal{M}' \models \exists x\varphi(x, a)$, where $a = (a_i)_{\mathcal{U}}$. We wish to show that $\mathcal{M} \models \exists x\varphi(x, a)$.

Since T_1 has the amalgamation property and $\mathcal{M} \subseteq \mathcal{M}'$ and $\mathcal{M} \subseteq \prod_{\mathcal{U}} F(\mathcal{M}_i)|L_1$, we have $\mathcal{Q} \models T_1$ and embeddings $i : \mathcal{M}' \hookrightarrow \mathcal{Q}$ and $j : \prod_{\mathcal{U}} F(\mathcal{M}_i)|L \hookrightarrow Q$ such that $i|\mathcal{M} = j|\mathcal{M}$. Without loss of generality, we may assume that j is an inclusion mapping and so $\prod_{\mathcal{U}} F(\mathcal{M}_i)|L \subseteq \mathcal{Q}$. We then have $\mathcal{R} := G(\mathcal{Q}, \prod_{\mathcal{U}} F(\mathcal{M}_i)) \models T_2$ such that $\mathcal{Q} \subseteq G(\mathcal{Q}, \mathcal{R})|L_1$ and $\prod_{\mathcal{U}} F(\mathcal{M}_i) \subseteq G(\mathcal{Q}, \mathcal{R})$. For each $i \in I$, let $\mathcal{P}_i \models T_2$ be e.c. with $F(\mathcal{M}_i) \subseteq \mathcal{P}_i$ (which is possible since T_2 is $\forall\exists$-axiomatizable), so $\prod_{\mathcal{U}} F(\mathcal{M}_i) \subseteq \mathcal{P} := \prod_{\mathcal{U}} \mathcal{P}_i$. By assumption, $\mathcal{P}$ is also e.c. Since T_2 has the amalgamation property, we can find $\mathcal{S} \models T_2$ and embeddings $b : G(\mathcal{Q}, \mathcal{R}) \hookrightarrow \mathcal{S}$ and $c : \mathcal{P} \hookrightarrow \mathcal{S}$ so that $b|\prod_{\mathcal{U}} F(\mathcal{M}_i) = c|\prod_{\mathcal{U}} F(\mathcal{M}_i)$.

We are now ready to conclude: Since $\mathcal{M}' \models \exists x\varphi(x, a)$, we have that $\mathcal{Q} \models \exists x\varphi(x, a)$ and so $G(\mathcal{Q}, \mathcal{R}) \models \exists x\varphi(x, a)$ and so $\mathcal{S} \models \exists x\varphi(b(a))$. Since $\mathcal{P}$ is e.c. and since $b(a) = c(a)$, we have that $\mathcal{P} \models \exists x\varphi(x, a)$. Consequently, for $\mathcal{U}$-almost all $i \in I$, we have $\mathcal{P}_i \models \exists x\varphi(x, a_i)$. Since $\mathcal{M}_i \subseteq \mathcal{P}_i|L_1$, we have that $\mathcal{M}_i \models \exists x\varphi(x, a_i)$ for these $i \in I$, and thus $\mathcal{M} \models \exists x\varphi(x, a)$, as desired. □

Lemma 5.9. *Suppose that* (T_1, T_2) *has the relative expansion property. Then for any e.c. model* $\mathcal{M}$ *of* T_2*, we have that* $\mathcal{M}|L_1$ *is an e.c. model of* T_1.

Proof. Take $\mathcal{N} \models T_1$ with $\mathcal{M}|L_1 \subseteq \mathcal{N}$ and an existential L_1-sentence σ with parameters from $\mathcal{M}$ such that $\mathcal{N} \models \sigma$. We then have $G(\mathcal{N}, \mathcal{M}) \models \sigma$ and $\mathcal{M} \subseteq G(\mathcal{N}, \mathcal{M})$; since $\mathcal{M}$ is e.c. then $\mathcal{M} \models \sigma$, as desired. □

Corollary 5.10. *Suppose that* T *is an* L*-theory. Further suppose that there is an increasing sequence* $(L_n)_{n\in\mathbb{N}}$ *of sublanguages of* L *with* $L = \bigcup_{n\in\mathbb{N}} L_n$ *such that* (T_n, T)

has the relative expansion property for all n*, where* $T_n := T|L_n$*. Further suppose that* T_n *has a model companion for each* n*. Then* T *also has a model companion.*

Proof. Fix a family $(\mathcal{M}_i)_{i\in I}$ of e.c. models of T and an ultrafilter $\mathcal{U}$ on I. Set $\mathcal{M} := \prod_{\mathcal{U}} \mathcal{M}_i$. Suppose that there is $\mathcal{M}' \models T$ with $\mathcal{M} \subseteq \mathcal{M}'$ and an existential L-sentence with parameters from $\mathcal{M}$ such that $\mathcal{M}' \models \sigma$. We want to show $\mathcal{M} \models \sigma$. Take $n \in \mathbb{N}$ such that σ is an L_n-sentence. By Lemma 5.9, $\mathcal{M}_i|L_n$ is an e.c. model of T_n for each $i \in I$; since T_n has a model companion, we have that $\prod_{\mathcal{U}}(\mathcal{M}_i|L_n)$ is an e.c. model of T_n. Since $\prod_{\mathcal{U}}(\mathcal{M}_i|L_n) \subseteq \mathcal{M}'|L_n$, we have that $\mathcal{M}|L_n = \prod_{\mathcal{U}}(\mathcal{M}_i|L_n) \models \sigma$, as desired. □

5.3. **Preservation properties for the existence of** T^*_Γ**.** The next two lemmas shows that, for any subgroup Λ of Γ, the pair (T_Λ, T_Γ) has both the expansion property and the relative expansion property.

Lemma 5.11. *Let* Γ *be a group with subgroup* Λ*. Then the pair* (T_Λ, T_Γ) *has the expansion property.*

Proof. This is an immediate consequence of the existence of the *coinduction* procedure (see, for example, [31, Section 10(G)]). □

The following is a special case of Epstein's construction of coinducing an action from a subequivalence relation, see [28, Section 3].

Lemma 5.12. *Let* Γ *be a group with subgroup* Λ*. Then the pair* (T_Λ, T_Γ) *has the relative expansion property. More precisely, let* $\Gamma \curvearrowright (X, \mu)$ *and* $\Lambda \curvearrowright (Y, \nu)$ *be p.m.p. actions on standard probability spaces and* $\phi : Y \to X$ *a* Λ*-equivariant factor map. Then there exists a p.m.p. action* $\Gamma \curvearrowright (\bar{Y}, \bar{\nu})$ *and a* Λ*-equivariant map* $\pi : \bar{Y} \to Y$ *such that* $\phi \circ \pi$ *is* Γ*-equivariant.*

Proof. Fix a choice of representatives $r : \Gamma/\Lambda \to \Gamma$ for the cosets of Λ in Γ with $r(\Lambda) = e$, and define the cocycle $\rho : \Gamma \times (\Gamma/\Lambda) \to \Lambda$ by $\rho(\gamma, a\Lambda) = r(\gamma a\Lambda)^{-1}\gamma r(a\Lambda)$. Define an action of Γ on $\bar{Y} := Y^{\Gamma/\Lambda}$ by

$$(\gamma \cdot \bar{y})(a\Lambda) = \rho(\gamma^{-1}, a\Lambda)^{-1}\bar{y}(\gamma^{-1}a\Lambda).$$

Let $\nu = \int \nu_x \, d\mu(x)$ be the **disintegration** of ν with respect to ϕ (that is, $x \mapsto \nu_x$ is the unique, up to agreement almost-everywhere, measurable function satisfying $\nu_x(\phi^{-1}(x)) = 1$ for μ-almost-every x and $\nu = \int \nu_x \, d\mu(x)$; see [22, Theorem A.7]). Define

$$\bar{\nu} = \int_X \prod_{a\Lambda \in \Gamma/\Lambda} \nu_{r(a\Lambda)^{-1}\cdot x} \, d\mu(x),$$

so a $\bar{\nu}$-random point $\bar{y}$ of $Y^{\Gamma/\Lambda}$ is obtained by choosing a μ-random point $x \in X$ and independently for each $a\Lambda \in \Gamma/\Lambda$ letting $\bar{y}(a\Lambda)$ be a $\nu_{r(a\Lambda)^{-1}\cdot x}$-random point in Y. Clearly if $\pi : Y^{\Gamma/\Lambda} \to Y$ is the evaluation map $\pi(\bar{y}) = \bar{y}(\Lambda)$, then π is measure-preserving, and it is also Λ-equivariant since

$$\pi(\lambda \cdot \bar{y}) = (\lambda \cdot \bar{y})(\Lambda) = \rho(\lambda^{-1}, \Lambda)^{-1}\bar{y}(\Lambda) = \lambda\bar{y}(\Lambda) = \lambda\pi(\bar{y}).$$

Also, for $\bar{\nu}$-almost-every $\bar{y} \in Y^{\Gamma/\Lambda}$ there is $x \in X$ satisfying $\phi(\bar{y}(a\Lambda)) = r(a\Lambda)^{-1}\cdot x$ for every $a\Lambda \in \Gamma/\Lambda$, and in this case

$$\begin{aligned}
\phi \circ \pi(\gamma \cdot \bar{y}) &= \phi((\gamma \cdot \bar{y})(\Lambda)) \\
&= \phi(\rho(\gamma^{-1}, \Lambda)^{-1}\bar{y}(\gamma^{-1}\Lambda)) \\
&= \rho(\gamma^{-1}, \Lambda)^{-1}\phi(\bar{y}(\gamma^{-1}\Lambda)) \\
&= \rho(\gamma^{-1}, \Lambda)^{-1}r(\gamma^{-1}\Lambda)^{-1} \cdot x \\
&= r(\Lambda)^{-1}\gamma \cdot x \\
&= \gamma \cdot x \\
&= \gamma \cdot \phi \circ \pi(\bar{y}).
\end{aligned}$$

Thus $\phi \circ \pi$ is Γ-equivariant on a $\bar{\nu}$-conull subset of $Y^{\Gamma/\Lambda}$.

To finish the proof, it only remains to check that the measure $\bar{\nu}$ is Γ-invariant. First notice that uniqueness of measure disintegration and the Λ-invariance of μ and ν imply that $\nu_x(\lambda \cdot E) = \nu_{\lambda^{-1}\cdot x}(E)$ for every $\lambda \in \Lambda$, every measurable set $E \subseteq Y$, and μ-almost-every $x \in X$. Now consider any collection of measurable sets $E_{a\Lambda} \subseteq Y$ for $a\Lambda \in \Gamma/\Lambda$, set $E = \prod_{a\Lambda\in\Gamma/\Lambda} E_{a\Lambda} \subseteq Y^{\Gamma/\Lambda}$, and let $\gamma \in \Gamma$. Notice that

$$\gamma \cdot E = \prod_{a\Lambda\in\Gamma/\Lambda} \rho(\gamma^{-1}, a\Lambda)^{-1}E_{\gamma^{-1}a\Lambda}.$$

We then have

$$\begin{aligned}
\bar{\nu}(\gamma \cdot E) &= \int_X \prod_{a\Lambda\in\Gamma/\Lambda} \nu_{r(a\Lambda)^{-1}\cdot x}(\rho(\gamma^{-1}, a\Lambda)^{-1}E_{\gamma^{-1}a\Lambda})\, d\mu(x) \\
&= \int_X \prod_{a\Lambda\in\Gamma/\Lambda} \nu_{\rho(\gamma^{-1},a\Lambda)r(a\Lambda)^{-1}\cdot x}(E_{\gamma^{-1}a\Lambda})\, d\mu(x) \\
&= \int_X \prod_{a\Lambda\in\Gamma/\Lambda} \nu_{r(\gamma^{-1}a\Lambda)^{-1}\gamma^{-1}\cdot x}(E_{\gamma^{-1}a\Lambda})\, d\mu(x) \\
&= \int_X \prod_{a\Lambda\in\Gamma/\Lambda} \nu_{r(\gamma^{-1}a\Lambda)^{-1}\cdot x}(E_{\gamma^{-1}a\Lambda})\, d\mu(x)
\end{aligned}$$

$$= \int_X \prod_{a\Lambda\in\Gamma/\Lambda} \nu_{r(a\Lambda)^{-1}\cdot x}(E_{a\Lambda})\, d\mu(x)$$
$$= \bar{\nu}(E),$$

where for the third-to-last equality we use the Γ-invariance of μ, and for the second-to-last equality we use the fact that left-multiplication by γ^{-1} permutes the set Γ/Λ. Since every measurable set can be approximated in $\bar{\nu}$-measure by sets that are finite disjoint unions of sets of the above form, it follows that $\bar{\nu}$ is indeed Γ-invariant. $\square$

Since both T_Γ and T_Λ have the amalgamation property (as witnessed by the existence of relatively independent joinings-see Lemma 5.23 below) and are $\forall\exists$-axiomatizable, Lemmas 5.7, 5.11, and 5.12 allow us to conclude:

Corollary 5.13. *The property of T_Γ having a model companion is inherited by subgroups: if Λ is a subgroup of Γ and T_Γ^* exists, then so does T_Λ^*.*

Lemmas 5.9 and 5.12 imply:

Corollary 5.14. *If $\Gamma \curvearrowright (X, \mu)$ is an e.c. action and $\Lambda \leq \Gamma$ is a subgroup, then the restricted action $\Lambda \curvearrowright (X, \mu)$ is e.c. as well.*

Corollary 5.10 and Lemma 5.12 imply:

Corollary 5.15. *The property of T_Γ having a model companion is a local property of groups: if Γ is a group such that T_Λ^* exists for every finitely generated subgroup Λ of Γ, then T_Γ^* also exists.*

The following corollary is an immediate consequence of Corollary 5.10; it was stated in the introduction of [5]:

Corollary 5.16. *If $\mathbb{F}_\infty$ is the free group on a countably infinite set of generators, then $T_{\mathbb{F}_\infty}^*$ exists.*

Note that it is unclear if having a model companion is closed under quotients. Indeed, by the previous corollary, if it is closed under quotients, then T_Γ^* exists for all countable groups Γ.

5.4. **Coamenable subgroups.** It is currently unknown if the class of groups for which T_Γ^* exists is closed under extensions, that is to say: if Γ is a countable group with normal subgroup Λ for which both T_Λ^* and $T_{\Gamma/\Lambda}^*$ exists, must T_Γ^* necessarily exist. In this subsection, we show that this is the case if we assume that Λ is a co-amenable normal subgroup of Γ, that is to say Γ/Λ is amenable.

Below, when Λ is a normal subgroup of Γ, we say that $\Gamma \curvearrowright (X, \mu)$ weakly contains a free p.m.p. action of Γ/Λ to mean that this action weakly contains some action $\Gamma \curvearrowright (Y, \nu)$ such that every point in Y has stabilizer Λ. In this case we can always choose (Y, ν) to be a standard Borel probability space since weak containment is transitive and every free p.m.p. action of Γ/Λ weakly contains a Bernoulli shift action of Γ/Λ [1].

The following is a consequence of the Ornstein-Weiss quasitiling lemma:

Lemma 5.17. *Suppose that Λ is a normal co-amenable subgroup of Γ and that $\Gamma \curvearrowright^a (X, \mu)$ is a p.m.p. action that weakly contains a free p.m.p. action of Γ/Λ. Then for every finite $S \subseteq \Gamma$ with $e \in S$ and every $\epsilon > 0$, there is a measurable map $c : X \to \Gamma/\Lambda$ having finite image such that $\mu(X') > 1 - \epsilon$, where*

$$X' = \{x \in X \ : \ c(s^{-1} \cdot x) = c(x)s \textit{ for all } s \in S\}.$$

Proof. Let (Y, ν) be a standard Borel probability space and $\Gamma \curvearrowright^b (Y, \nu)$ an action that is weakly contained in the action $\Gamma \curvearrowright (X, \mu)$ and has the property that every $y \in Y$ has stabilizer Λ. Since Γ/Λ is amenable, the orbit equivalence relation

$$E_\Gamma^Y = \{(y, \gamma \cdot y) : y \in Y, \ \gamma \in \Gamma\}$$

must be ν-hyperfinite [36], meaning there is a sequence of Borel equivalence relations $E_n \subseteq E_\Gamma^Y$ such that each class of each E_n is finite, $E_n \subseteq E_{n+1}$ for all n, and $\bigcup_n E_n$ coincides with E_Γ^Y on a Γ-invariant conull set. Set $F = E_n$ for a sufficiently large value of n so that the set

$$Y' = \{y \in Y : (y, s^{-1} \cdot y) \in F \text{ for all } s \in S\}$$

has measure greater than $1 - \epsilon$. Since F is a Borel equivalence relation whose classes are all finite, there exists a Borel set $D \subseteq Y$ that contains precisely one point from every F-class [29, Thm 12.16]. Let $d : Y \to \Gamma/\Lambda$ be the function that sends $y \in Y$ to the Λ-coset $\gamma\Lambda$ for any (equivalently every) $\gamma \in \Gamma$ satisfying $\gamma y \in D$ and $(y, \gamma y) \in F$. We observe that d is measurable since for every $\gamma \in \Gamma$ the set of $y \in Y$ satisfying $y \ F \ \gamma \cdot y$ is equal to the Borel set $(\mathrm{id} \times b(\gamma))^{-1}(F)$. It is immediate from these definitions that the set of $y \in Y$ satisfying $d(s^{-1} \cdot y) = d(y)s$ for all $s \in S$ is precisely Y' and thus has measure larger than $1 - \epsilon$. Finally, since the map d is described by the countable measurable partition $\{d^{-1}(\gamma\Lambda) : \gamma\Lambda \in \Gamma/\Lambda\}$ and the translated maps $d(s^{-1} \cdot)$, $s \in S$, are described by the S-translates of that partition, the fact that $\Gamma \curvearrowright (X, \mu)$ weakly contains $\Gamma \curvearrowright (Y, \nu)$ immediately implies that a function $c : X \to \Gamma/\Lambda$ with the desired property exists. □

Theorem 5.18. *Let Γ be a countable group and Λ a normal co-amenable subgroup of Γ. If $\Gamma \curvearrowright (X, \mu)$ is a p.m.p. action that weakly contains a free p.m.p. action of Γ/Λ and if the restricted action $\Lambda \curvearrowright (X, \mu)$ is existentially closed, then the action $\Gamma \curvearrowright (X, \mu)$ is existentially closed.*

Proof. We use the criterion for being e.c. established in Proposition 3.6 (and use the notation established right before the statement of that proposition). Let $\Gamma \curvearrowright (Y, \nu)$ be a p.m.p. action and let $\phi : Y \to X$ be a Γ-equivariant factor map. Fix $p, q \in \mathbb{N}$, $\alpha : Y \to p$ and $\beta : X \to q$ measurable maps, $S \subseteq \Gamma$ be finite with $e \in S$, and $\epsilon > 0$. Fix a choice $r : \Gamma/\Lambda \to \Gamma$ of representatives for the cosets of Λ in Γ and let $\rho : (\Gamma/\Lambda) \times \Gamma \to \Lambda$ be the cocycle $\rho(a\Lambda, \gamma) = r(a\Lambda)\gamma r(a\gamma\Lambda)^{-1}$.

By applying Lemma 5.17 and composing with the function r, we obtain a measurable map $c : X \to r(\Gamma/\Lambda)$ having finite image and satisfying $\mu(X') > 1 - \frac{\epsilon}{2}$, where

$$X' = \{x \in X : \ c(s^{-1} \cdot x) = r(c(x)s\Lambda) \text{ for all } s \in S\}.$$

Define the finite set $W = c(X) \subseteq r(\Gamma/\Lambda) \subseteq \Gamma$ and for $w \in W$ set $X_w = c^{-1}(w)$. Also set $W' = \{w \in W : X_w \cap X' \neq \varnothing\}$ and notice that whenever $w \in W'$ and $s \in S$, we have $r(ws\Lambda) \in W$ since $r(ws\Lambda) = c(s^{-1} \cdot x)$ whenever $x \in X_w \cap X'$.

Now consider the functions α_{wS} for $w \in W$. We observe that we always have $w \cdot \alpha_S(y) = \alpha_{wS}(w \cdot y)$ since for $s \in S$, we have

$$(w \cdot \alpha_S(y))(ws) = \alpha_S(y)(s) = \alpha(s^{-1} \cdot y) = \alpha(s^{-1}w^{-1}w \cdot y) = \alpha_{wS}(w \cdot y)(ws).$$

Therefore for all $w \in W$ and $\pi \in p^S$, we have $w \cdot \alpha_S^{-1}(\pi) = \alpha_{wS}^{-1}(w \cdot \pi)$. In particular, we have

$$w \cdot \Big(\alpha_S^{-1}(\pi) \cap \phi^{-1}(X_w)\Big) = \alpha_{wS}^{-1}(w \cdot \pi) \cap w \cdot \phi^{-1}(X_w). \tag{6}$$

Similarly, since $e \in S$, for $w \in W'$, $s \in S$, and $y \in Y$ we have

$$\begin{aligned}\alpha_{wS}(y)(ws) = \alpha(s^{-1}w^{-1} \cdot y) &= \alpha(r(ws\Lambda)^{-1}\rho(w\Lambda, s)^{-1} \cdot y) \\ &= \alpha_{r(ws\Lambda)S}(\rho(w\Lambda, s)^{-1} \cdot y)(r(ws\Lambda)).\end{aligned}$$

So for every $w \in W'$, we have

$$\bigcup_{s \in S} \big\{y \in Y : \alpha_{r(ws\Lambda)S}(\rho(w\Lambda, s)^{-1} \cdot y)(r(ws\Lambda)) \neq \alpha_{wS}(y)(ws)\big\} = \emptyset. \tag{7}$$

Since $\Lambda \curvearrowright (X, \mu)$ is existentially closed, we can find measurable functions $\gamma_w : w \cdot X_w \to p^{wS}$ for $w \in W$ satisfying the following two conditions. First, relative to each of the sets $w \cdot (\beta^{-1}(j) \cap X_w)$, the γ_w's will have identical distribution in measure to the functions α_{wS}, meaning that for all $w \in W$, $j \in q$, and $\pi \in p^S$, we have

$$\mu\Big(\gamma_w^{-1}(w \cdot \pi) \cap w \cdot \beta^{-1}(j)\Big) = \nu\Big(\alpha_{wS}^{-1}(w \cdot \pi) \cap \phi^{-1}(w \cdot \beta^{-1}(j) \cap w \cdot X_w)\Big). \tag{8}$$

(We point out that taking an intersection with $w \cdot X_w$ on the left would be redundant since the domain of (γ_w) is $w \cdot X_w$). Second, we control how the functions

γ_w relate to the action of Λ and demand, in view of (7), that $\mu(D_w) < \frac{\epsilon}{2|W|}$ for all $w \in W'$, where

$$D_w = \bigcup_{s\in S} \left\{x \in w \cdot X_w : \gamma_{r(ws\Lambda)}(\rho(w\Lambda, s)^{-1} \cdot x)(r(ws\Lambda)) \neq \gamma_w(x)(ws)\right\}.$$

Define $\tilde{\alpha} : X \to p$ by setting $\tilde{\alpha}(x) = \gamma_w(w \cdot x)(w)$ when $w \in W$ and $x \in X_w$. Notice that when $x \in (X_w \setminus w^{-1} \cdot D_w) \cap X'$, we have $w \cdot \tilde{\alpha}_S(x) = \gamma_w(w \cdot x)$, since for any $s \in S$, we have $s^{-1} \cdot x \in X_{r(ws\Lambda)}$ since $x \in X'$ and therefore

$$\begin{aligned}(w \cdot \tilde{\alpha}_S(x))(ws) = \tilde{\alpha}_S(x)(s) = \tilde{\alpha}(s^{-1} \cdot x) &= \gamma_{r(ws\Lambda)}(r(ws\Lambda)s^{-1} \cdot x)(r(ws\Lambda)) \\ &= \gamma_{r(ws\Lambda)}(\rho(w\Lambda, s)^{-1} w \cdot x)(r(ws\Lambda))\end{aligned}$$

and the final term above is equal to $\gamma_w(w \cdot x)(ws)$ since $w \cdot x \notin D_w$. As we additionally have that $X_w \cap X' = \emptyset$ when $w \in W \setminus W'$, we conclude that for all $w \in W$, we have

$$\left(w \cdot (\tilde{\alpha}_S^{-1}(\pi) \cap X_w)\right) \triangle \left(\gamma_w^{-1}(w \cdot \pi) \cap w \cdot X_w\right) \subseteq D_w \cup w \cdot (X_w \setminus X'). \tag{9}$$

For $\pi \in p^S$ and $j \in q$, equation (6) implies that

$$\nu(\alpha_S^{-1}(\pi) \cap \phi^{-1}(\beta^{-1}(j))) = \sum_{w\in W} \nu(\alpha_{wS}^{-1}(w \cdot \pi) \cap w \cdot \phi^{-1}(\beta^{-1}(j) \cap X_w)) \tag{10}$$

and equation (9) implies

$$\left|\mu(\tilde{\alpha}_S^{-1}(\pi) \cap \beta^{-1}(j)) - \sum_{w\in W} \mu(\gamma_w^{-1}(w \cdot \pi) \cap w \cdot (\beta^{-1}(j) \cap X_w))\right| < \epsilon. \tag{11}$$

Since the sums over $w \in W$ in (10) and (11) are equal by (8), it follows that

$$\left|\nu(\alpha_S^{-1}(\pi) \cap \phi^{-1}(\beta^{-1}(j))) - \mu(\tilde{\alpha}_S^{-1}(\pi) \cap \beta^{-1}(j))\right| < \epsilon.$$

We conclude that the action $\Gamma \curvearrowright (X, \mu)$ is existentially closed. □

Corollary 5.19. *Suppose that Γ is a group containing a normal coamenable subgroup Λ for which T_Λ^* exists. Then T_Γ^* exists as well.*

Proof. Let $(\mathcal{M}_{a_i})_{i\in I}$ be a family of e.c. models of T_Γ and fix an ultrafilter $\mathcal{U}$ on I; it suffices to show that $\mathcal{M}_a := \prod_{\mathcal{U}} \mathcal{M}_{a_i}$ is also an e.c. model of T_Γ. For each $i \in I$, let $\mathcal{M}_{b_i} := \mathcal{M}_{a_i}|L_\Lambda \models T_\Lambda$ denote the restricted action and set $\mathcal{M}_b := \prod_{\mathcal{U}} \mathcal{M}_{b_i} = \mathcal{M}_a|L_\Lambda$. By Corollary 5.14, each $\mathcal{M}_{b_i}$ is an e.c. model of T_Λ. Since T_Λ^* exists, $\mathcal{M}_b$ is also an e.c. model of T_Λ. Since each $\mathcal{M}_{a_i}$ is e.c. it is also locally universal, whence so is $\mathcal{M}_a$. By Theorem 5.18, $\mathcal{M}_a$ is e.c. as desired. □

Remark 5.20. Under the assumptions of the previous corollary, the axioms for T_Γ^* are the axioms for T_Λ^* together with the axioms for $T_{\Gamma,max}$.

Recall from the introduction that a group Γ is universally free if it embeds into an ultrapower of a free group.

Corollary 5.21. *Suppose that Γ is a universally free group. Then T^*_Γ exists.*

Proof. By Corollary 5.15, we may assume that Γ is finitely generated, that is, that Γ is a limit group. By the aforemtnioned result of Kochloukova [34], Γ has a normal coamenable (not necessarily finitely generated) free subgroup Λ. By Corollary 5.16, T^*_Λ exists. Thus, by Corollary 5.19, we have that T^*_Γ exists. □

5.5. **An open mapping characterization for the existence of T^*_Γ.** In this subsection, we give an ergodic-theoretic characterization of the existence of T^*_Γ, which also yields axioms for the model companion when it exists. First, we need a few preparatory lemmas.

Lemma 5.22. *For every $n \in \mathbb{N}$, let $\Gamma \curvearrowright (X_n, \mu_n)$ be a p.m.p. action. Then there exists a p.m.p. action $\Gamma \curvearrowright (X, \mu)$ which is e.c. and which factors onto $\Gamma \curvearrowright (X_n, \mu_n)$ for every n.*

Proof. Let $\Gamma \curvearrowright (Y, \nu)$ be any pmp action factoring onto each $\Gamma \curvearrowright (X_n, \mu_n)$ and then let $\Gamma \curvearrowright (X, \mu)$ be an e.c. action factoring onto $\Gamma \curvearrowright (Y, \nu)$. □

Lemma 5.23. *Let $\Gamma \curvearrowright (X, \mu)$ and $\Gamma \curvearrowright (Y, \nu)$ be p.m.p. actions that both factor onto the p.m.p. action $\Gamma \curvearrowright (Z, \eta)$ via the factor maps ϕ and ψ, respectively. If (Z, η) is a standard probability space then there is a Γ-invariant measure λ on $X \times Y$ having marginals μ on X and ν on Y, respectively, and satisfying $\phi(x) = \psi(y)$ for λ-almost-every $(x, y) \in X \times Y$.*

Proof. For each set $A \in \mathcal{B}_X$ denote by f_A the Radon–Nikodym derivative of the measure $C \in \mathcal{B}_Z \mapsto \mu(A \cap \phi^{-1}(C))$ with respect to η, and similarly for $B \in \mathcal{B}_Y$ define g_B to be the Radon–Nikodym derivative of $C \mapsto \nu(B \cap \psi^{-1}(C))$ with respect to η.

Define a function λ on the set of all measurable rectangles $A \times B$ ($A \in \mathcal{B}_X$, $B \in \mathcal{B}_Y$) by

$$\lambda(A \times B) = \int f_A \cdot g_B \, d\eta.$$

Suppose that $A \times B$ is the disjoint union of $A_n \times B_n$, $n \in \mathbb{N}$. Since (Z, η) is standard, we can pick countably generated σ-algebras $\Sigma_X \subseteq \mathcal{B}_X$ and $\Sigma_Y \subseteq \mathcal{B}_Y$ containing $\phi^{-1}(\mathcal{B}_Z)$ and $\psi^{-1}(\mathcal{B}_Z)$, respectively, with $A_n \in \Sigma_X$ and $B_n \in \Sigma_Y$ for all n. Disintegrate $\mu \restriction \Sigma_X$ with respect to ϕ and $\nu \restriction \Sigma_Y$ with respect to ψ to obtain almost-everywhere unique measurable maps $z \mapsto \mu_z$ and $z \mapsto \nu_z$, where μ_z and ν_z are probability measures on Σ_X and Σ_Y respectively, satisfying $\mu_z(\phi^{-1}(z)) =$

$1 = \nu_z(\psi^{-1}(z))$ for η-almost every $z \in Z$, $\mu \upharpoonright \Sigma_X = \int_Z \mu_z \, d\eta$ and $\nu \upharpoonright \Sigma_Y = \int_Z \nu_z \, d\eta$. Then for $A' \in \Sigma_X$ we have

$$\int \mu_z(A') \cdot 1_C \, d\eta = \mu(A' \cap \phi^{-1}(C)) = \int f_{A'} \cdot 1_C \, d\eta$$

for all $C \in \mathcal{B}_Z$ and therefore $f_{A'}(z) = \mu_z(A')$ for a.e. z. Similarly $g_{B'}(z) = \nu_z(B')$ for every $B' \in \Sigma_Y$ and a.e. z. It follows that the function λ and the measure $\int \mu_z \times \nu_z \, d\eta$ coincide when restricted to measurable rectangles in $\Sigma_X \times \Sigma_Y$ and therefore $\sum_{n \in \mathbb{N}} \lambda(A_n \times B_n) = \lambda(A \times B)$.

The previous paragraph shows that λ is a probability premeasure on the algebra of finite unions of measurable rectangles, so by Caratheodory's theorem λ has a unique extension to a probability measure on $\mathcal{B}_X \times \mathcal{B}_Y$. Since λ is Γ-invariant it follows from the uniqueness of the extension that it must be Γ-invariant as well. Moreover, it is immediately seen that this extension, which we denote by λ as well, has marginals μ and ν on X and Y respectively.

Lastly, since Z is standard we have that $Z \times Z \setminus \{(z, z) : z \in Z\}$ is a countable union of measurable rectangles. Since every measurable rectangle $C \times D \subseteq Z \times Z$ that is disjoint with the diagonal satisfies

$$\lambda((\phi \times \psi)^{-1}(C \times D)) = \int_Z 1_C \cdot 1_D \, d\eta = \eta(C \cap D) = 0,$$

we conclude that $\phi(x) = \psi(y)$ for λ-almost-every (x, y). $\square$

Lemma 5.24. *Let $(\Gamma \curvearrowright (X_i, \mu_i))_{i \in I}$ be a collection of p.m.p. actions, let $\mathcal{U}$ be an ultrafilter on I, let $\Gamma \curvearrowright \prod_{\mathcal{U}}(X_i, \mu_i)$ be the ultraproduct action, and let $q \in \mathbb{N}$.*

(1) *Given any measurable map $\alpha : \prod_{\mathcal{U}} X_i \to q$, there exist measurable maps $\alpha^i : X_i \to q$ such that $\alpha([x_i]_{\mathcal{U}}) = \lim_{i \to \mathcal{U}} \alpha^i(x_i)$ for almost-every $[x_i]_{\mathcal{U}} \in \prod_{\mathcal{U}} X_i$.*
(2) *If $\alpha_i : X_i \to q$ is measurable for each $i \in I$ and $\alpha : \prod_{\mathcal{U}} X_i \to q$ is defined by $\alpha([x_i]_{\mathcal{U}}) = \lim_{i \to \mathcal{U}} \alpha^i(x_i)$, then*
   (a) *α is measurable,*
   (b) *$\alpha_\Gamma([x_i]_{\mathcal{U}}) = \lim_{i \to \mathcal{U}} \alpha^i_\Gamma([x_i]_{\mathcal{U}})$ for every $[x_i]_{\mathcal{U}} \in \prod_{\mathcal{U}} X_i$, and*
   (c) *$(\alpha_\Gamma)_*(\prod_{\mathcal{U}} \mu_i) = \lim_{i \to \mathcal{U}} (\alpha^i_\Gamma)_*(\mu_i)$.*

Proof. (1). Since α is measurable, for each $k \in q - 1$ there is a collection of measurable sets $A^i_k \subseteq X_i$ such that $\alpha^{-1}(k) = [A^i_k]_{\mathcal{U}}$ up to a $\prod_{\mathcal{U}} \mu_i$-null set. Define $\alpha^i : X_i \to q$ by

$$\alpha^i(x) = \begin{cases} k & \text{if } k \in q - 1 \text{ is least with } x \in A^i_k \\ q - 1 & \text{if } x \notin A^i_0 \cup \cdots \cup A^i_{q-2}. \end{cases}$$

Then α^i is measurable and it is easy to check by induction on $k \in q$ that for almost-every $[x_i]_{\mathcal{U}} \in \alpha^{-1}(k)$ we have $\alpha([x_i]_{\mathcal{U}}) = \lim_{i \to \mathcal{U}} \alpha^i(x_i)$.

(2). It is immediate that α is measurable since $\alpha^{-1}(k) = [(\alpha^i)^{-1}(k)]_{\mathcal{U}}$. Also, for every $\gamma \in \Gamma$ we have

$$\alpha_\Gamma([x_i]_{\mathcal{U}})(\gamma) = \alpha(\gamma^{-1} \cdot [x_i]_{\mathcal{U}}) = \alpha([\gamma^{-1}x_i]_{\mathcal{U}}) = \lim_{i \to \mathcal{U}} \alpha^i(\gamma^{-1}x_i) = \lim_{i \to \mathcal{U}} \alpha^i_\Gamma(x_i)(\gamma).$$

This establishes (a) and (b). Next consider any clopen set $C \subseteq q^\Gamma$. For every $[x_i]_{\mathcal{U}} \in \prod_{\mathcal{U}} X_i$ we have that $\alpha_\Gamma([x_i]_{\mathcal{U}}) = \lim_{i\to\mathcal{U}} \alpha^i_\Gamma(x_i)$ belongs to C if and only if (since C is both open and closed) $\{i \in I : \alpha^i_\Gamma(x_i) \in C\} \in \mathcal{U}$, or equivalently $[x_i]_{\mathcal{U}} \in [(\alpha^i_\Gamma)^{-1}(C)]_{\mathcal{U}}$. Therefore $(\alpha_\Gamma)^{-1}(C) = [(\alpha^i_\Gamma)^{-1}(C)]_{\mathcal{U}}$ and

$$(\alpha_\Gamma)_*(\prod_{\mathcal{U}} \mu_i)(C) = \lim_{\mathcal{U}} (\alpha^\Gamma_i)_*(\mu_i)(C).$$

Since $\lim_{i\to\mathcal{U}}(\alpha^i_\Gamma)_*(\mu_i)$ is a probability measure on q^Γ and all Borel probability measures on q^Γ are uniquely determined by their values on clopen sets, (c) follows. □

We now prove a lemma providing an ergodic-theoretic characterization of e.c. factor maps. To state it, given any $q \in \mathbb{N}$, we let $\mathrm{Prob}_\Gamma(q^\Gamma)$ denote the set of probability measures on q^Γ preserved by the natural action of Γ on q^Γ. Given another integer p, we let $\mathrm{Prob}_\Gamma(q^\Gamma \times p^\Gamma)$ have the analogous meaning. We view each of these spaces as equipped with their weak*-topologies. We also let $\pi : q^\Gamma \times p^\Gamma \to q^\Gamma$ denote the canonical projection map, which induces a push-forward map $\pi_* : \mathrm{Prob}_\Gamma(q^\Gamma \times p^\Gamma) \to \mathrm{Prob}_\Gamma(q^\Gamma)$.

Lemma 5.25. *The action $\Gamma \curvearrowright^a (X, \mu)$ is e.c. if and only if: for any $p, q \in \mathbb{N}$, any weak*-open subset U of $\mathrm{Prob}_\Gamma(q^\Gamma \times p^\Gamma)$, and any $\beta : X \to q$ for which $(\beta_\Gamma)_*(\mu) \in \pi_*(U)$, there is $\gamma : X \to p$ such that $((\beta \times \gamma)_\Gamma)_*(\mu) \in U$.*

Proof. First suppose that $\Gamma \curvearrowright^a (X, \mu)$ is e.c. Take a weak*-open subset U of $\mathrm{Prob}_\Gamma(q^\Gamma \times p^\Gamma)$ and a function $\beta : X \to q$ for which $(\beta_\Gamma)_*(\mu) \in \pi_*(U)$. By assumption, there is $\lambda \in U$ such that $\pi_*(\lambda) = (\beta_\Gamma)_*(\mu)$. By Lemma 5.23, there is a Γ-invariant probability measure ω on $X \times q^\Gamma \times p^\Gamma$ that has marginal μ on X and λ on $q^\Gamma \times p^\Gamma$ and satisfies $\beta_\Gamma(x) = y$ for ω-almost-every $(x, y, z) \in X \times q^\Gamma \times p^\Gamma$. By assumption, the factor map given by the projection $\phi : X \times q^\Gamma \times p^\Gamma \to X$ is e.c. and we have that $(((\beta \circ \phi) \times \alpha)_\Gamma)_*\omega = \lambda \in U$, where $\alpha : X \times q^\Gamma \times p^\Gamma \to p$ is the map $\alpha(x, y, z) = z(e)$. Therefore the desired γ is obtained by applying Proposition 3.6 with this α together with a finite set $S \subseteq \Gamma$ and an $\epsilon > 0$ that are suitable for ensuring membership in the open set U containing λ.

We now prove the converse. Towards this end, fix a factor map $\phi : Y \to X$; we wish to show that this map is e.c. using the criterion of Proposition 3.6. We thus take measurable maps $\beta : X \to q$ and $\gamma : Y \to p$, finite $S \subseteq \Gamma$, and $\epsilon > 0$. Note then that $((\beta \circ \phi)_\Gamma \times \gamma_\Gamma)_*(\nu) \in \mathrm{Prob}_\Gamma(q^\Gamma \times p^\Gamma)$ and that S and ϵ determine an open neighborhood U of this measure. Moreover,

$$(\beta_\Gamma)_*(\mu) = (\pi_\Gamma)_*((\beta \circ \phi)_\Gamma \times \gamma_\Gamma)_*(\nu) \in \pi_*(U).$$

By hypothesis, there is $\tilde{\gamma} : X \to p$ such that $((\beta \times \tilde{\gamma})_\Gamma)_*(\mu) \in U$, which verifies the criterion of Proposition 3.6. ☐

The following theorem is the main result of this subsection and offers an ergodic-theoretic characterization of the existence of T^*_Γ.

Theorem 5.26. T^*_Γ *exists if and only if for every* $p, q \in \mathbb{N}$*, the push-forward map* $\pi_* : \mathrm{Prob}_\Gamma(q^\Gamma \times p^\Gamma) \to \mathrm{Prob}_\Gamma(q^\Gamma)$ *is an open map with respect to the weak* topologies.*

Proof. ($\Rightarrow$) Fix $q, p \in \mathbb{N}$ and let $\pi : q^\Gamma \times p^\Gamma \to q^\Gamma$ be the projection map. Consider any measure $\lambda \in \mathrm{Prob}_\Gamma(q^\Gamma \times p^\Gamma)$ and any weak* neighborhood $W \subseteq \mathrm{Prob}_\Gamma(q^\Gamma \times p^\Gamma)$ of λ. Set $\nu = \pi_*(\lambda)$ and consider any sequence $\nu_n \in \mathrm{Prob}_\Gamma(q^\Gamma)$ that converges weak* to ν and a nonprincipal ultrafilter $\mathcal{U}$ on $\mathbb{N}$. It will suffice to show that

$$\{n \in \mathbb{N} : \nu_n \in \pi_*(W)\} \in \mathcal{U}.$$

By Lemma 5.22, we may take an e.c. action $\Gamma \curvearrowright (X, \mu)$ having the property that it factors onto $\Gamma \curvearrowright (q^\Gamma, \nu_n)$ for every $n \in \mathbb{N}$. For each $n \in \mathbb{N}$, pick a measurable map $\alpha^n : X \to q$ satisfying $(\alpha^n_\Gamma)_*(\mu) = \nu_n$. Consider the ultrapower action $\Gamma \curvearrowright (X, \mu)_\mathcal{U}$ and define $\alpha : X_\mathcal{U} \to q$ by setting $\alpha([x_n]_\mathcal{U}) = \lim_\mathcal{U} \alpha^n(x_n)$. Then α is measurable and $(\alpha_\Gamma)_*(\mu_\mathcal{U}) = \nu$.

Apply Lemma 5.23 to get a Γ-invariant probability measure ω on $X_\mathcal{U} \times q^\Gamma \times p^\Gamma$ that has marginal $\mu_\mathcal{U}$ on $X_\mathcal{U}$, marginal λ on $q^\Gamma \times p^\Gamma$, and satisfies $\alpha_\Gamma(x) = \pi(y, z)$ for ω-a.e. $(x, y, z) \in X_\mathcal{U} \times q^\Gamma \times p^\Gamma$. Define $\bar{\alpha}$ and $\bar{\beta}$ on $X \times q^\Gamma \times p^\Gamma$ by setting $\bar{\alpha}(x, y, z) = \alpha(x) \in q$ and $\bar{\beta}(x, y, z) = z(e) \in p$. By our choice of ω, we have that $\bar{\alpha}_\Gamma(x, y, z) = \alpha_\Gamma(x)$ is equal to y almost-everywhere. Therefore $(\bar{\alpha}_\Gamma \times \bar{\beta}_\Gamma)(x, y, z) = (y, z)$ almost-everywhere and hence $(\bar{\alpha}_\Gamma \times \bar{\beta}_\Gamma)_*(\omega) = \lambda \in W$.

Since T^*_Γ exists, the action $\Gamma \curvearrowright (X, \mu)_\mathcal{U}$ is e.c. Consequently, based on the last sentence of the previous paragraph, there must exist a measurable map $\beta : X_\mathcal{U} \to p$ satisfying $(\alpha_\Gamma \times \beta_\Gamma)_*(\mu_\mathcal{U}) \in W$. Let $\beta^n : X \to p$ be a sequence of measurable maps satisfying $\beta([x_n]_\mathcal{U}) = \lim_\mathcal{U} \beta^n(x_n)$. Then we have $(\alpha \times \beta)([x_n]_\mathcal{U}) = \lim_\mathcal{U}(\alpha_n(x_n), \beta_n(x_n))$ and therefore

$$\lim_{n \to \mathcal{U}} (\alpha^n_\Gamma \times \beta^n_\Gamma)_*(\mu) = (\alpha_\Gamma \times \beta_\Gamma)_*(\mu_\mathcal{U}) \in W.$$

Applying π_* to both sides we obtain $\{n \in \mathbb{N} : \nu_n \in \pi_*(W)\} \in \mathcal{U}$ as claimed.

($\Leftarrow$) Assuming the open mapping condition, we see that the characterization of being e.c. in the previous lemma is first-order. Indeed, first note that it suffices to assume that U is a basic open subset of $\mathrm{Prob}_\Gamma(q^\Gamma \times p^\Gamma)$. Second, by assumption,

$\pi_*(U)$ is an open subset of $\mathrm{Prob}_\Gamma(q^\Gamma)$, and it then suffices to consider basic open subsets of $\mathrm{Prob}_\Gamma(q^\Gamma)$ contained in $\pi_*(U)$. □

Remark 5.27. As explained in the proof of the previous theorem, the characterization for being e.c. given in Lemma 5.25 does indeed yield an axiomatization of T_Γ^* when it exists.

6. A concrete axiomatization

In this section, we show that, for strongly treeable groups, the model companion T_Γ^* exists and has a concrete set of axioms that are ergodic-theoretic in nature. We additionally obtain the same result for treeable groups but with axioms that, while still ergodic-theoretic in nature, are slightly less concrete.

Our axioms for the model companion will rely on two properties we introduce: the definable cocycle property and the extension-MD property. These properties are discussed in the first two subsections below, and in the third subsection we will describe the concrete axiomatization. Lastly, in the final three subsections of this section we will verify that these two properties hold for suitable actions of (strongly) treeable groups.

6.1. **Model companions and the definable cocycle property.** Building upon our observations about cocycles for e.c. actions in Subsection 3.4 (specifically Lemma 3.11), we first note that the mere existence of the model companion immediately yields a remarkable feature of cochains mapping to finite groups that are close to satisfying the conditions for being a cocycle. If such cochains are said to be "almost-cocycles," then the next result says that if the model companion exists, then every almost-cocycle is near an actual cocycle in some extension (equivalently, it is near a coboundary in some extension).

Proposition 6.1. *Suppose that T_Γ^* exists. Then for every $\epsilon > 0$, finite set $S \subseteq \Gamma$, and finite group K, there is $\delta > 0$ such that, for all actions $\Gamma \curvearrowright^a (X, \mu)$ and measurable maps $\sigma : \Gamma \times X \to K$, if $\mathrm{Cocy}_K^{\mathcal{M}_a}(B_\sigma) < \delta$, then there is an action $\Gamma \curvearrowright (Y, \nu)$, a factor map $\phi : Y \to X$, and a measurable map $\alpha : Y \to K$ such that*

$$\nu(\{y \in Y \,:\, \alpha(sy) = \sigma(s, \phi(y))\alpha(y) \textit{ for every } s \in S\}) > 1 - \epsilon.$$

Proof. Suppose, towards a contradiction, that the above condition does not hold for some ϵ, S and K. For each $n \in \mathbb{N}$, take a p.m.p. action $\Gamma \curvearrowright^{a_n} (X_n, \mu_n)$ and a map $\sigma_n : \Gamma \times X_n \to K$ such that $\mathrm{Cocy}_K^{\mathcal{M}_{a_n}}(B_{\sigma_n}) < \frac{1}{n}$ and yet, for every extension Y of X_n and $\alpha : Y \to K$, the set of y satisfying $\alpha(sy) = \sigma_n(s, \phi(y))\alpha(y)$ for all $s \in S$ has measure at most $1 - \epsilon$. For each $n \in \mathbb{N}$, take an e.c. action $\Gamma \curvearrowright^{b_n} (Z_n, \eta_n)$ that factors onto $\Gamma \curvearrowright^{a_n} (X_n, \mu_n)$, say via the map ϕ_n. Let $\mathcal{U}$ be a nonprincipal ultrafilter on $\mathbb{N}$ and write $\Gamma \curvearrowright^b (Z, \eta)$ for the ultraproduct of the actions $\Gamma \curvearrowright^{b_n}$

(Z_n, η_n). Define $\sigma : \Gamma \times Z \to K$ by setting $\sigma_\gamma^{-1}(k) := [\phi_n^{-1} \circ (\sigma_n)_\gamma^{-1}(k)]_{\mathcal{U}}$ for each $k \in K$. Then $\mathrm{Cocy}_K^{\mathcal{M}_b}(B_\sigma) = 0$ and therefore σ is a cocycle for the ultraproduct action $\Gamma \curvearrowright^b Z$. Since T_Γ^* exists, $\Gamma \curvearrowright^b (Z, \eta)$ is e.c. whence Lemma 3.11 implies that there is a measurable map $\alpha : Z \to K$ such that the measure of the set of z with $\alpha(sz) = \sigma(s,z)\alpha(z)$ for all $s \in S$ has measure strictly greater than $1-\epsilon$. Pick a sequence of maps $\alpha_n : Z_n \to K$ such that α is the ultralimit of the α_n's, that is, that $\alpha^{-1}(k) = [\alpha_n^{-1}(k)]_{\mathcal{U}}$ for all $k \in K$. It follows that there is $n \in \mathbb{N}$ such that the set of $z \in Z_n$ satisfying $\alpha_n(sz) = \sigma_n(s, \phi_n(z))\alpha_n(z)$ for all $s \in S$ has measure greater than $1 - \epsilon$, contradicting the choice of the action $\Gamma \curvearrowright^{b_n} (X_n, \mu_n)$. □

The above proposition provides the impetus for a new definition, something we call the definable cocycle property. This property will be used to ensure that statements about cocycles mapping into finite groups are actually first-order, contributing part of our concrete axiomatization of the model companion. We first note some equivalences.

Lemma 6.2. *Let Γ be a countable group, K a finite group, and fix an L_Γ-theory T extending T_Γ (such as T_Γ itself, $T_{\Gamma,\mathrm{free}}$, $T_{\Gamma,\max}$, or T_Γ^*). Let ρ_1 and ρ_2 be metrics on $C^1(\Gamma, K^\Gamma)$ and $C^2(\Gamma, K^\Gamma)$ respectively that are compatible with their product topologies. Then the following are equivalent:*

(1) *For any $\epsilon > 0$, there is a $\delta > 0$ so that, for any $\mathcal{M}_a \models T$ and any $B \in \mathcal{M}_a^{\Gamma \times K}$, if $\mathrm{Cocy}_K^{\mathcal{M}_a}(B) < \delta$, then there is a cocycle σ of a such that $d(B, B_\sigma) \le \epsilon$.*
(2) *For any $\epsilon > 0$, there is $\delta > 0$ so that, for any action $\Gamma \curvearrowright^a (X, \mu)$ with $\mathcal{M}_a \models T$ and any equivariant measurable map $c : X \to C^1(\Gamma, K^\Gamma)$, if*

$$\int \rho_2(\partial c(x), e_{C^2(\Gamma, K^\Gamma)})\, d\mu < \delta,$$

then there is a measurable equivariant map $z : X \to Z^1(\Gamma, K^\Gamma)$ such that

$$\int \rho_1(c(x), z(x))\, d\mu < \epsilon.$$

(3) *For any family $(\mathcal{M}_{a_i})_{i \in I}$ of models of T and any ultrafilter $\mathcal{U}$ on I, setting $\mathcal{M} := \prod_{\mathcal{U}} \mathcal{M}_{a_i}$, we have $Z(\mathrm{Cocy}_K^{\mathcal{M}}) = \prod_{\mathcal{U}} Z(\mathrm{Cocy}_K^{\mathcal{M}_{a_i}})$.*
(4) *For any T-formula $\Phi(x, y)$, with x ranging over sort $\mathcal{M}_a^{\Gamma \times K}$, the T-functors $\sup_{B \in Z(\mathrm{Cocy}_K)} \Phi(B, y)$ and $\inf_{B \in Z(\mathrm{Cocy}_K)} \Phi(B, y)$ are T-formulae again.*

Proof. The equivalence of (1) and (2) is immediate from the definitions. The equivalence of (1), (3) and (4) is a special case of [24, Theorem 2.13]. Note that in [24, Theorem 2.13], there was no cardinality restriction on the index set (recall Convention 2.2 above); however, for separable theories (such as the T considered here), it is an artifact of the proof that one only needs to assume the preservation under countable ultraproducts. □

Note that condition (3) of the previous lemma is equivalent to the following: given any family $(\Gamma \curvearrowright^{a_i} (X_i, \mu_i))_{i \in I}$ of models of T and any ultrafilter $\mathcal{U}$ on I, setting $\Gamma \curvearrowright^a (X, \mu)$ to be the ultraproduct action, if $\sigma : \Gamma \times X \to K$ is a cocycle, then there are cocycles $\sigma^i : \Gamma \times X_i \to K$ such that $\sigma_\gamma^{-1}(k) = [(\sigma_\gamma^i)^{-1}(k)]_\mathcal{U}$ for all $\gamma \in \Gamma$ and $k \in K$. In other words, if we let $\sigma^\mathcal{U} : \Gamma \times X \to K$ denote the corresponding **ultraproduct cocycle**, that is, the cocycle of the ultraproduct action given by the formula $(\sigma_\mathcal{U})_\gamma^{-1}(k) := [(\sigma_\gamma^i)^{-1}(k)]_\mathcal{U}$ for all $\gamma \in \Gamma$ and $k \in K$, then condition (2) states that every cocycle of the product action is an ultraproduct cocyle.

When the equivalent conditions of Lemma 6.2 hold, we say that Cocy_K is a **T-definable set**. If Cocy_K is a T-definable set for all finite groups K, then we say that T has the **definable cocycle property**.

Corollary 6.3. *Suppose that T_Γ^* exists. Then T_Γ^* has the definable cocycle property.*

Proof. This follows immediately from Proposition 6.1 and the definition of e.c. action. □

We will soon establish that $T_{\Gamma,\mathrm{free}}$ (resp. $T_{\Gamma,\max}$) has the definable cocycle property when Γ is strongly treeable (resp. Γ is treeable). For now we make the following simple observation:

Lemma 6.4. *If $\mathbb{F}$ is any free group, then $T_\mathbb{F}$ has the definable cocycle property.*

Proof. We verify condition (3) of Lemma 6.2. Suppose that $\mathbb{F} \curvearrowright^a X$ is the ultraproduct of the actions $\mathbb{F} \curvearrowright^{a_i} X_i$ and suppose that $\sigma : \mathbb{F} \times X \to K$ is a cocycle for a. If S is a free generating set for $\mathbb{F}$, then for each $s \in S$ and $k \in K$, we can write $\pi_{s,k}(B_\sigma) = [X_{i,s,k}]_\mathcal{U}$, where, for each $i \in I$ and $s \in S$, $(X_{i,s,k})_{k \in K}$ is a measurable partition of X_i. If we define $\sigma_{i,0} : S \times X_i \to K$ by setting $\sigma_{i,0}(s, x) := k$ when $x \in X_{i,s,k}$, then $\sigma_{i,0}$ extends uniquely to a cocycle $\sigma_i : \mathbb{F} \times X_i \to K$ in such a way that $[\sigma_i]_\mathcal{U} = \sigma$. □

6.2. **Finite-to-one extensions and the extension-MD property.** In the last subsection, we observed that when T has the definable cocycle property, the K-valued cocycles form a definable set. We also know that cocycles to finite groups correspond to finite-to-one extensions (via the skew product construction and Rokhlin's skew product theorem). This leads us to consider the nature of finite-to-one extensions and their relationship with e.c. actions. We start with the following theorem, which characterizes those actions that are e.c. for finite-to-one extensions.

Theorem 6.5. *An action $\Gamma \curvearrowright (X, \mu)$ is e.c. for finite-to-one extensions if and only if: for every integer k, every cocycle $\sigma : \Gamma \times X \to \text{Sym}(k)$, every finite set $F \subseteq \Gamma$, every*

$\epsilon > 0$, *every integer* q, *and every map* $\beta : X \to q$, *there exists a map* $\alpha : X \to k$ *such that:*

(1) $\alpha_*\mu$ *is the normalized counting measure on* k.
(2) α *is* ϵ*-independent with* β, *that is, for all* $i < k$ *and all* $j < q$, *we have*

$$\left|\mu(\alpha^{-1}(i) \cap \beta^{-1}(j)) - \mu(\alpha^{-1}(i))\mu(\beta^{-1}(j))\right| \leq \epsilon.$$

(3) $\mu(\{x \in X\ : \alpha(fx) = \sigma(f,x)(\alpha(x))$ *for every* $f \in F\}) \geq 1 - \epsilon$.

Proof. First assume that $\Gamma \curvearrowright (X, \mu)$ is e.c. for finite-to-one extensions and fix k, σ, F, ϵ, q, and β as above. Let u_k denote the normalized counting measure on k and let $(Y, \nu) = (X \times_\sigma k, \mu \times u_k)$ be the skew-product extension of $\Gamma \curvearrowright (X, \mu)$ with respect to σ. Let $\phi : Y \to X$ and $\hat{\alpha} : Y \to k$ be the coordinate projection maps. Notice that $\hat{\alpha}_*\nu = u_k$ and that $\hat{\alpha}$ is independent with $\beta \circ \phi$. Also notice that $\hat{\alpha}(f \cdot y) = \sigma(f, \phi(y))(\hat{\alpha}(y))$ for every $f \in F$ and $y \in Y$, or equivalently, writing $S_{f,\tau} = \{x \in X : \sigma(f,x) = \tau\}$,

$$Y = \bigcup_{i \in k} \bigcap_{f \in F} \bigcup_{\tau \in \mathrm{Sym}(k)} \hat{\alpha}^{-1}(i) \cap \phi^{-1}(S_{f,\tau}) \cap f^{-1} \cdot \hat{\alpha}^{-1}(\tau(i)).$$

Since the action of $\Gamma \curvearrowright (X, \mu)$ is e.c. for finite-to-one extensions, there exists a map $\alpha : X \to k$ such that $\alpha_*\mu$ is as close to u_k as desired, α is $(\epsilon/2)$-independent with β, and the set

$$\bigcup_{i \in k} \bigcap_{f \in F} \bigcup_{\tau \in \mathrm{Sym}(k)} \alpha^{-1}(i) \cap S_{f,\tau} \cap f^{-1} \cdot \alpha^{-1}(\tau(i)),$$

that is, the set of $x \in X$ satisfying $\alpha(fx) = \sigma(f,x)(\alpha(x))$ for all $f \in F$, has measure greater than $1 - (\epsilon/2)$. If we choose α so that $\alpha_*\mu$ is sufficiently close to u_k, then we can perturb α so as to satisfy items (1)-(3) above.

Now assume that $\Gamma \curvearrowright (X, \mu)$ satisfies items (1)-(3) for any choice of k, σ, F, ϵ, q, and β as in the statement of the proposition. We show that $\Gamma \curvearrowright (X, \mu)$ is e.c. for finite-to-one extensions. By the Rohklin skew-product theorem and the ergodic decomposition, it suffices to show that $\Gamma \curvearrowright (X, \mu)$ is e.c. for skew-product extensions associated with finite groups. Thus, fix a cocycle $\sigma : \Gamma \times X \to \mathrm{Sym}(k)$, let u_k be normalized counting measure on k, and consider the associated skew-product extension $(Y, \nu) = (X \times_\sigma k, \mu \times u_k)$. We wish to show that the projection map $\phi : Y \to X$ is e.c. Let $p, q \in \mathbb{N}$, let $\gamma : Y \to p$ and $\beta : X \to q$ be measurable maps, let $\epsilon > 0$, and let $F \subseteq \Gamma$ be finite. Our goal is to find a measurable map $\hat{\gamma} : X \to p$ satisfying

$$|\mu(\hat{\gamma}_F^{-1}(\pi) \cap \beta^{-1}(j)) - \nu(\gamma_F^{-1}(\pi) \cap \phi^{-1}(\beta^{-1}(j))| < \epsilon \tag{12}$$

for all $\pi \in p^F$ and all $j \in q$.

For $\pi \in p^F$ and $i \in k$, let $D_{\pi,i}$ be the set of $x \in X$ satisfying $\gamma_F(x, i) = \pi$. Then

$$\nu(\gamma_F^{-1}(\pi) \cap \phi^{-1}(\beta^{-1}(j))) = \sum_{i \in k} \frac{1}{k} \mu(D_{\pi,i} \cap \beta^{-1}(j)). \tag{13}$$

By our assumption on the action $\Gamma \curvearrowright (X, \mu)$, we can pick a measurable map $\alpha : X \to k$ satisfying $\mu(A) > 1 - \frac{\epsilon}{2}$, where

$$A = \{x \in X : \forall f \in F\ \alpha(f^{-1}x) = \sigma(f^{-1}, x)(\alpha(x)),$$

and satisfying

$$\left| \sum_{i \in k} \mu(\alpha^{-1}(i) \cap D_{\pi,i} \cap \beta^{-1}(j)) - \sum_{i \in k} \frac{1}{k} \mu(D_{\pi,i} \cap \beta^{-1}(j) \right| < \frac{\epsilon}{2} \tag{14}$$

for every $\pi \in p^F$ and $j \in q$. Define $\hat{\gamma} : X \to p$ by $\hat{\gamma}(x) = \gamma(x, \alpha(x))$. Then for $x \in A$, we have

$$\begin{aligned}
\hat{\gamma}_F(x)(f) &= \hat{\gamma}(f^{-1}x) \\
&= \gamma(f^{-1}x, \alpha(f^{-1}x)) \\
&= \gamma(f^{-1}x, \sigma(f^{-1}, x)(\alpha(x))) \\
&= \gamma(f^{-1} \cdot (x, \alpha(x))) \\
&= \gamma_F(x, \alpha(x))(f).
\end{aligned}$$

As a result, for $x \in A$ we have $\hat{\gamma}_F(x) = \pi$ if and only if there is $i \in k$ with $x \in \alpha^{-1}(i) \cap D_{\pi,i}$. Therefore

$$\left| \mu(\hat{\gamma}_F^{-1}(\pi) \cap \beta^{-1}(j)) - \sum_{i \in k} \mu(\alpha^{-1}(i) \cap D_{\pi,i} \cap \beta^{-1}(j)) \right| < \mu(X \setminus A) < \frac{\epsilon}{2}. \tag{15}$$

Combining equations (13), (14), and (15) shows that (12) holds. □

Let $\Gamma \curvearrowright^a (X, \mu)$ be a p.m.p. action, let λ be Lebesgue measure, and let $\Gamma \curvearrowright^{\mathrm{id}} [0, 1]$ be the trivial action fixing every point. We call $\Gamma \curvearrowright^{a \times \mathrm{id}} (X \times [0, 1], \mu \times \lambda)$ the trivial extension of $\Gamma \curvearrowright^a (X, \mu)$ having atomless fibers.

Corollary 6.6. *Let $\Gamma \curvearrowright^a (X, \mu)$ be a p.m.p. action, and suppose that $\Gamma \curvearrowright^{a \times \mathrm{id}} (X \times [0, 1], \mu \times \lambda)$ is an e.c. extension of $\Gamma \curvearrowright^a (X, \mu)$. Then $\Gamma \curvearrowright^a (X, \mu)$ is e.c. for finite-to-one extensions if and only if $B^1(a, \mathrm{Sym}(k))$ is dense in $Z^1(a, \mathrm{Sym}(k))$ for every $k \in \mathbb{N}$.*

Proof. We will apply Theorem 6.5. So let $k, q \in \mathbb{N}$, let $\sigma \in Z^1(a, \mathrm{Sym}(k))$, let $F \subseteq \Gamma$ be finite, let $\epsilon > 0$, and let $\beta : X \to q$ be measurable.

Let u_k be the normalized counting measure on $k = \{0, \dots, k-1\}$. Define $(\tilde{X}, \tilde{\mu}) = (X \times k, \mu \times u_k)$ and define $\tilde{\beta} : \tilde{X} \to q$ and $\tilde{\sigma} : \Gamma \times \tilde{X} \to \mathrm{Sym}(k)$ by

$$\tilde{\beta}(x, i) = \beta(x) \quad \text{and} \quad \tilde{\sigma}(\gamma, (x, i)) = \sigma(\gamma, x).$$

We know that $\Gamma \curvearrowright^{a \times \mathrm{id}} (\tilde{X}, \tilde{\mu})$ is an e.c. extension of $\Gamma \curvearrowright^{a} (X, \mu)$ since this extension is intermediary to the trivial extension having atomless fibers. Therefore it suffices to find a measurable map $\tilde{\alpha} : \tilde{X} \to k$ satisfying conditions (1), (2), and (3) of Theorem 6.5 with $X, \mu, \sigma, \alpha, \beta$ replaced by $\tilde{X}, \tilde{\mu}, \tilde{\sigma}, \tilde{\alpha}, \tilde{\beta}$.

Since we are assuming $B^1(a, \mathrm{Sym}(k))$ is dense in $Z^1(a, \mathrm{Sym}(k))$, there is a measurable map $\alpha : X \to \mathrm{Sym}(k)$ such that the set

$$Y = \{x \in X : \sigma(f, x) = \alpha(f^a \cdot x)\alpha(x)^{-1} \text{ for all } f \in F\}$$

satisfies $\mu(Y) > 1 - \epsilon$. Now define $\tilde{\alpha} : \tilde{X} \to k$ by

$$\tilde{\alpha}(x, i) = \alpha(x)(i).$$

Then we have

$$\tilde{\alpha}_* \tilde{\mu} = \int \alpha(x)_* u_k \, d\mu = \int u_k \, d\mu = u_k,$$

so (1) is satisfied, and

$$\begin{aligned} \tilde{\mu}(\tilde{\alpha}^{-1}(i) \cap \tilde{\beta}^{-1}(j)) &= \tilde{\mu}(\{(x, \alpha(x)^{-1}(i)) : x \in \beta^{-1}(j)\}) \\ &= \frac{1}{k} \cdot \mu(\beta^{-1}(j)) \\ &= \tilde{\mu}(\tilde{\alpha}^{-1}(i))\tilde{\mu}(\tilde{\beta}^{-1}(j)), \end{aligned}$$

so (2) is satisfied. Finally, $\tilde{\mu}(Y \times k) = \mu(Y) > 1 - \epsilon$ and for every $(x, i) \in Y \times k$ and $f \in F$ we have

$$\tilde{\alpha}(f^{a \times \mathrm{id}}(x, i)) = \alpha(f^a \cdot x)(i) = \sigma(f, x) \circ \alpha(x)(i) = \tilde{\sigma}(f, (x, i))(\tilde{\alpha}(x, i)).$$

Thus (3) is satisfied. □

The previous results motivate the consideration of actions having the property that all of their extensions can be approximated by their finite-to-one extensions. Drawing a parallel with property MD for groups, we make the following definition.

Definition 6.7. We say that a p.m.p. action $\Gamma \curvearrowright^{a} (X, \mu)$ on a standard probability space is **extension-MD** if for any (equivalently, every) non-atomic standard probability space (Y, ν), the set of finite-to-one a-extensions are dense in $F_a(\Gamma, Y, X)$ (as defined in Subsection 2.2). If every free p.m.p. action of Γ on a standard probability space is extension-MD, then we say that Γ has the **extension-MD property**.

Similar to Proposition 3.6, we will rely on the following characterization of this property.

Proposition 6.8. *The action* $\Gamma \curvearrowright^{a} (X, \mu)$ *is extension-MD if and only if* (X, μ) *is a standard probability space and, whenever* $\Gamma \curvearrowright^{b} (Y, \nu)$ *is a p.m.p. action extending* $\Gamma \curvearrowright^{a} (X, \mu)$*, say via* $\phi : Y \to X$*,* $p, q \in \mathbb{N}$*,* $\alpha : Y \to p$*, and* $\beta : X \to q$ *are measurable,* $F \subseteq \Gamma$ *is finite, and* $\epsilon > 0$*, there is a p.m.p. action* $\Gamma \curvearrowright^{b'} (Y', \nu')$ *extending* $\Gamma \curvearrowright^{a} (X, \mu)$*, say via* $\phi' : Y' \to X$*, and a measurable map* $\alpha' : Y' \to p$ *such that* ϕ' *is almost-everywhere finite-to-one and*

$$\left|\nu\Big(\alpha_F^{-1}(\pi) \cap \phi^{-1}(\beta^{-1}(j))\Big) - \nu'\Big((\alpha')_F^{-1}(\pi) \cap (\phi')^{-1}(\beta^{-1}(j))\Big)\right| < \epsilon \tag{16}$$

for all $\pi \in p^F$ *and* $j \in q$*.*

Proof. If $\Gamma \curvearrowright^{a} (X, \mu)$ is extension-MD, then (X, μ) is a standard probability space and it is easy to see that it is then enough to check the above condition in the cases where (Y, ν) is a standard non-atomic probability space. In this case, every (ϕ', b') in some open neighborhood of (ϕ, b) will satisfy (16) using the same map $\alpha : Y \to p$ (in this case, take care to note that α_F depends on the action being considered and can be written $\alpha_{b(F)}$ and $\alpha_{b'(F)}$ for clarity). The assumption that the action a is extension-MD implies that this open set contains a finite-to-one extension of a. Conversely, suppose the above condition holds. Consider a standard non-atomic probability space (Y, ν), an a-extension $(\phi, b) \in F_a(\Gamma, Y, X)$, and an open neighborhood $U_a^{\mathcal{A},\mathcal{B},F,\epsilon}(\phi, b)$. Choose $\alpha : Y \to p$ and $\beta : X \to q$ so that $\mathcal{A} = \{\alpha^{-1}(i) : i \in p\}$ and $\mathcal{B} = \{\beta^{-1}(j) : j \in q\}$. Let $\Gamma \curvearrowright^{b'} (Y', \nu')$, $\phi' : Y' \to X$, and $\alpha' : Y' \to p$ satisfy (16) for all $\pi \in p^F$ and $j \in q$. We can assume that (Y', ν') is non-atomic, and by passing to a factor of Y' for which ϕ' and α' remain measurable, we may further assume that (Y', ν') is a standard probability space. Notice that the sets appearing in (16) partition Y and Y' as $\pi \in p^F$ and $j \in q$ vary. Since (16) implies that the measures of these pieces are within ϵ of one another, there is an isomorphism of probability spaces $S : (Y', \nu') \to (Y, \nu)$ that matches the respective pieces of the partitions, for each $\pi \in p^F$ and $j \in q$, up to an error (symmetric difference) of measure ϵ. The resulting pair $(S \cdot \phi', S \cdot b')$ will belong to $U_a^{\mathcal{A},\mathcal{B},F,\epsilon}(\phi, b)$, and since ϕ' is finite-to-one and S is an isomorphism, $S \cdot \phi'$ will be finite-to-one as well. □

The above proposition allows us to extend the definition of extension-MD to actions on non-standard probability spaces: we say that a p.m.p. action $\Gamma \curvearrowright^{a} (X, \mu)$ is **extension-MD** if it satisfies the condition of the above proposition. It is easily seen then that if Γ has the extension-MD property as defined in Definition 6.7 then all free p.m.p. actions of Γ (on both standard and non-standard spaces) are extension-MD.

A trivial consequence of the above proposition and Proposition 3.6 is that every e.c. action is automatically extension-MD. The following is also clear.

Lemma 6.9. *An extension-MD action is e.c. if and only if it is e.c. for finite-to-one extensions.*

Before ending this subsection, we make an additional observation about extension-MD actions and show that free groups have the extension-MD property.

Lemma 6.10. *Let* $\Gamma \curvearrowright^a (X, \mu)$ *be a p.m.p. action. Let* $G = \mathrm{Aut}([0,1], \lambda)$, *where* λ *is Lebesgue measure, equipped with the weak topology, and let* ρ_1 *be a metric on* $C^1(\Gamma, G^\Gamma)$ *compatible with its product topology. Write* $Z^1(a, G)_{\mathrm{fin}}$ *for the set of all measurable cocycles on* a *that map to a finite subgroup of* G. *Then the following are equivalent:*

(1) $Z^1(a, G)_{\mathrm{fin}}$ *is dense in* $Z^1(a, G)$;
(2) *for every measurable cocycle* $\sigma : \Gamma \times X \to G$, *every finite set* $F \subseteq \Gamma$, *every* $\epsilon > 0$, *and every finite tuple* $(C_i)_{i \in n}$ *of Borel subsets of* $[0,1]$, *there is a finite subgroup* $H \leq G$ *and a measurable cocycle* $\sigma' : \Gamma \times X \to H$ *such that, for all* $\gamma \in F$ *and all* $i \in n$:
$$\int \lambda\big(\sigma(\gamma, x)^{-1}(C_i) \triangle \sigma'(\gamma, x)^{-1}(C_i)\big)\, d\mu < \epsilon;$$
(3) *for every measurable equivariant map* $z : X \to Z^1(\Gamma, G^\Gamma)$ *and every* $\epsilon > 0$, *there is a finite subgroup* $H \leq G$ *and a measurable equivariant map* $z' : X \to Z^1(\Gamma, H^\Gamma)$ *satisfying* $\int \rho_1(z(x), z'(x))\, d\mu < \epsilon$.

Moreover, the above properties imply that the action $\Gamma \curvearrowright^a (X, \mu)$ *is extension-MD.*

Proof. The equivalence of (1), (2), and (3) is immediate from the definitions. So it will be enough to show that (2) implies that $\Gamma \curvearrowright^a (X, \mu)$ is extension-MD. We will do this by checking the condition in Proposition 6.8.

Let $\Gamma \curvearrowright^b (Y, \nu)$ be a p.m.p. extension of $\Gamma \curvearrowright^a (X, \mu)$, say via the map $\phi : (Y, \nu) \to (X, \mu)$. By disintegrating ν with respect to ϕ we obtain a measurable map $x \in X \mapsto \nu_x \in \mathrm{Prob}(Y)$ satisfying $\nu_x(\phi^{-1}(x)) = 1$ for μ-almost-every x and $\int \nu_x\, d\mu = \nu$. Since, in verifying the criterion in Proposition 6.8, we could let Γ act trivially on $([0,1], \lambda)$, where λ is the Lebesgue measure, and lift any $\alpha : Y \to p$ to the direct product action $\Gamma \curvearrowright (Y \times [0,1], \nu \times \lambda)$, we see that without loss of generality we may assume that ν_x is non-atomic for μ-almost-every $x \in X$. Then by the Rokhlin skew-product theorem we can assume that $(Y, \nu) = (X \times [0,1], \mu \times \lambda)$, that ϕ is the projection map to X, and that there is a measurable cocycle $\sigma : \Gamma \times X \to \mathrm{Aut}([0,1], \lambda)$ so that the action b is the skew-product action given by the formula
$$\gamma^b \cdot (x, r) = (\gamma^a \cdot x, \sigma(\gamma, x)(r)).$$

Now consider a pair of measurable maps $\alpha : X \times [0,1] \to p$ and $\beta : X \to q$, a finite $F \subseteq \Gamma$ and an $\epsilon > 0$. Since $\mathrm{MALG}([0,1],\lambda)$ is complete, we can find a finite algebra $\mathcal{C}$ of Borel subsets of $[0,1]$ and a measurable function $\tilde{\alpha} : X \times [0,1] \to p$ satisfying $\{r \in [0,1] : \tilde{\alpha}(x,r) = i\} \in \mathcal{C}$ for every $x \in X$ and $(\mu \times \lambda)(\{(x,r) : \alpha(x,r) \neq \tilde{\alpha}(x,r)\}) < \epsilon/(2|F|)$. Notice that for every $\pi \in p^F$ and $j \in q$

$$|(\mu \times \lambda)(\alpha_F^{-1}(\pi) \cap (\beta \circ \phi)^{-1}(j)) - (\mu \times \lambda)(\tilde{\alpha}_F^{-1}(\pi) \cap (\beta \circ \phi)^{-1}(j)) < \frac{\epsilon}{2} \tag{17}$$

Define $A_i^x = \{r \in [0,1] : \tilde{\alpha}(x,r) = i\} \in \mathcal{C}$ for each $x \in X$ and $i \in p$.

Let $\sigma' : \Gamma \times X \to G$ be a measurable cocycle that takes values in a finite subgroup of G and satisfies

$$\sum_{\gamma \in F^{-1}} \sum_{C \in \mathcal{C}} \int \lambda\big(\sigma(\gamma,x)^{-1}(C) \triangle \sigma'(\gamma,x)^{-1}(C)\big)\, d\mu < \frac{\epsilon}{2}.$$

Let b' be the skew-product action of Γ on $(X \times [0,1], \mu \times \lambda)$ given by σ', that is, $\gamma^{b'} \cdot (x,r) = (\gamma^a \cdot x, \sigma'(\gamma,x)(r))$, and regard b' as an extension of a via the projection map ϕ. We will write $\tilde{\alpha}_{b(F)}$ and $\tilde{\alpha}_{b'(F)}$ in place of $\tilde{\alpha}_F$ in order to clarify the action being considered. Then for every $\pi \in p^F$ and $j \in q$ we have

$$\begin{aligned}
&\big|(\mu \times \lambda)\big(\tilde{\alpha}_{b(F)}^{-1}(\pi) \cap (\beta \circ \phi)^{-1}(j)\big) - (\mu \times \lambda)\big(\tilde{\alpha}_{b'(F)}^{-1}(\pi) \cap (\beta \circ \phi)^{-1}(j)\big)\big| \\
&\leq (\mu \times \lambda)\big((\tilde{\alpha}_{b(F)}^{-1}(\pi) \triangle \tilde{\alpha}_{b'(F)}^{-1}(\pi)) \cap (\beta \circ \phi)^{-1}(j)\big) \\
&= \int_{\beta^{-1}(j)} \lambda\left(\left(\bigcap_{\gamma \in F^{-1}} \sigma(\gamma,x)^{-1}(A_{\pi(\gamma^{-1})}^{\gamma^a \cdot x})\right) \triangle \left(\bigcap_{\gamma \in F^{-1}} \sigma'(\gamma,x)^{-1}(A_{\pi(\gamma^{-1})}^{\gamma^a \cdot x})\right)\right)\, d\mu \\
&\leq \sum_{\gamma \in F^{-1}} \int_X \lambda\Big(\sigma(\gamma,x)^{-1}(A_{\pi(\gamma^{-1})}^{\gamma^a \cdot x}) \triangle \sigma'(\gamma,x)^{-1}(A_{\pi(\gamma^{-1})}^{\gamma^a \cdot x})\Big)\, d\mu < \frac{\epsilon}{2}.
\end{aligned}$$

Consequently, for every $\pi \in p^F$ and $j \in q$ we have

$$|(\mu \times \lambda)(\alpha_{b(F)}^{-1}(\pi) \cap (\beta \circ \phi)^{-1}(j)) - (\mu \times \lambda)(\tilde{\alpha}_{b'(F)}^{-1}(\pi) \cap (\beta \circ \phi)^{-1}(j)) < \epsilon.$$

Finally, say σ' takes values in the finite subgroup $H \leq G$, and let $\mathcal{C}'$ be the finite H-invariant algebra generated by $\mathcal{C}$. Let (U, ρ) be a finite probability space and $\psi : ([0,1], \lambda) \to (U, \rho)$ a measure-preserving map such that the ψ-preimage of the powerset of U is $\mathcal{C}'$. Since H descends to the group of measure-preserving automorphisms of (U, ρ), the skew-product action b' descends to an intermediary extension $\Gamma \curvearrowright^c (X \times U, \mu \times \rho)$ of a, and the map $\tilde{\alpha}$ descends as well. Since U is finite, this is a finite-to-one extension of a that, together with the descended map $\tilde{\alpha}$, approximates the extension b and the map α relative to the parameters F and ϵ, in accordance with Proposition 6.8. □

We remark that we do not know if the converse of the previous lemma holds. If $\Gamma \curvearrowright^a (X, \mu)$ is extension-MD, then one can show that, for every measurable cocycle $\sigma : \Gamma \times X \to \mathrm{Aut}([0,1], \lambda)$, every finite $F \subseteq \Gamma$, every $\epsilon > 0$, every finite tuple $(C_i)_{i \in n}$ of Borel subsets of $[0,1]$, and every finite tuple $(A_k)_{k \in m}$ of Borel subsets of X, there is a finite subgroup $H \leq \mathrm{Aut}([0,1], \lambda)$, and a measurable cocycle $\sigma' : \Gamma \times X \to H$ such that for all $\gamma \in F$, $i, j \in n$ and $k \in m$

$$\left| \int_{A_k} \lambda(C_j \cap \sigma(\gamma, x)^{-1}(C_i)) \, d\mu - \int_{A_k} \lambda(C_j \cap \sigma'(\gamma, x)^{-1}(C_i)) \, d\mu \right| < \epsilon.$$

Compare this with (2) above.

Lemma 6.11. *If $\mathbb{F}$ is a free group, then $\mathbb{F}$ has the extension-MD property.*

Proof. Suppose that $\mathbb{F}$ is freely generated by the set S, and let $\mathbb{F} \curvearrowright^a (X, \mu)$ be a free p.m.p. action. Every measurable cocycle $\sigma : \mathbb{F} \times X \to \mathrm{Aut}([0,1], \lambda)$ is uniquely determined by its restriction to $S \times X$, and conversely every measurable map from $S \times X$ to $\mathrm{Aut}([0,1], \lambda)$ uniquely determines a cocycle. Moreover, if we topologize the collection of measurable functions from $S \times X$ to $\mathrm{Aut}([0,1], \lambda)$ so that $f_n \to f$ if and only if $f_n(s, \cdot)$ converges to $f(s, \cdot)$ in measure for every $s \in S$, then this correspondence is a homeomorphism. Since $\mathrm{Aut}([0,1], \lambda)$ admits an increasing sequence of finite subgroups whose union is dense (that is, consider the subgroup H_n of automorphisms that fix 1 and permute the intervals $[\frac{i}{2^n}, \frac{i+1}{2^n})$ via order-preserving isometries), it is immediate that the measurable maps from $S \times X$ to finite subgroups of $\mathrm{Aut}([0,1], \lambda)$ are dense in the space of all measurable maps from $S \times X$ to $\mathrm{Aut}([0,1], \lambda)$. Therefore condition (1) of Lemma 6.10 is satisfied. □

6.3. **The axiomatization.** In this section, we combine the ideas from the previous subsections to show that, for a certain class of groups, the model companion T^*_Γ exists and has a concrete set of axioms that are ergodic-theoretic in nature.

Theorem 6.12. *Suppose that Γ has the extension-MD property and that $T_{\Gamma,\mathrm{free}}$ has the definable cocycle property. Then T^*_Γ exists.*

Proof. For each $k, q \geq 1$ and each finite $F \subseteq \Gamma$, let $\theta_{k,q,F}$ be the $T_{\Gamma,\mathrm{free}}$-sentence

$$\sup_{A \in \mathrm{Part}_q} \sup_{B \in Z(\mathrm{Cocy}_{\mathrm{Sym}(k)})} \inf_{C \in \mathrm{Part}_k} \max(\varphi_1(C), \varphi_2(A, C), \varphi_3(B, C)),$$

where:

(1) $\varphi_1(C)$ is the formula

$$\max_{i=1,\ldots,k} \left| \mu(C_i) - \frac{1}{k} \right|,$$

(2) $\varphi_2(A, C)$ is the formula

$$\max_{i=1,\ldots,q}\max_{j=1,\ldots,k}|\mu(A_i\cap C_j)-\mu(A_i)\mu(C_j)|,$$

(3) $\varphi_3(B, C)$ is the formula

$$\max_{\gamma\in F}\max_{i=1,\ldots,k}\max_{\rho\in\mathrm{Sym}(k)} d\left(\pi_{\gamma,\rho}(B)\cap C_i,\pi_{\gamma,\rho}(B)\cap\gamma^{-1}C_{\rho(i)}\right)$$

Here, Part_k denotes the T_Γ-definable set of k-tuples that form a partition of the measure space. Since $T_{\Gamma,\mathrm{free}}$ has the definable cocycle property, the above sentences are indeed (equivalent modulo $T_{\Gamma,\mathrm{free}}$ to) $T_{\Gamma,\mathrm{free}}$-sentences.

Set T_Γ^* to be $T_{\Gamma,\mathrm{free}}$ together with all of the sentences $\theta_{k,q,F}$. Theorem 6.5, together with Lemma 6.9, shows that T_Γ^* does indeed axiomatize the class of e.c. models of T_Γ. □

Recalling Lemmas 6.4 and 6.11, we see that the concrete axioms in the previous theorem yield an alternative axiomatization for $T_{\mathbb{F}}^*$ for free groups $\mathbb{F}$.

The exact same proof as that of Theorem 6.12 yields the following theorem:

Theorem 6.13. *If $T_{\Gamma,\max}$ has the definable cocycle property and every action of Γ that is maximal with respect to weak containment is extension-MD, then T_Γ^* exists. Moreover, the axioms for T_Γ^* are given by the axioms for $T_{\Gamma,\max}$ together with the sentences $\theta_{k,q,F}$ from the proof of Theorem 6.12.*

In the remaining three subsections, we will show that if Γ is strongly treeable, then Γ has the extension-MD property and $T_{\Gamma,\mathrm{free}}$ has the definable cocycle property. Thus Theorem 6.12 yields a concrete axiomatization for T_Γ^* when Γ is strongly treeable. We will additionally show that when Γ is only assumed to be treeable, $T_{\Gamma,\max}$ has the definable cocycle property and all actions of Γ that are maximal with respect to weak containment are extension-MD. As a result, Theorem 6.13 provides an axiomatization for T_Γ^* when Γ is treeable.

6.4. **Trees and group cohomology.** We now turn our attention towards the goal of (partially) generalizing Lemmas 6.4 and 6.11 to (strongly) treeable groups. The proofs of these prior results for a free group $\mathbb{F}$ were based on the simple observation that cocycles $\sigma:\mathbb{F}\times X\to G$ are in one-to-one correspondence with measurable maps $S\times X\to G$, where S is a free generating set for $\mathbb{F}$. In fact, the map that restricts cochains $\theta:\mathbb{F}\times X\to G$ to the domain $S\times X$, when combined with the one-to-one correspondence just mentioned, provides a retraction from the space of cochains to the space of cocycles. (Recall that if Y is a topological space and $A\subseteq Y$, then a map $r:Y\to A$ is called a retraction if it is continuous and restricts to the identity map on A.)

The lemma below generalizes this retraction phenomena to other groups by replacing the role of the free generating set S with the set of edges of a tree $T \in \mathcal{T}(\Gamma)$.

Lemma 6.14. *Let Γ be a countable group and G a Polish group. There is a map assigning each tree $T \in \mathcal{T}(\Gamma)$ a retraction $r_T : C^1(\Gamma, G^\Gamma) \to Z^1(\Gamma, G^\Gamma)$ such that:*

(1) $r_{\gamma^d \cdot T}(\gamma^t \cdot c) = \gamma^t \cdot r_T(c)$ *for all* $T \in \mathcal{T}(\Gamma)$, $c \in C^1(\Gamma, G^\Gamma)$, *and* $\gamma \in \Gamma$; *and*
(2) *for every* $\alpha, \beta \in \Gamma$ *there is a clopen partition* $\mathcal{U}$ *of* $\mathcal{T}(\Gamma)$ *so that* $r_T(c)(\beta)(\alpha) = r_{T'}(c)(\beta)(\alpha)$ *for all* $c \in C^1(\Gamma, G^\Gamma)$ *when* T *and* T' *belong to a common* $V \in \mathcal{V}$.

Proof. For $(T, c) \in \mathcal{T}(\Gamma) \times C^1(\Gamma, G^\Gamma)$ define a function $g^c_T : T \cup \bar{T} \to G$ by setting, for each $(v, u) \in T$,

$$g^c_T(v, u) = c(u^{-1}v)(u)$$

and

$$g^c_T(u, v) = c(u^{-1}v)(u)^{-1}.$$

Clearly the map $c \mapsto g^c_T(v, u)$ is continuous for fixed T. Also note that for any $\gamma \in \Gamma$ we have that $(v, u) \in T$ if and only if $(\gamma v, \gamma u) \in \gamma^d \cdot T$; moreover when this occurs we have

$$g^{\gamma^t \cdot c}_{\gamma^d \cdot T}(\gamma v, \gamma u) = (\gamma^t \cdot c)(u^{-1}v)(\gamma u) = c(u^{-1}v)(u) = g^c_T(v, u).$$

Similarly $g^{\gamma^t \cdot c}_{\gamma^d \cdot T}(\gamma v, \gamma u) = g^c_T(v, u)$ when $(v, u) \in \bar{T}$.

For $(T, c) \in \mathcal{T}(\Gamma) \times C^1(\Gamma, G^\Gamma)$ we define $r_T(c) \in Z^1(\Gamma, G^\Gamma)$ as follows. For each $\alpha, \beta \in \Gamma$, let $v_0, \ldots, v_n$ be the sequence of vertices in the geodesic path from α to $\alpha\beta$ in $T \cup \bar{T}$ (meaning $v_0 = \alpha$, $(v_{i+1}, v_i) \in T \cup \bar{T}$ for every i, and $v_n = \alpha\beta$) and set

$$r_T(c)(\beta)(\alpha) = g^c_T(v_n, v_{n-1}) g^c_T(v_{n-1}, v_{n-2}) \cdots g^c_T(v_1, v_0). \tag{18}$$

Based on the previous paragraph, it is immediate that $c \mapsto r_T(c)$ is continuous for fixed T. Additionally, $r_T(c)(\beta)(\alpha)$ only depends on T in so far as T determines the path $v_0, \ldots, v_n$; hence (2) holds. Note that since T is a tree and $g^c_T(u, v) = g^c_T(v, u)^{-1}$ for all $(v, u) \in T \cup \bar{T}$, in the above formula one can use any path in $T \cup \bar{T}$ from α to $\alpha\beta$, not necessarily the geodesic path. Consequently, if $u_0 = \gamma, \ldots, u_\ell = \gamma\alpha$ is a path from γ to $\gamma\alpha$ in $T \cup \bar{T}$ and $v_0 = \gamma\alpha, \ldots, v_m = \gamma\alpha\beta$ is a path from $\gamma\alpha$ to $\gamma\alpha\beta$ in $T \cup \bar{T}$, then $u_0, \ldots, u_\ell, v_1, \ldots, v_m$ is a path in $T \cup \bar{T}$ from γ to $\gamma\alpha\beta$ and it from equation (18) that, for all $\gamma \in \Gamma$:

$$r_T(c)(\beta)(\gamma\alpha) r_T(c)(\alpha)(\gamma) = r_T(c)(\alpha\beta)(\gamma).$$

Thus

$$r_T(c)(\beta)^\alpha r_T(c)(\alpha) = r_T(c)(\alpha\beta)$$

and $r_T(c) \in Z^1(\Gamma, G^\Gamma)$ as required.

To see that r_T is a retraction, consider any $z \in Z^1(\Gamma, G^\Gamma)$. For all $\alpha, \beta, \gamma \in \Gamma$ we have $\partial z(\alpha, \beta)(\gamma) = e_G$, meaning

$$z(\beta)(\gamma\alpha)z(\alpha)(\gamma) = z(\alpha\beta)(\gamma).$$

By making the substitutions $\alpha = u$, $\beta = u^{-1}v$, and $\gamma = e$ we obtain

$$z(u^{-1}v)(u)z(u)(e) = z(v)(e).$$

Thus when $(v, u) \in T$ we have

$$g^z_T(v, u) = z(u^{-1}v)(u) = z(v)(e)z(u)(e)^{-1}$$

and

$$g^z_T(u, v) = g^z_T(v, u)^{-1} = z(u)(e)z(v)(e)^{-1}.$$

It now immediately follows from the formula (18) that for all $\alpha, \beta \in \Gamma$

$$r_T(z)(\beta)(\alpha) = z(\alpha\beta)(e)z(\alpha)(e)^{-1} = z(\beta)(\alpha),$$

and thus $r_T(z) = z$.

Lastly, when $v_0 = \gamma^{-1}\alpha, \ldots, v_n = \gamma^{-1}\alpha\beta$ is a path in $T \cup \bar{T}$ from $\gamma^{-1}\alpha$ to $\gamma^{-1}\alpha\beta$, we have $\gamma v_0, \ldots, \gamma v_n$ is a path in $\gamma^d \cdot (T \cup \bar{T})$ from α to $\alpha\beta$. Therefore

$$\begin{aligned} r_{\gamma^d \cdot T}(\gamma^t \cdot c)(\beta)(\alpha) &= g^{\gamma^t \cdot c}_{\gamma^d \cdot T}(\gamma v_n, \gamma v_{n-1}) \cdots g^{\gamma^t \cdot c}_{\gamma^d \cdot T}(\gamma v_1, \gamma v_0) \\ &= g^c_T(v_n, v_{n-1}) \cdots g^c_T(v_1, v_0) \\ &= r_T(c)(\beta)(\gamma^{-1}\alpha) \\ &= (\gamma^t \cdot r_T(c))(\beta)(\alpha), \end{aligned}$$

and $r_{\gamma^d \cdot T}(\gamma^t \cdot c) = \gamma^t \cdot r_T(c)$ as required by (1). □

Note that if H is a subgroup of G then $C^1(\Gamma, H^\Gamma)$ and $Z^1(\Gamma, H^\Gamma)$ are subsets of $C^1(\Gamma, G^\Gamma)$ and $Z^1(\Gamma, G^\Gamma)$, respectively; moreover if r_T is the map defined in the previous lemma with respect to G, then r_T maps the subset $C^1(\Gamma, H^\Gamma)$ to $Z^1(\Gamma, H^\Gamma)$.

When $\Gamma \curvearrowright^a (X, \mu)$ is a free p.m.p. treeable action there is a measurable equivariant map $x \in X \mapsto T(x) \in \mathcal{T}(\Gamma)$ from $\Gamma \curvearrowright^a (X, \mu)$ to $\Gamma \curvearrowright^d (\mathcal{T}(\Gamma), \tau)$ for some invariant Borel probability measure $\tau = T_*\mu$. In this case the above lemma provides a technique for turning cochains into cocycles (recall Lemma 2.3): given any measurable equivariant map $\theta : X \to C^1(\Gamma, G^\Gamma)$ we obtain a measurable equivariant map $\phi : X \to Z^1(\Gamma, G^\Gamma)$ via the formula $\phi(x) = r_{T(x)}(\theta(x))$. Although this does closely resemble the technique we used for verifying the definable cocycle and extension-MD properties for free groups, this new variant of the technique has a significant limitation in that the measure $\tau \in \mathrm{Prob}_\Gamma(\mathcal{T}(\Gamma))$ may vary as one considers different treeable actions of Γ. The constraint of factoring onto $\Gamma \curvearrowright^d (\mathcal{T}(\Gamma), \tau)$ for some particular τ poses an obstruction that, for the moment, prevents us from proving that $T_{\Gamma,\mathrm{free}}$ has the definable cocyle property when Γ is strongly treeable.

6.5. **Weak containment of treeings.** The purpose of this section is to overcome the limitation described at the end of the previous subsection. We will do this by relaxing the requirement of factoring onto $\Gamma \curvearrowright^{d} (\mathcal{T}(\Gamma), \tau)$ to merely weakly containing the action $\Gamma \curvearrowright^{d} (\mathcal{T}(\Gamma), \tau)$. We will show that in this case the mapping $T \mapsto r_T$ from Lemma 6.14 is still usable in an approximate form.

We begin with a technical lemma.

Lemma 6.15. *Let* X *and* Z *be compact Hausdorff spaces, let* $\pi_X : X \times Z \to X$ *be the projection map, let* μ *be a Borel probability measure on* X*, and let* $Y \subseteq Z$ *be Borel. Set*

$$\Omega = \{\omega \in \mathrm{Prob}(X \times Z) : \omega(X \times Y) = 1,\ (\pi_X)_*\omega = \mu\},$$

and equip Ω *with the relative topology inherited from the weak* topology on* $\mathrm{Prob}(X \times Z)$*. If* $f : X \times Y \to [0,1]$ *is a measurable function having the property that for every* $\epsilon > 0$ *there is a relatively clopen partition* $\mathcal{V}$ *of* Y *satisfying*

$$\sup_{x \in X} |f(x,y) - f(x,y')| < \epsilon$$

whenever y *and* y' *belong to a common* $V \in \mathcal{V}$*, then the map* $\omega \in \Omega \mapsto \int f \, d\omega$ *is continuous.*

Proof. Fix $\epsilon > 0$ and let $\mathcal{V}$ be as described. Also fix a point $\omega_0 \in \Omega$. Choose a finite subcollection $\mathcal{V}' \subseteq \mathcal{V}$ so that $\omega_0(Y \setminus \bigcup \mathcal{V}') < \epsilon$, and choose a clopen set $W \subseteq Z$ with $W \cap Y = Y \setminus \bigcup \mathcal{V}'$. For each $V \in \mathcal{V}'$ pick a point $y_V \in V$ and choose a continuous function $g_V : X \to [0,1]$ satisfying

$$\int |g_V(x) - f(x, y_V)| \, d\mu < \frac{\epsilon}{|\mathcal{V}'|}.$$

Pick a collection $\mathcal{U}'$ of pairwise disjoint clopen subsets of Z having nonempty intersection with Y and satisfying $\{U \cap Y : U \in \mathcal{U}'\} = \mathcal{V}'$, and define a continuous function $g : X \times Z \to [0,1]$ by

$$g(x,z) = \sum_{U \in \mathcal{U}'} g_{U \cap Y}(x) 1_U(z).$$

For every $\omega \in \Omega$ we have

$$\begin{aligned} \int |f - g| \, d\omega &\leq \omega(W) + \sum_{V \in \mathcal{V}'} \int_{X \times V} |f(x,y) - g_V(x)| \, d\omega \\ &\leq \omega(W) + \epsilon + \sum_{V \in \mathcal{V}'} \int_{X \times V} |f(x,y_V) - g_V(x)| \, d\omega \end{aligned}$$

$$\leq \omega(W) + \epsilon + \sum_{V \in \mathcal{V}'} \int_{X \times Z} |f(x, y_V) - g_V(x)| \, d\omega$$
$$= \omega(W) + \epsilon + \sum_{V \in \mathcal{V}'} \int |f(x, y_V) - g_V(x)| \, d\mu$$
$$< \omega(W) + 2\epsilon.$$

The set

$$\left\{ \omega \in \Omega : \omega(W) < \epsilon, \left| \iint g \, d\omega - \int g \, d\omega_0 \right| < \epsilon \right\}$$

is a relatively open neighborhood of ω_0, and every ω in this set satisfies

$$\left| \iint f \, d\omega - \int f \, d\omega_0 \right| < \omega(W) + 2\epsilon + \omega_0(W) + 2\epsilon + \epsilon = 7\epsilon. \qquad \square$$

For the lemma below, fix metrics ρ_1 and ρ_2 on $C^1(\Gamma, G^\Gamma)$ and $C^2(\Gamma, G^\Gamma)$, respectively, that map to $[0, 1]$, are compatible with the product topologies, and satisfy the following uniform condition: for every $\epsilon > 0$ there is a finite set $F \subseteq \Gamma$ such that $\forall c, c' \in C^1(\Gamma, G^\Gamma)$:

$$\big(c(\beta)(\alpha) = c'(\beta)(\alpha) \text{ for all } \alpha, \beta \in F \big) \Rightarrow \rho_1(c, c'), \rho_2(\partial c, \partial c') < \epsilon.$$

When G is compact this uniform condition automatically holds for all metrics that are compatible with the topology. In any case, such metrics ρ_1 and ρ_2 can be constructed by, for example, fixing an enumeration $(\gamma_n)_{n \in \mathbb{N}}$ of Γ, picking a $[0, 1]$-valued metric ρ_0 on G compatible with its topology, and defining

$$\rho_1(c, c') = \sum_{n,m \in \mathbb{N}} 2^{-n-m} \rho_0\big(c(\gamma_n)(\gamma_m), c'(\gamma_n)(\gamma_m)\big)$$

and

$$\rho_2(c, c') = \sum_{n,m,k \in \mathbb{N}} 2^{-n-m-k} \rho_0\big(c(\gamma_n, \gamma_m)(\gamma_k), c'(\gamma_n, \gamma_m)(\gamma_k)\big).$$

Lemma 6.16. *Let τ be an invariant Borel probability measure on $\mathcal{T}(\Gamma)$, let X be a standard Borel space, and let $\Gamma \curvearrowright^a (X, \mu)$ be a p.m.p. action that weakly contains $\Gamma \curvearrowright^d (\mathcal{T}(\Gamma), \tau)$. Then there is an invariant Borel probability measure ω on $X \times \mathcal{T}(\Gamma)$ that pushes forward to μ and τ under the projection maps to X and $\mathcal{T}(\Gamma)$, respectively, and satisfies the following: for every measurable equivariant map $\theta : X \to C^1(\Gamma, G^\Gamma)$ and $\epsilon > 0$ there is a measurable equivariant map $\theta' : X \to C^1(\Gamma, G^\Gamma)$ satisfying*

$$\int \rho_1(\theta(x), \theta'(x)) \, d\mu < \int \rho_1(\theta(x), r_T(\theta(x))) \, d\omega + \epsilon$$

and

$$\int \rho_2(\partial\theta'(x), e_{C^2(\Gamma, G^\Gamma)}) \, d\mu < \epsilon.$$

Proof. We let Γ act on $\mathrm{Pow}(\Gamma\times\Gamma)^\Gamma$ by the usual left-shift action

$$(\gamma^s\cdot y)(\alpha)=y(\gamma^{-1}\alpha)\qquad y\in\mathrm{Pow}(\Gamma\times\Gamma)^\Gamma,\ \alpha,\gamma\in\Gamma$$

and we define an equivariant lifting map $\ell:\mathcal{T}(\Gamma)\to\mathcal{T}(\Gamma)^\Gamma\subseteq\mathrm{Pow}(\Gamma\times\Gamma)^\Gamma$ via the formula

$$\ell(T)(\gamma)=(\gamma^{-1})^d\cdot T.$$

We will first construct a sequence of measurable equivariant maps $\psi_n:X\to\mathcal{T}(\Gamma)^\Gamma$ such that $(\psi_n)_*\mu$ converges to $\ell_*\tau$ in the weak* topology on $\mathrm{Prob}(\mathrm{Pow}(\Gamma\times\Gamma)^\Gamma)$. Since $\mathrm{Pow}(\Gamma\times\Gamma)$ is 0-dimensional, we can pick an increasing sequence $\mathcal{U}_n$ of finite clopen partitions of $\mathrm{Pow}(\Gamma\times\Gamma)$ such that $\mathcal{U}_{n+1}$ is finer than $\mathcal{U}_n$ for every n and such that $\bigcup_n\mathcal{U}_n$ is a base for the topology on $\mathrm{Pow}(\Gamma\times\Gamma)$. Also fix an increasing sequence of finite sets $F_n\subseteq\Gamma$ having the property $\bigcup_n F_n=\Gamma$. For each n write $\mathcal{U}_n=\{U_n^i : i\in k_n\}$. Define the clopen partition $\mathcal{V}_n=\{V_{n,\pi}:\pi\in k_n^{F_n}\}$ of $\mathrm{Pow}(\Gamma\times\Gamma)^\Gamma$ where

$$V_{n,\pi}=\prod_{\gamma\in F_n}U_n^{\pi(\gamma)}.$$

Then $(\mathcal{V}_n)$ is a sequence of clopen partitions of $\mathrm{Pow}(\Gamma\times\Gamma)^\Gamma$ with $\mathcal{V}_{n+1}$ finer than $\mathcal{V}_n$ for every n and with $\bigcup_n\mathcal{V}_n$ a base for the topology on $\mathrm{Pow}(\Gamma\times\Gamma)^\Gamma$. It easily follows from these properties, as well as the compactness of $\mathrm{Pow}(\Gamma\times\Gamma)^\Gamma$, that the set of all finite linear combinations of characteristic functions of sets in $\bigcup_n\mathcal{V}_n$ is uniformly dense in the space of all real-valued continuous functions on $\mathrm{Pow}(\Gamma\times\Gamma)^\Gamma$. So it will be enough for our maps ψ_n to satisfy $(\psi_n)_*\mu(V)\to\ell_*\tau(V)$ as $n\to\infty$ for every $V\in\bigcup_m\mathcal{V}_m$.

Since $\Gamma\curvearrowright^a(X,\mu)$ weakly contains $\Gamma\curvearrowright(\mathcal{T}(\Gamma),\tau)$, for every n there is a partition $\mathcal{Q}_n=\{Q_n^i : i\in k_n\}$ of X such $Q_n^i=\varnothing$ when $U_n^i\cap\mathcal{T}(\Gamma)=\varnothing$ and satisfying

$$\sum_{\pi\in k_n^{F_n}}\left|\mu\left(\bigcap_{\gamma\in F_n}(\gamma^{-1})^a\cdot Q_n^{\pi(\gamma)}\right)-\tau\left(\bigcap_{\gamma\in F_n}(\gamma^{-1})^d\cdot U_n^{\pi(\gamma)}\right)\right|<\frac{1}{2^n}. \tag{19}$$

For every n and every $i\in k_n$ with $U_n^i\cap\mathcal{T}(\Gamma)\neq\varnothing$, pick a point $u_n^i\in U_n^i\cap\mathcal{T}(\Gamma)$. Define $\psi_n:X\to\mathcal{T}(\Gamma)^\Gamma$ by setting $\psi_n(x)(\gamma)=u_n^i$ when $(\gamma^{-1})^a\cdot x\in Q_n^i$. Then we have that for every n

$$\begin{aligned}&\sum_{\pi\in k_n^{F_n}}\left|(\psi_n)_*\mu(V_{n,\pi})-\ell_*\tau(V_{n,\pi})\right|\\&=\sum_{\pi\in k_n^{F_n}}\left|(\psi_n)_*\mu\left(\prod_{\gamma\in F_n}U_n^{\pi(\gamma)}\right)-\ell_*\tau\left(\prod_{\gamma\in F_n}U_n^{\pi(\gamma)}\right)\right|\end{aligned}$$

$$= \sum_{\pi \in k_n^{F_n}} \left| \mu\left(\bigcap_{\gamma \in F_n} (\gamma^{-1})^a \cdot Q_n^{\pi(\gamma)} \right) - \tau\left(\bigcap_{\gamma \in F_n} (\gamma^{-1})^d \cdot U_n^{\pi(\gamma)} \right) \right| < \frac{1}{2^n}.$$

It follows that $(\psi_n)_*\mu$ weak* converges to $\ell_*\tau$ as desired.

We will now define ω. Without loss of generality, we can assume X is uncountable. Since X is a standard Borel space, we can pick a compact Hausdorff topology on X that is compatible with its Borel σ-algebra. Since $\mathrm{Pow}(\Gamma \times \Gamma)^\Gamma$ is also compact, the space of all Borel probability measures on $X \times \mathrm{Pow}(\Gamma \times \Gamma)^\Gamma$ is compact in the weak* topology. So there is a Borel probability measure $\widehat{\omega}$ on $X \times \mathrm{Pow}(\Gamma \times \Gamma)^\Gamma$ that is a subsequential limit of the measures $(\mathrm{id} \times \psi_n)_*\mu$. Since each ψ_n is equivariant, $\widehat{\omega}$ is Γ-invariant, and since the projection map from $X \times \mathrm{Pow}(\Gamma \times \Gamma)^\Gamma$ to $\mathrm{Pow}(\Gamma \times \Gamma)^\Gamma$ is continuous, the pushforward of $\widehat{\omega}$ with respect to the projection must be equal to $\ell_*\tau$. Also note that $\widehat{\omega}$ and every measure $(\mathrm{id} \times \psi_n)_*\mu$ belong to the set Ω of Borel probability measures on $X \times \mathrm{Pow}(\Gamma \times \Gamma)^\Gamma$ that assign measure 1 to $X \times \mathcal{T}(\Gamma)^\Gamma$ and pushforward to μ under the projection map to X.

We let ω be the measure obtained as the pushforward of $\widehat{\omega}$ under the map $(x, y) \mapsto (x, y(e))$, and note that the pushforwards of ω to X and $\mathcal{T}(\Gamma)$ are μ and τ. Also notice that $\widehat{\omega}$ is similarly obtained as the pushforward of ω under the map $(x, T) \mapsto (x, \ell(T))$.

Finally, let $\theta : X \to C^1(\Gamma, G^\Gamma)$ be any equivariant measurable map. Let $T \mapsto r_T$ be the map from Lemma 6.14. For $y \in \mathcal{T}(\Gamma)^\Gamma$ and $c \in C^1(\Gamma, G^\Gamma)$ define $\hat{r}_y(c) \in C^1(\Gamma, G^\Gamma)$ by the formula

$$\hat{r}_y(c)(\beta)(\alpha) = r_{y(\alpha)}((\alpha^{-1})^t \cdot c)(\beta)(e).$$

We claim that, using $Z = \mathrm{Pow}(\Gamma \times \Gamma)^\Gamma$ and $Y = \mathcal{T}(\Gamma)^\Gamma$, both of the functions

$$(x, y) \mapsto \rho_1(\theta(x), \hat{r}_y(\theta(x))) \quad \text{and} \quad (x, y) \mapsto \rho_2(\partial \hat{r}_y(\theta(x)), e_{C^2(\Gamma, G^\Gamma)})$$

satisfy the assumptions stated in Lemma 6.15. To see this, let $\epsilon > 0$. Pick a finite set $F \subseteq \Gamma$ such that, for all $c, c' \in C^1(\Gamma, G^\Gamma)$:

$$\big(c(\beta)(\alpha) = c'(\beta)(\alpha) \text{ for all } \alpha, \beta \in F \big) \Rightarrow \rho_1(c, c'), \rho_2(\partial c, \partial c') < \epsilon.$$

By Lemma 6.14 there is a clopen partition $\mathcal{U}$ of $\mathcal{T}(\Gamma)$ so that for every $U \in \mathcal{U}$ we have $r_T(c)(\beta)(e) = r_{T'}(c)(\beta)(e)$ for all $c \in C^1(\Gamma, G^\Gamma)$, $\beta \in F$, and $T, T' \in U$. Letting $\mathcal{V}$ be the clopen partition of $\mathcal{T}(\Gamma)^\Gamma$ whereby $y, y' \in \mathcal{T}(\Gamma)^\Gamma$ belong to the same piece of $\mathcal{V}$ if and only if $y(\alpha)$ and $y'(\alpha)$ both belong to a common piece of $\mathcal{U}$ for every $\alpha \in F$, it follows that when $y, y' \in V \in \mathcal{V}$ we have that for all $c \in C^1(\Gamma, G^\Gamma)$

$$\hat{r}_y(c)(\beta)(\alpha) = r_{y(\alpha)}((\alpha^{-1})^t \cdot c)(\beta)(e) = r_{y'(\alpha)}((\alpha^{-1})^t \cdot c)(\beta)(e) = \hat{r}_{y'}(c)(\beta)(\alpha)$$

for all $\alpha, \beta \in F$ and hence $\rho_1(\hat{r}_y(c), \hat{r}_{y'}(c)),\ \rho_2(\partial\hat{r}_y(c), \partial\hat{r}_{y'}(c)) < \epsilon$. Therefore when $y, y' \in V \in \mathcal{V}$ we have that

$$\sup_{x\in X} \left|\rho_1(\theta(x), \hat{r}_y(\theta(x))) - \rho_1(\theta(x), \hat{r}_{y'}(\theta(x)))\right| \leq \epsilon$$

and

$$\sup_{x\in X} \left|\rho_2(\partial\hat{r}_y(\theta(x)), e_{C^2(\Gamma,G^\Gamma)}) - \rho_2(\partial\hat{r}_{y'}(\theta(x)), e_{C^2(\Gamma,G^\Gamma)})\right| \leq \epsilon.$$

This verifies our claim. Thus Lemma 6.15 applies to these two functions and the measures $\widehat{\omega}$ and $(\mathrm{id} \times \psi_n)_*\mu$.

Notice that the map $(y, c) \mapsto \hat{r}_y(c)$ is jointly equivariant since

$$\begin{aligned} \hat{r}_{\gamma^s\cdot y}(\gamma^t \cdot c)(\beta)(\alpha) &= r_{(\gamma^s\cdot y)(\alpha)}((\alpha^{-1}\gamma)^t \cdot c)(\beta)(e) \\ &= r_{y(\gamma^{-1}\alpha)}((\alpha^{-1}\gamma)^t \cdot c)(\beta)(e) \\ &= \hat{r}_y(c)(\beta)(\gamma^{-1}\alpha) \\ &= (\gamma^t \cdot \hat{r}_y(c))(\beta)(\alpha). \end{aligned}$$

Therefore $x \mapsto \hat{r}_{\psi_n(x)}(\theta(x))$ is a measurable equivariant map from X to $C^1(\Gamma, G^\Gamma)$ for every n. Additionally, for every $T \in \mathcal{T}(\Gamma)$ we have

$$\begin{aligned} \hat{r}_{\ell(T)}(c)(\beta)(\alpha) &= r_{\ell(T)(\alpha)}((\alpha^{-1})^t \cdot c)(\beta)(e) \\ &= r_{(\alpha^{-1})^d\cdot T}((\alpha^{-1})^t \cdot c)(\beta)(e) \\ &= ((\alpha^{-1})^t \cdot r_T(c))(\beta)(e) \\ &= r_T(c)(\beta)(\alpha), \end{aligned}$$

and thus $\hat{r}_{\ell(T)} = r_T$. Therefore

$$\int \rho_1(\theta(x), r_T(\theta(x)))\ d\omega = \int \rho_1(\theta(x), \hat{r}_y(\theta(x)))\ d\widehat{\omega}$$

and, since each r_T maps to $Z^1(\Gamma, G^\Gamma)$,

$$0 = \int \rho_2(\partial r_T(\theta(x)), e_{C^2(\Gamma,G^\Gamma)})\ d\omega = \int \rho_2(\partial\hat{r}_y(\theta(x)), e_{C^2(\Gamma,G^\Gamma)})\ d\widehat{\omega}.$$

By applying Lemma 6.15 we conclude that for any $\epsilon > 0$ there is an n with

$$\int \rho_1(\theta(x), \hat{r}_y(\theta(x)))\ d(\mathrm{id} \times \psi_n)_*\mu < \int \rho_1(\theta(x), r_T(\theta(x)))\ d\omega + \epsilon$$

and

$$\int \rho_2(\partial\hat{r}_y(\theta(x)), e_{C^2(\Gamma,G^\Gamma)})\ d(\mathrm{id} \times \psi_n)_*\mu < \epsilon.$$

Defining $\theta'(x) = \hat{r}_{\psi_n(x)}(\theta(x))$ for any such value of n completes the proof. □

6.6. **Cocycles on actions of treeable groups.** The next proposition is a "definable cocycle property" for the class of actions that weakly contain a particular treeable action.

Proposition 6.17. *Let Γ be a treeable group, let τ be an invariant Borel probability measure on $\mathcal{T}(\Gamma)$, let G be a compact metrizable group, and let ρ_1 and ρ_2 be metrics on $C^1(\Gamma, G^\Gamma)$ and $C^2(\Gamma, G^\Gamma)$, respectively, that are compatible with their product topologies and map to $[0,1]$.*

Then for every $\epsilon > 0$ there is $\delta(\epsilon) > 0$ with the following property: whenever $\Gamma \curvearrowright^a (X, \mu)$ is a p.m.p. action that weakly contains $\Gamma \curvearrowright^d (\mathcal{T}(\Gamma), \tau)$ and $\theta : X \to C^1(\Gamma, G^\Gamma)$ is an equivariant measurable map satisfying

$$\int \rho_2(\partial\theta(x), e_{C^2(\Gamma,G^\Gamma)})\, d\mu < \delta(\epsilon),$$

there is an equivariant measurable map $\phi : X \to Z^1(\Gamma, G^\Gamma)$ satisfying

$$\int \rho_1(\theta(x), \phi(x))\, d\mu < \epsilon.$$

Proof. If needed, we can replace $\Gamma \curvearrowright^a (X, \mu)$ with a factor that is an action on a standard Borel space such that θ descends to this factor and this factor still weakly contains $\Gamma \curvearrowright^d (\mathcal{T}(\Gamma), \tau)$. So without loss of generality, throughout the proof we always assume that X is a standard Borel space.

We first show that under the stated assumptions there is $\delta(\epsilon) > 0$ so that whenever $\Gamma \curvearrowright^a (X, \mu)$ is a p.m.p. action that weakly contains $\Gamma \curvearrowright^d (\mathcal{T}(\Gamma), \tau)$ and $\theta : X \to C^1(\Gamma, G^\Gamma)$ is a measurable equivariant map with

$$\int \rho_2(\partial\theta(x), e_{C^2(\Gamma,G^\Gamma)})\, d\mu < \delta(\epsilon),$$

then there is an $\epsilon' \in (0, \epsilon)$ so that for every $\delta' > 0$ there is an equivariant measurable map $\theta' : X \to C^1(\Gamma, G^\Gamma)$ satisfying

$$\int \rho_1(\theta(x), \theta'(x))\, d\mu < \epsilon' \quad \text{and} \quad \int \rho_2(\partial\theta'(x), e_{C^2(\Gamma,G^\Gamma)})\, d\mu < \delta'.$$

Let P be the set of all invariant Borel probability measures on $\mathrm{Pow}(\Gamma \times \Gamma) \times C^1(\Gamma, G^\Gamma)$ whose pushforward under the projection map to $\mathrm{Pow}(\Gamma \times \Gamma)$ is equal to τ. Since G is compact and the pushforward map associated with the projection map is continuous, it follows that P is a compact subset of $\mathrm{Prob}(\mathrm{Pow}(\Gamma \times \Gamma) \times C^1(\Gamma, G^\Gamma))$.

For each $T \in \mathcal{T}(\Gamma)$ let $r_T : C^1(\Gamma, G^\Gamma) \to Z^1(\Gamma, G^\Gamma)$ be the retraction described in Lemma 6.14, and note that for every $\epsilon > 0$ there is a clopen partition $\mathcal{W}$ of $\mathcal{T}(\Gamma)$

with

$$\sup_{c\in C^1(\Gamma,G^\Gamma)} \left|\rho_1(c, r_T(c)) - \rho_1(c, r_{T'}(c))\right| < \epsilon$$

for all $W \in \mathcal{W}$ and $T, T' \in W$. For $\epsilon, \delta > 0$ set

$$V_\epsilon = \left\{\omega \in P : \int \rho_1(c, r_T(c))\ d\omega < \epsilon\right\}$$

and

$$U_\delta = \left\{\omega \in P : \int \rho_2(\partial c, e_{C^2(\Gamma,G^\Gamma)})\ d\omega \leq \delta\right\}.$$

Since each r_T is a retraction to $Z^1(\Gamma, G^\Gamma)$, we have that $\bigcap_{\delta>0} U_\delta \subseteq V_\epsilon$. Additionally, using $C^1(\Gamma, G^\Gamma)$, $\mathrm{Pow}(\Gamma\times\Gamma)$, and $\mathcal{T}(\Gamma)$ for X, Z, Y in Lemma 6.15, respectively, we see that V_ϵ is an open subset of P. Also, each U_δ is compact since $\rho_2(\partial c, e_{C^2(\Gamma,G^\Gamma)})$ is a continuous function on $\mathrm{Pow}(\Gamma\times\Gamma)\times C^1(\Gamma, G^\Gamma)$. It follows that for every $\epsilon > 0$ there is some $\delta(\epsilon) > 0$ such that $U_{\delta(\epsilon)} \subseteq V_\epsilon$, and since V_ϵ is covered by the open sets $V_{\epsilon'}$, $\epsilon' < \epsilon$, there is an $\epsilon' < \epsilon$ with $U_{\delta(\epsilon)} \subseteq V_{\epsilon'}$.

Now fix $\epsilon > 0$ and suppose that $\Gamma \curvearrowright^a (X, \mu)$ is a p.m.p. action that weakly contains $\Gamma \curvearrowright^d (\mathcal{T}(\Gamma), \tau)$ and that $\theta : X \to C^1(\Gamma, G^\Gamma)$ is a measurable equivariant map satisfying

$$\int \rho_2(\partial\theta(x), e_{C^2(\Gamma,G^\Gamma)})\ d\mu < \delta(\epsilon).$$

Letting $\omega \in \mathrm{Prob}(X\times\mathcal{T}(\Gamma))$ be the measure obtained from Lemma 6.16, we have

$$(\pi_{\mathcal{T}(\Gamma)} \times \theta)_*\omega \in U_{\delta(\epsilon)} \subseteq V_{\epsilon'}$$

for some $\epsilon' < \epsilon$, where $\pi_{\mathcal{T}(\Gamma)}$ denotes the projection map. It follows from Lemma 6.16 that for every $\delta' > 0$ there is a measurable equivariant map $\theta' : X \to C^1(\Gamma, G^\Gamma)$ satisfying

$$\int \rho_1(\theta(x), \theta'(x))\ d\mu < \epsilon' \quad \text{and} \quad \int \rho_2(\partial\theta'(x), e_{C^2(\Gamma,G^\Gamma)})\ d\mu < \delta'.$$

This establishes our claim. We will now use this claim to complete the proof.

Let $\epsilon > 0$, let $\Gamma \curvearrowright^a (X, \mu)$ be a p.m.p. action that weakly contains $\Gamma \curvearrowright^d (\mathcal{T}(\Gamma), \tau)$ and let $\theta : X \to C^1(\Gamma, G^\Gamma)$ be a measurable equivariant map satisfying

$$\int \rho_2(\partial\theta(x), e_{C^2(\Gamma,G^\Gamma)})\ d\mu < \delta(\epsilon).$$

Set $\theta_0 = \theta$ and $\epsilon_0 = \epsilon$. Inductively suppose that θ_n and ϵ_n have been defined and satisfy

$$\int \rho_2(\partial\theta_n(x), e_{C^2(\Gamma,G^\Gamma)})\ d\mu < \min(\delta(\epsilon_n), 2^{-n+1})$$

(this holds when $n = 0$ since ρ_2 is bounded by 1). Applying our above claim gives us an $\epsilon' \in (0, \epsilon_n)$. Choose any $0 < \epsilon_{n+1} < \min(\epsilon_n - \epsilon', 2^{-n})$. Then by applying our above claim with $\delta' = \min(\delta(\epsilon_{n+1}), 2^{-(n+1)+1})$ we obtain a measurable equivariant map $\theta_{n+1} : X \to C^1(\Gamma, G^\Gamma)$ satisfying

$$\int \rho_1(\theta_n(x), \theta_{n+1}(x))\ d\mu < \epsilon' < \epsilon_n - \epsilon_{n+1} < \epsilon_n$$

and

$$\int \rho_2(\partial\theta_{n+1}(x), e_{C^2(\Gamma, G^\Gamma)})\ d\mu < \delta' = \min(\delta(\epsilon_{n+1}), 2^{-(n+1)+1}).$$

This allows the inductive construction to proceed.

Since $\sum_n \epsilon_n < \infty$, the sequence (θ_n) is Cauchy with respect to the metric $\int \rho_1(\cdot,\cdot)\ d\mu$. Since G is Polish, $C^1(\Gamma, G^\Gamma)$ is Polish as well and therefore both ρ_1 and $\int \rho_1(\cdot,\cdot)\ d\mu$ are complete. So (θ_n) converges μ-almost-everywhere to a measurable function $\phi : X \to C^1(\Gamma, G^\Gamma)$. Since each θ_n is equivariant, ϕ is as well (if needed, we can redefine ϕ to have value $e_{C^1(\Gamma, G^\Gamma)}$ on the null set of orbits for which it fails to be equivariant). Also, since the coboundary map ∂ is continuous we have that $(\partial\theta_n)$ converges to $\partial\phi$ almost-everywhere. So $\int \rho_2(\partial\phi(x), e_{C^2(\Gamma, G^\Gamma)})\ d\mu = 0$ and by redefining ϕ on a Γ-invariant null set if necessary we have that ϕ is a measurable equivariant map from X to $Z^1(\Gamma, G^\Gamma)$. Finally,

$$\int \rho_1(\theta(x), \phi(x))\ d\mu \leq \sum_{n=0}^{\infty} \int \rho_1(\theta_n(x), \theta_{n+1}(x))\ d\mu < \sum_{n=0}^{\infty} \epsilon_n - \epsilon_{n+1} = \epsilon_0 = \epsilon. \quad \square$$

We can now conclude the following, generalizing Lemma 6.4:

Theorem 6.18.

(1) *If* Γ *is strongly treeable, then* $T_{\Gamma,\mathrm{free}}$ *has the definable cocycle property.*
(2) *If* Γ *is treeable, then* $T_{\Gamma,\max}$ *has the definable cocycle property.*

Proof. The theorem follows immediately from the previous proposition by noting that, in either case, a model of the respective theory weakly contains $(\mathcal{T}(\Gamma), \tau)$ for some invariant Borel probability measure τ on $\mathcal{T}(\Gamma)$. Indeed, this is immediate in the second case by assumption. In the first case, by a result of Abert-Weiss [1], every free p.m.p. action of Γ weakly contains the Bernoulli shift action $\Gamma \curvearrowright^s ([0,1], \lambda)$. Since Γ is strongly treeable, the action $\Gamma \curvearrowright^s ([0,1], \lambda)$ must be treeable, meaning it admits a factor map to $\Gamma \curvearrowright^d (\mathcal{T}(\Gamma), \tau)$ for some invariant Borel probability measure τ. It follows that $\Gamma \curvearrowright^s ([0,1], \lambda)$ weakly contains $\Gamma \curvearrowright^d (\mathcal{T}(\Gamma), \tau)$, and since weak containment is transitive, every free p.m.p. action of Γ weakly contains $\Gamma \curvearrowright (\mathcal{T}(\Gamma), \tau)$. $\square$

We similarly generalize Lemma 6.11:

Theorem 6.19. *If Γ is strongly treeable, then Γ has the extension-MD property. More generally, if Γ is a treeable group, and $\Gamma \curvearrowright^a (X, \mu)$ is a p.m.p. action that weakly contains a free treeable p.m.p. action of Γ, then $\Gamma \curvearrowright^a (X, \mu)$ is extension-MD.*

Proof. Recalling that every free p.m.p. action of a strongly treeable group weakly contains a free treeable p.m.p. action (see the proof of Theorem 6.18), it suffices to prove the second statement. So let τ be an invariant Borel probability measure on $\mathcal{T}(\Gamma)$ and suppose that $\Gamma \curvearrowright^a (X, \mu)$ is a p.m.p. action that weakly contains $\Gamma \curvearrowright^d (\mathcal{T}(\Gamma), \tau)$. It will suffice to verify condition (3) of Lemma 6.10. So let $z : X \to Z^1(\Gamma, G^\Gamma)$ be a measurable equivariant map, where $G = \mathrm{Aut}([0, 1], \lambda)$ equipped with the weak topology, and let $\epsilon > 0$. Also let ρ_1 and ρ_2 be metrics on $C^1(\Gamma, G^\Gamma)$ and $C^2(\Gamma, G^\Gamma)$ satisfying the conditions stated prior to Lemma 6.16. As in the proof of Proposition 6.17, without loss of generality we will assume that X is a standard Borel space.

For any subgroup $H \leq G$ we can build an equivariant map $\theta : X \to C^1(\Gamma, H^\Gamma)$ approximating z by choosing, independently for each $\beta \in \Gamma$, a measurable map $x \in X \mapsto \theta(x)(\beta)(e) \in H$ that approximates $x \mapsto z(x)(\beta)(e)$, and then define $\theta(x)(\beta)(\alpha) = \theta((\alpha^{-1})^a \cdot x)(\beta)(e)$ for $\alpha, \beta \in \Gamma$. It is easily checked that any such θ will indeed be equivariant. Upon noting that the map $c \in C^1(\Gamma, G^\Gamma) \mapsto \rho_1(c, r_T(c))$ is continuous for every $T \in \mathcal{T}(\Gamma)$, that $\rho_1(z(x), r_T(z(x))) = 0$ for ω-almost-every (x, T), and that G admits a dense subset that is an increasing union of finite subgroups, we see that we can find a finite subgroup H and construct a measurable equivariant map $\theta : X \to C^1(\Gamma, H^\Gamma)$ that approximates z sufficiently well so that

$$\int \rho_1(z(x), \theta(x))\, d\mu < \epsilon/3 \quad \text{and} \quad \int \rho_1(\theta(x), r_T(\theta(x)))\, d\omega < \epsilon/3.$$

Let $\delta(\epsilon/3) > 0$ be as given by Proposition 6.17 for τ and the compact group H. Now apply first Lemma 6.16 to get a measurable equivariant map $\theta' : X \to C^1(\Gamma, H^\Gamma)$ satisfying

$$\int \rho_1(\theta(x), \theta'(x))\, d\mu < \epsilon/3$$

and

$$\int \rho_2(\partial\theta'(x), e_{C^2(\Gamma, G^\Gamma)})\, d\mu < \delta(\epsilon/3),$$

and next apply Proposition 6.17 to θ' to obtain a measurable equivariant map $z' : X \to Z^1(\Gamma, H^\Gamma)$ with $\int \rho_1(\theta'(x), z'(x))\, d\mu < \epsilon/3$. Since

$$\rho_1(z(x), z'(x)) \leq \rho_1(z(x), \theta(x)) + \rho_1(\theta(x), \theta'(x)) + \rho_1(\theta'(x), z'(x)),$$

we have $\int \rho_1(z(x), z'(x))\, d\mu < \epsilon$. □

The following is the main conclusion of this section:

Theorem 6.20. *If Γ is treeable, then T_Γ^* exists and the axioms for T_Γ^* are given by the axioms for $T_{\Gamma,\max}$ together with the sentences $\theta_{k,q,F}$ from the proof of Theorem 6.12. If, in addition, Γ is strongly treeable, then we may replace the axioms for $T_{\Gamma,\max}$ by the axioms for $T_{\Gamma,\mathrm{free}}$.*

Furthermore, we obtain a natural-to-state characterization of the e.c. actions of treeable groups.

Theorem 6.21. *Let Γ be a treeable group. Then a p.m.p. action $\Gamma \curvearrowright^a (X,\mu)$ is e.c. if and only if all of the following hold:*

(1) *$\Gamma \curvearrowright^a (X,\mu)$ weakly contains a free treeable action of Γ;*
(2) *$\Gamma \curvearrowright^{a\times \mathrm{id}} (X\times[0,1],\mu\times\lambda)$ is an e.c. extension of $\Gamma \curvearrowright^a (X,\mu)$;*
(3) *$B^1(a,\mathrm{Sym}(k))$ is dense in $Z^1(a,\mathrm{Sym}(k))$ for every $k\in\mathbb{N}$.*

Notice that when Γ is strongly treeable (1) can be replaced with the requirement that the action a is free.

Proof of Theorem 6.21. The forward implication follows from Lemmas 3.8 and 3.11 and the definition of being e.c., and the reverse implication follows from Corollary 6.6, Lemma 6.9 and Theorem 6.19. □

Since conditions (1) and (2) are obviously necessary but not sufficient, the key condition in the above characterization is (3). This reveals that being e.c. (for treeable groups) is closely tied to cohomological properties of the action. We also remark that when $\Gamma \curvearrowright^a (X,\mu)$ is ergodic, condition (2) is equivalent to $\Gamma \curvearrowright^a (X,\mu)$ not being strongly ergodic (the forward implication holds since a weakly contains the non-ergodic action $a\times\mathrm{id}$, and the reverse implication can be seen from the (very short) proof of [1, Theorem 3]).

7. Approximately treeability and existence of the model companion

In this final section, we seek to establish the existence of T_Γ^* for as large a class of groups as we are able, but without insisting on finding explicit axioms for T_Γ^* (though the general set of axioms described in Subsection 5.5 will always apply). Ultimately, we use the open mapping characterization of the existence of the model companion (Theorem 5.26) to show that T_Γ^* exists whenever Γ is an approximately treeable group.

7.1. **Approximately treeable groups.** We first give some constructions for and examples of approximately treeable groups. Recall that every treeable group is necessarily approximately treeable.

Lemma 7.1. *If Γ is an increasing union of approximately treeable groups, then Γ is approximately treeable.*

Proof. Say $\Gamma = \bigcup_n \Gamma_n$, with $\Gamma_n \leq \Gamma_{n+1}$ and Γ_n approximately treeable for every n. Let $H \subseteq \Gamma$ be finite and let $\epsilon > 0$. Pick any n with $H \subseteq \Gamma_n$. Let ν be a Γ_n-invariant Borel probability measure on $\mathcal{F}(\Gamma_n)$ with $\nu(\{F \in \mathcal{F}(\Gamma_n) : H \times H \not\subseteq E_F\}) < \epsilon$. Viewing $\mathcal{F}(\Gamma_n)$ as a subset of $\mathcal{F}(\Gamma)$, the pushforward measure $\gamma^d_* \nu$ depends only on the coset $\gamma\Gamma_n \in \Gamma/\Gamma_n$ and not on the particular representative γ. We can view the product $\prod_{\gamma\Gamma_n \in \Gamma/\Gamma_n} \gamma^d \cdot \mathcal{F}(\Gamma_n)$ as a subset of $\mathcal{F}(\Gamma)$, and by letting μ be the measure on $\mathcal{F}(\Gamma)$ obtained as the product of the measures $\gamma^d_*\nu$ for $\gamma\Gamma_n \in \Gamma/\Gamma_n$, we see that μ is a Γ-invariant measure satisfying $\mu(\{F \in \mathcal{F}(\Gamma) : H \times H \not\subseteq E_F\}) < \epsilon$. Thus Γ is approximately treeable. □

Lemma 7.2. *If $\Lambda \lhd \Gamma$, Λ is approximately treeable, and Γ/Λ is amenable, then Γ is approximately treeable.*

Proof. Fix a choice of representatives $r : \Gamma/\Lambda \to \Gamma$ for the cosets of Λ in Γ with $r(\Lambda) = e$. Define the cocycle $\rho : \Gamma \times (\Gamma/\Lambda) \to \Lambda$ by $\rho(\gamma, a\Lambda) = r(\gamma a\Lambda)^{-1}\gamma r(a\Lambda)$. Let $\Lambda \curvearrowright^b (Y, \nu)$ be a free approximately treeable p.m.p. action. Define the action $\Gamma \curvearrowright^{b'} (Y^{\Gamma/\Lambda}, \nu^{\Gamma/\Lambda})$ by

$$(\gamma^{b'} \cdot \bar{y})(a\Lambda) = (\rho(\gamma^{-1}, a\Lambda)^{-1})^b \cdot \bar{y}(\gamma^{-1}a\Lambda).$$

Since $\Lambda \curvearrowright^b (Y, \nu)$ is free and measure-preserving, it is easily seen that $\Gamma \curvearrowright^{b'} (Y^{\Gamma/\Lambda}, \nu^{\Gamma/\Lambda})$ is free and measure-preserving as well.

Let m denote Lebesgue measure on $[0, 1]$ and let $\Gamma \curvearrowright^c ([0, 1]^{\Gamma/\Lambda}, m^{\Gamma/\Lambda})$ be the left-shift action given by

$$(\gamma^c \cdot z)(a\Lambda) = z(\gamma^{-1}a\Lambda).$$

Let (X, μ) be the product $(Y^{\Gamma/\Lambda} \times [0, 1]^{\Gamma/\Lambda}, \nu^{\Gamma/\Lambda} \times m^{\Gamma/\Lambda})$ and let $\Gamma \curvearrowright^a (X, \mu)$ be the direct product action $a = b' \times c$. Since the action $\Gamma \curvearrowright^a (X, \mu)$ is free and measure-preserving, it suffices to show that it is approximately treeable. Towards this end, fix a finite set $H_0 \subseteq \Gamma$ and an $\epsilon > 0$.

Since Λ is normal in Γ, almost-every $z \in [0, 1]^{\Gamma/\Lambda}$ has stabilizer equal to Λ. This means that $\Gamma \curvearrowright^c ([0, 1]^{\Gamma/\Lambda}, m^{\Gamma/\Lambda})$ descends to a free action of Γ/Λ. Since Γ/Λ is amenable, the orbit equivalence relation $\mathcal{R}_c = \{(z, \gamma^c \cdot z) : z \in [0, 1]^{\Gamma/\Lambda}, \gamma \in \Gamma\}$ must be $m^{\Gamma/\Lambda}$-hyperfinite [36], meaning there is a sequence of measurable equivalence relations $E_n \subseteq \mathcal{R}_c$ such that each class of each E_n is finite, $E_n \subseteq E_{n+1}$ for all n, and $\bigcup_n E_n$ coincides with $\mathcal{R}_c$ on a Γ/Λ-invariant conull set. This implies that we can pick n large enough so that $F = E_n$ satisfies $m^{\Gamma/\Lambda}(Z_0) > 1-\epsilon/3$ where

$$Z_0 = \{z \in Z : \forall h \in H_0 \ (h^c \cdot z, z) \in F\}.$$

Since F is a Borel equivalence relation whose classes are all finite, there exists a Borel set $D \subseteq [0,1]^{\Gamma/\Lambda}$ that contains precisely one point from every F-class [29, Theorem 12.16]. Let $d : [0,1]^{\Gamma/\Lambda} \to \Gamma$ be a function satisfying $d(z)^c \cdot z \in D$ and $(d(z)^c \cdot z, z) \in F$ for all $z \in Z$. By fixing an enumeration of the elements of Γ and letting $d(z)$ be the least element satisfying these conditions, we have that d is measurable. Pick a finite set $H_1 \subseteq \Gamma$ large enough that $m^{\Gamma/\Lambda}(Z_1) > 1 - \epsilon/3$ where

$$Z_1 = \{z \in [0,1]^{\Gamma/\Lambda} : \forall h \in H_0 \ d(h^c \cdot z) \in H_1\}.$$

Using the fact that Λ is approximately treeable, pick a measurable directed graph $\mathcal{H} \subseteq \mathcal{R}_b$ having no cycles such that the equivalence relation $\mathcal{R}_{\mathcal{H}}$ given by the $\mathcal{H}$-connected components satisfies $\nu(Y_1) > 1 - \epsilon/(3|H_1|)$ where

$$Y_1 = \{y \in Y : \forall h \in H_1 H_0 H_1^{-1} \cap \Lambda \ (h^b \cdot y, y) \in \mathcal{R}_{\mathcal{H}}\}.$$

Now define a measurable directed graph $\mathcal{H}' \subseteq \{((\bar{y}, z), (\lambda^a \cdot \bar{y}, z)) : \bar{y} \in Y^{\Gamma/\Lambda},\ z \in D,\ \lambda \in \Lambda\}$ via the rule

$$((\bar{y}_1, z), (\bar{y}_2, z)) \in \mathcal{H}' \Leftrightarrow z \in D \wedge (\bar{y}_1(e), \bar{y}_2(e)) \in \mathcal{H}.$$

Since the map $\pi : Y^{\Gamma/\Lambda} \to Y$ given by $\pi(\bar{y}) = y(e)$ is Λ-equivariant and since the action $\Lambda \curvearrowright^b (Y, \nu)$ is free, we see that $\mathcal{H}'$ has no cycles.

Finally, define

$$\mathcal{G} = \mathcal{H}' \cup \{((\bar{y}, z), d(z)^a \cdot (\bar{y}, z)) : \bar{y} \in Y^{\Gamma/\Lambda},\ z \in [0,1]^{\Gamma/\Lambda} \setminus D\}.$$

When $(\bar{y}, z) \in Y^{\Gamma/\Lambda} \times ([0,1]^{\Gamma/\Lambda} \setminus D)$ we have that $((\bar{y}, z), d(z)^a \cdot (\bar{y}, z))$ is the only edge in $\mathcal{G}$ leaving $(\bar{y}, z)$ and there are no edges in $\mathcal{G}$ pointing towards $(\bar{y}, z)$. Therefore $\mathcal{G}$ remains acyclic.

Now suppose $(\bar{y}, z) \in X_0$ where

$$X_0 = \left(\bigcap_{h \in H_1} (h^{-1})^{b'} \cdot \pi^{-1}(Y_1) \right) \times (Z_0 \cap Z_1),$$

and let $h_0 \in H_0$. Then $(h_0^c \cdot z, z) \in F$ which implies that $z' = d(z)^c \cdot z \in D$ is equal to $(d(h_0^c \cdot z)h_0)^c \cdot z$. Also note that

$$((\bar{y}, z), d(z)^a \cdot (\bar{y}, z)) \in \mathcal{G} \quad \text{and} \quad (h_0^a \cdot (\bar{y}, z), (d(h_0^c \cdot z)h_0)^a \cdot (\bar{y}, z) \in \mathcal{G}.$$

Since $z \in Z_1$ we have that $h_1 = d(h_0^c \cdot z)h_0 d(z)^{-1}$ belongs to $H_1 H_0 H_1^{-1}$. Moreover, $h_1^c \cdot z' = z'$. Since the stabilizer of z' must be Λ (excluding a null set), we have $h_1 \in H_1 H_0 H_1^{-1} \cap \Lambda$. Finally, since $\pi(d(z)^{b'} \cdot \bar{y}) \in Y_1$ we have that the points

$$d(z)^a \cdot (\bar{y}, z) = (d(z)^{b'} \cdot \bar{y}, z')$$

and

$$(d(h_0^c \cdot z)h_0)^a \cdot (\bar{y}, z) = (h_1 d(z))^a \cdot (\bar{y}, z) = h_1^a \cdot (d(z)^{b'} \cdot \bar{y}, z')$$

belong to the same connected component of $\mathcal{H}'$. Thus $h_0^a \cdot (\bar{y}, z)$ and $(\bar{y}, z)$ belong to the same connected component of $\mathcal{G}$. This completes the proof since $\mu(X_0) > 1 - \epsilon$. □

Corollary 7.3. *Universally free groups are approximately treeable.*

Proof. By Lemma 7.1 we may assume that Γ is finitely generated, that is, that Γ is a limit group. By the aforementioned theorem of Kochloukova [34], Γ has a normal coamenable free subgroup Λ. Since Λ is a free group, it is strongly treeable. Therefore Γ is approximately treeable by Lemma 7.2. □

7.2. **Outline of the proof.** Recall from Subsection 2.5 that Γ is approximately treeable if for every weak* open neighborhood U of the point-mass $\delta_{\Gamma\times\Gamma} \in \mathrm{Prob}(\mathcal{E}(\Gamma))$, there is an invariant Borel probability measure μ on $\mathcal{F}(\Gamma)$ so that the pushforward of μ under the map $F \in \mathcal{F}(\Gamma) \mapsto E_F \in \mathcal{E}(\Gamma)$ belongs to U.

For any set A, we let s denote the left-shift action of Γ on A^Γ given by the formula $(\gamma^s \cdot x)(\delta) = x(\gamma^{-1}\delta)$ for $\gamma, \delta \in \Gamma$ and $x \in A^\Gamma$. When $C \subseteq \Gamma$ and $\gamma \in \Gamma$, we also write γ^s to denote the map from A^C to $A^{\gamma C}$ given by the same formula (where now $\delta \in \gamma C$).

The following is the main theorem of this section.

Theorem 7.4. *If Γ is an approximately treeable group, then T_Γ^* exists.*

We provide a brief sketch of the proof. As mentioned earlier, our proof of this theorem will use the open mapping characterization of the existence of the model companion given in Theorem 5.26. Towards that end, consider $p, q \in \mathbb{N}$, nonempty open $\Lambda \subseteq \mathrm{Prob}_\Gamma(q^\Gamma \times p^\Gamma)$, and $\lambda \in \Lambda$. Let $\nu \in \mathrm{Prob}_\Gamma(q^\Gamma)$ be the pushforward of λ under the canonical projection map, and let $y \in q^\Gamma \mapsto \lambda_y \in \mathrm{Prob}(p^\Gamma)$ be the disintegration of λ over ν, so that $\lambda = \int \delta_y \times \lambda_y \, d\nu$ where δ_y is the point-mass measure at y.

For $y \in q^\Gamma$ and $F \in \mathcal{F}(\Gamma)$, we can construct a Borel probability measure $\phi(\lambda_y, E_F)$ on p^Γ by taking the independent product over the classes $C \in \Gamma/E_F$ of the pushforward of λ_y with respect to the projection $p^\Gamma \to p^C$. In other words, $\phi(\lambda_y, E_F)$ is obtained from λ_y by independently re-randomizing λ_y over each of the classes $C \in \Gamma/E_F$. A simple but important observation is that if $E_F = \Gamma \times \Gamma$ is the indiscrete equivalence relation then $\phi(\lambda_y, E_F) = \lambda_y$, and moreover if E_F is sufficiently close to $\Gamma \times \Gamma$ then $\phi(\lambda_y, E_F)$ will be close to λ_y. The fact that Γ is approximately treeable allows us to pick an invariant measure $\mu \in \mathrm{Prob}(\mathcal{F}(\Gamma))$ so that E_F is close to $\Gamma \times \Gamma$ with probability close to 1. This can be done so that when we average the measures $\delta_y \times \phi(\lambda_y, E_F)$ over ν and μ we obtain an invariant probability measure contained in Λ.

In the next step in the proof, which is the more technically challenging portion, we choose a continuous function $\kappa : q^\Gamma \to \mathrm{Prob}(p^\Gamma)$ that approximates the function $y \in q^\Gamma \mapsto \lambda_y$ sufficiently well in ν-measure. For $y \in q^\Gamma$ and $F \in \mathcal{F}(\Gamma)$ we build a measure on $(p^\Gamma)^\Gamma$ whose projection to $(p^\Gamma)^{\{\gamma\}}$ is $\kappa((\gamma^{-1})^s \cdot y) \approx \lambda_{(\gamma^{-1})^s \cdot y}$ for each $\gamma \in \Gamma$. When $(\delta, \gamma) \in F$ the projections of the measure to $(p^\Gamma)^{\{\delta\}}$ and $(p^\Gamma)^{\{\gamma\}}$ will be coupled via an isomorphism from $(p^\Gamma, \kappa((\gamma^{-1})^s \cdot y))$ to $(p^\Gamma, \kappa((\delta^{-1})^s \cdot y))$ that is close to the shift map $(\delta^{-1}\gamma)^s$ with probability close to 1. Additionally, as in the previous paragraph the measure will have independent projections to the sets $(p^\Gamma)^C$ for $C \in \Gamma/E_F$. Using the map $z \in (p^\Gamma)^\Gamma \mapsto z(\cdot)(e) \in p^\Gamma$ we pushforward this measure to p^Γ and call the resulting measure $\theta(\kappa_y, F)$. The fact that the couplings associated with the edges in F are close to the respective shift maps, together with the fact that E_F is close to $\Gamma \times \Gamma$ with probability close to 1, will imply that $\theta(\kappa_y, F)$ is close to $\kappa(y)$ with probability close to 1. Finally, after taking the average of $\theta(\kappa_y, F)$ with respect to ν and μ we obtain an invariant measure close to the one we constructed before and therefore still belonging to Λ.

The final step is to observe that the construction of the previous paragraph can be repeated for any invariant probability measure ν' on q^Γ. Moreover, since κ and the overall construction are continuous, the measure obtained will depend continuously on ν'. Therefore if ν' is sufficiently close to ν the construction will yield an invariant measure in Λ whose projection to q^Γ is ν', completing the proof.

We now proceed to develop all of the necessary details. Throughout the discussion below, fix $p \in \mathbb{N}$.

7.3. **Map-measure pairs.** Let M be the set of all pairs (h, ω) where $h : p^\Gamma \to p^\Gamma$ is Borel measurable and ω is a Borel probability measure on p^Γ. We equip M with the weakest topology making the map $(h, \omega) \in M \mapsto (h \times \mathrm{id})_* \omega \in \mathrm{Prob}(p^\Gamma \times p^\Gamma)$ continuous. Equivalently, since $p^\Gamma \times p^\Gamma$ is compact and zero-dimensional, the topology on M is the weakest topology making the maps $(h, \omega) \mapsto \omega(A \cap h^{-1}(B))$ continuous for all pairs of clopen sets $A, B \subseteq p^\Gamma$.

We call a pair $((h_1, \omega_1), (h_0, \omega_0))$ of elements of M *composable* if $\omega_1 = (h_0)_* \omega_0$. We write M_2 for the set of all composable pairs and define a composition operation from M_2 to M, which we denote simply by $\cdot$, by the rule $(h_1, \omega_1) \cdot (h_0, \omega_0) = (h_1 \circ h_0, \omega_0)$.

Lemma 7.5. *The composition function* $((h_1, \omega_1), (h_0, \omega_0)) \in M_2 \mapsto (h_1 \circ h_0, \omega_0) \in M$ *is continuous.*

Proof. Let $((h_1, \omega_1), (h_0, \omega_0)) \in M_2$, let $A, B \subseteq p^\Gamma$ be clopen, and let $\epsilon > 0$. Pick a clopen set $B_1 \subseteq p^\Gamma$ such that $\omega_1(B_1 \triangle h_1^{-1}(B)) < \epsilon$. Now consider any

$((g_1,\zeta_1),(g_0,\zeta_0)) \in M_2$ that is close to $((h_1,\omega_1),(h_0,\omega_0))$ in the sense that

$$\zeta_1(B_1 \triangle g_1^{-1}(B)) < \epsilon \text{ and } |\zeta_0(A \cap g_0^{-1}(B_1)) - \omega_0(A \cap h_0^{-1}(B_1))| < \epsilon.$$

Since each pair is composable, we have that $\zeta_1 = (g_0)_*\zeta_0$ and $\omega_1 = (h_0)_*\omega_0$. Therefore

$$\begin{aligned}
&|\zeta_0(A \cap g_0^{-1}(g_1^{-1}(B))) - \omega_0(A \cap h_0^{-1}(h_1^{-1}(B)))| \\
&\leq |\zeta_0(A \cap g_0^{-1}(B_1)) - \omega_0(A \cap h_0^{-1}(B_1))| \\
&\qquad + \zeta_0(g_0^{-1}(B_1 \triangle g_1^{-1}(B))) + \omega_0(h_0^{-1}(B_1 \triangle h_1^{-1}(B))) \\
&= |\zeta_0(A \cap g_0^{-1}(B_1)) - \omega_0(A \cap h_0^{-1}(B_1))| + \zeta_1(B_1 \triangle g_1^{-1}(B)) + \omega_1(B_1 \triangle h_1^{-1}(B)) \\
&< 3\epsilon.
\end{aligned}$$

□

Let $C \subseteq \Gamma$ and write $M^{(C)}$ for the set of all pairs $((h_\gamma)_{\gamma\in C}, \omega)$ where $h_\gamma : p^\Gamma \to p^\Gamma$ is a Borel measurable function for every $\gamma \in C$ and ω is a Borel probability measure on p^Γ. We give $M^{(C)}$ the weakest topology so that the map $((h_\gamma)_{\gamma\in C}, \omega) \in M^{(C)} \mapsto (h_\gamma, \omega) \in M$ is continuous for every $\gamma \in C$. For $((h_\gamma)_{\gamma\in C}, \omega) \in M^{(C)}$ we write $\prod_{\gamma\in C} h_\gamma$ for the function from p^Γ to $(p^\Gamma)^C$ given by

$$(\prod_{\gamma\in C} h_\gamma)(x)(\delta)(\beta) = h_\delta(x)(\beta)$$

for $x \in p^\Gamma$, $\delta \in C$, and $\beta \in \Gamma$.

Lemma 7.6. *The map*

$$((h_\gamma)_{\gamma\in C}, \omega) \in M^{(C)} \mapsto \left(\prod_{\gamma\in C} h_\gamma\right)_* \omega \in \mathrm{Prob}((p^\Gamma)^C)$$

is continuous.

Proof. Choose any cylinder set of $(p^\Gamma)^C$, say $A = \prod_{\gamma\in C} B_\gamma$ where $B_\gamma \subseteq p^\Gamma$ is nonempty and clopen for every $\gamma \in C$ and where, for some finite set $D \subseteq C$, $B_\gamma = p^\Gamma$ for all $\gamma \in C \setminus D$. Fix any $((h_\gamma)_{\gamma\in C}, \omega) \in M^{(C)}$ and $\epsilon > 0$. For each $\gamma \in D$ pick a clopen set $B'_\gamma \subseteq p^\Gamma$ satisfying $\omega(B'_\gamma \triangle h_\gamma^{-1}(B_\gamma)) < \epsilon/(2|D|+2)$. Then, for any $((g_\gamma)_{\gamma\in C}, \zeta) \in M^{(C)}$ satisfying $|\zeta(\bigcap_{\gamma\in D} B'_\gamma) - \omega(\bigcap_{\gamma\in D} B'_\gamma)| < \epsilon/(|D|+1)$ and $\zeta(B'_\gamma \triangle g_\gamma^{-1}(B_\gamma)) < \epsilon/(2|D|+2)$ for every $\gamma \in D$, we have

$$\begin{aligned}
&\left| \left(\prod_{\gamma\in C} g_\gamma\right)_* \zeta(A) - \left(\prod_{\gamma\in C} h_\gamma\right)_* \omega(A) \right| \\
&= \left| \zeta\left(\bigcap_{\gamma\in D} g_\gamma^{-1}(B_\gamma)\right) - \omega\left(\bigcap_{\gamma\in D} h_\gamma^{-1}(B_\gamma)\right) \right|
\end{aligned}$$

$$\leq \left| \zeta\left(\bigcap_{\gamma\in D} B'_\gamma\right) - \omega\left(\bigcap_{\gamma\in D} B'_\gamma\right)\right| + |D|\frac{\epsilon}{|D|+1} < \epsilon. \qquad \square$$

Let P^* denote the set of all pairs (κ_1, κ_0) of Borel probability measures on p^Γ having the property that either $\kappa_1 = \kappa_0$ or else both κ_0 and κ_1 are non-atomic. We equip P^* with the subspace topology inherited from the product space $\mathrm{Prob}(p^\Gamma)\times \mathrm{Prob}(p^\Gamma)$.

Lemma 7.7. *There exists a function ψ assigning to each pair $(\kappa_1, \kappa_0) \in P^*$ a Borel measurable function $\psi(\kappa_1, \kappa_0) : p^\Gamma \to p^\Gamma$ such that*

(1) $\kappa_1 = \psi(\kappa_1, \kappa_0)_*\kappa_0$;
(2) $\psi(\kappa_0, \kappa_1) \circ \psi(\kappa_1, \kappa_0)$ *is equal to the identity κ_0-almost-everywhere;*
(3) *the map* $(\kappa_1, \kappa_0) \in P^* \mapsto (\psi(\kappa_1, \kappa_0), \kappa_0) \in M$ *is continuous; and*
(4) $(\psi(\kappa_1, \kappa_0), \kappa_0) \to (\mathrm{id}, \kappa)$ *as* $(\kappa_1, \kappa_0) \to (\kappa, \kappa)$.

Proof. Let $\preceq$ denote the (non-strict) lexicographical ordering on p^Γ obtained from some fixed enumeration of the elements of Γ, and for $z \in p^\Gamma$ set $L_z = \{z' \in p^\Gamma : z' \preceq z\}$. For $(\kappa_1, \kappa_0) \in P^*$ define $\psi(\kappa_1, \kappa_0) : p^\Gamma \to p^\Gamma$ by

$$\psi(\kappa_1, \kappa_0)(z) = \inf_{\prec}\{z' \in p^\Gamma : \kappa_1(L_{z'}) \geq \kappa_0(L_z)\}.$$

The above set will always be nonempty since the function on Γ having constant value $p-1$ is $\preceq$-maximal. Consequently ψ is well-defined.

The function $\psi(\kappa_1, \kappa_0) : p^\Gamma \to p^\Gamma$ is Borel measurable since it is $\prec$-monotone increasing. Specfically, the sets $L_{z'}$, $z' \in p^\Gamma$, generate the Borel σ-algebra of p^Γ and each preimage $\psi(\kappa_1, \kappa_0)^{-1}(L_{z'})$ is necessarily Borel (it is either empty or else equal to L_z or $L_z \setminus \{z\}$ where $z = \sup_{\prec} \psi(\kappa_1, \kappa_0)^{-1}(L_{z'})$).

For $\kappa \in \mathrm{Prob}(p^\Gamma)$ set

$$Z_\kappa = \{z \in p^\Gamma : \forall z' \prec z\ \kappa(L_z \setminus L_{z'}) > 0\}.$$

Since p^Γ contains a countable set that is dense in the $\prec$-ordering, we have that $\kappa(Z_\kappa) = 1$. From these definitions it is clear that $\psi(\kappa, \kappa)$ restricts to the identity map on Z_κ. Thus conditions (1) and (2) hold when $(\kappa_1, \kappa_0) = (\kappa, \kappa)$. Additionally, since $\psi(\kappa, \kappa)$ is κ-almost-everywhere equal to id, no open set in M can separate the points $(\psi(\kappa, \kappa), \kappa)$ and (id, κ) and therefore condition (4) will be implied by condition (3). When $\kappa_1 \neq \kappa_0$ are non-atomic it is easy to see from the definitions that $\psi(\kappa_1, \kappa_0)$ maps Z_{κ_0} bijectively onto Z_{κ_1} and $\psi(\kappa_0, \kappa_1) \circ \psi(\kappa_1, \kappa_0)$ restricts to the identity on Z_{κ_0} (in fullfillment of condition (2)). Finally, since Borel probability measures on p^Γ are uniquely determined from their values on the sets L_z, $z \in p^\Gamma$, condition (1) easily follows.

To conclude we verify condition (3). First observe that, while L_z is always closed, it is open precisely for those countably many $z \in p^\Gamma$ which evaluate to $p-1$ at all but finitely many elements of Γ. Write $L_\varnothing = \varnothing$ and define Z to be the set of all $z \in p^\Gamma \cup \{\varnothing\}$ for which L_z is clopen. Extend $\prec$ to Z by declaring $\varnothing$ to be $\prec$-minimum. For any nonempty clopen sets $A, B \subseteq p^\Gamma$ we can choose $\prec$-increasing sequences $z_0, \ldots, z_{2n-1} \in Z$ and $z'_0, \ldots, z'_{2m-1} \in Z$ so that A and B are the disjoint unions

$$A = \bigsqcup_{k \in n} L_{z_{2k+1}} \setminus L_{z_{2k}} \quad \text{and} \quad B = \bigsqcup_{\ell \in m} L_{z'_{2\ell+1}} \setminus L_{z'_{2\ell}}.$$

Then, by a simple inclusion-exclusion computation, $\kappa_0(A \cap \psi(\kappa_1, \kappa_0)^{-1}(B))$ is equal to

$$\sum_{(k,\ell) \in n \times m} \sum_{i,j \in 2} (-1)^{i+j} \kappa_0 \left(L_{z_{2k+i}} \cap \psi(\kappa_1, \kappa_0)^{-1}(L_{z'_{2\ell+j}}) \right).$$

Since $\psi(\kappa_1, \kappa_0)$ is $\prec$-monotone increasing, one of the two sets $\psi(\kappa_1, \kappa_0)^{-1}(L_{z'_{2\ell+j}})$ and $L_{z_{2k+i}}$ must contain the other, meaning the measure of their intersection is the smaller of their two measures. Combining this observation with condition (1), we obtain

$$\kappa_0(A \cap \psi(\kappa_1, \kappa_0)^{-1}(B)) = \sum_{(k,\ell) \in n \times m} \sum_{i,j \in 2} (-1)^{i+j} \min \left(\kappa_0(L_{z_{2k+i}}), \kappa_1(L_{z'_{2\ell+j}}) \right).$$

Since each of the sets $L_{z_{2k+i}}$ and $L_{z'_{2\ell+j}}$ are clopen, the above is a continuous function of $(\kappa_0, \kappa_1) \in P^*$. □

7.4. **Measure constructions.** Recall the space $\mathcal{E}(\Gamma)$ of equivalence relations on Γ and the space $\mathcal{F}(\Gamma)$ of directed forests on Γ from Subsection 2.5.

Lemma 7.8. *There is a map* $\phi : \mathrm{Prob}(p^\Gamma) \times \mathcal{E}(\Gamma) \to \mathrm{Prob}(p^\Gamma)$ *such that:*

(1) ϕ *is continuous;*
(2) $\phi(\omega, \Gamma \times \Gamma) = \omega$ *for all* $\omega \in \mathrm{Prob}(p^\Gamma)$;
(3) $\gamma^s_* \phi(\omega, E) = \phi(\gamma^s_* \omega, \gamma^d \cdot E)$ *for all* $\gamma \in \Gamma$, $\omega \in \mathrm{Prob}(p^\Gamma)$ *and* $E \in \mathcal{E}(\Gamma)$.

Proof. For $C \subseteq \Gamma$ let $\pi_C : p^\Gamma \to p^C$ be the projection map. Notice that $\beta^s \circ \pi_C = \pi_{\beta C} \circ \beta^s$ for $\beta \in \Gamma$.

For $E \in \mathcal{E}(\Gamma)$ and $\omega \in \mathrm{Prob}(p^\Gamma)$ define

$$\phi(\omega, E) = \prod_{C \in \Gamma/E} (\pi_C)_* \omega.$$

While the above is formally a product measure indexed by the E-classes in Γ, since $(\pi_C)_*\omega$ is a probability measure on p^C and the sets $C \in \Gamma/E$ partition Γ we will identify $\phi(\omega, E)$ as a Borel probability measure on p^Γ.

Clause (2) follows immediately from the definition. Also (3) holds since for $\beta \in \Gamma$ we have

$$\begin{aligned}\beta^s_* \phi(\omega, E) &= \beta^s_* \prod_{C \in \Gamma/E} (\pi_C)_* \omega \\ &= \prod_{\beta C \in \Gamma/\beta^d \cdot E} \beta^s_* \circ (\pi_C)_* \omega \\ &= \prod_{\beta C \in \Gamma/\beta^d \cdot E} (\pi_{\beta C})_* \beta^s_* \omega = \phi(\beta^s_* \omega, \beta^d \cdot E).\end{aligned}$$

Lastly, suppose that $\varnothing \neq B_\gamma \subseteq p$ for $\gamma \in \Gamma$ and that $B_\gamma = p$ for all $\gamma \in \Gamma \setminus D$ where $D \subseteq \Gamma$ is finite. Consider the cylinder set $A = \prod_{\gamma \in \Gamma} B_\gamma$. Then $\phi(\omega, E)(A)$ is a continuous function of (ω, E) since the family of sets $\mathcal{D}_E = \{D \cap C : C \in \Gamma/E\}$ is a locally constant function of E and when $\mathcal{D}_E = \mathcal{D}$

$$\phi(\omega, E)(A) = \prod_{C' \in \mathcal{D}} \omega \left(\prod_{\gamma \in C'} B_\gamma \times p^{\Gamma \setminus C'} \right)$$

is a continuous function of ω. It follows that ϕ is continuous, in fulfillment of (1). □

Recall that P^* denotes the set of all pairs $(\kappa_1, \kappa_0) \in \mathrm{Prob}(p^\Gamma) \times \mathrm{Prob}(p^\Gamma)$ such that either $\kappa_1 = \kappa_0$ or else both κ_1 and κ_0 are non-atomic. Let $P^{(\Gamma)}$ be the set of all $\omega \in \mathrm{Prob}(p^\Gamma)^\Gamma$ satisfying:

(1) $\omega(\gamma)$ is non-atomic for every $\gamma \in \Gamma$; or
(2) $\gamma^s_* \omega(\gamma) = \omega(e)$ for every $\gamma \in \Gamma$.

In other words, $\omega \in \mathrm{Prob}(p^\Gamma)^\Gamma$ belongs to $P^{(\Gamma)}$ if and only if $((\delta^{-1}\gamma)^s_* \omega(\gamma), \omega(\delta)) \in P^*$ for all $\delta, \gamma \in \Gamma$.

Lemma 7.9. *There is a map* $\theta : P^{(\Gamma)} \times \mathcal{F}(\Gamma) \to \mathrm{Prob}(p^\Gamma)$ *such that:*

(*a*) $\theta(\omega, F)$ *is a Borel function of* (ω, F) *and is a continuous function of* ω;
(*b*) $\theta(\omega, F) = \phi(\omega(e), E_F)$ *whenever* $\omega \in P^{(\Gamma)}$ *satisfies* $\gamma^s_* \omega(\gamma) = \omega(e)$ *for all* $\gamma \in \Gamma$;
(*c*) $\gamma^s_* \theta(\omega, F) = \theta(\gamma^s \cdot \omega, \gamma^d \cdot F)$ *for all* $\gamma \in \Gamma$, $\omega \in P^{(\Gamma)}$ *and* $F \in \mathcal{F}(\Gamma)$.

Proof. Let ψ be the function described in Lemma 7.7. For $\delta, \gamma \in \Gamma$ and $\omega \in P^{(\Gamma)}$ define Borel functions $g^\omega_{\delta,\gamma}, \bar{g}^\omega_{\delta,\gamma} : p^\Gamma \to p^\Gamma$ by

$$g^\omega_{\delta,\gamma} = \psi(\omega(\delta), (\delta^{-1}\gamma)^s_* \omega(\gamma)) \circ (\delta^{-1}\gamma)^s$$

and

$$\bar{g}^{\omega}_{\delta,\gamma} = (\delta^{-1}\gamma)^s \circ \psi((\gamma^{-1}\delta)^s_*\omega(\delta), \omega(\gamma)).$$

The definition of $P^{(\Gamma)}$ ensures that ψ is defined in both of the expressions above. With these definitions, $g^{\omega}_{\delta,\gamma}$ first performs the shift on p^{Γ} by $\delta^{-1}\gamma$ and then, according to ψ, applies a map from p^{Γ} to itself that pushes $(\delta\gamma^{-1})_*\omega(\gamma)$ forward to $\omega(\delta)$. Similarly, $\bar{g}^{\omega}_{\delta,\gamma}$ first applies a map from p^{Γ} to itself that pushes $\omega(\gamma)$ forward to $(\gamma^{-1}\delta)^s_*\omega(\delta)$ and then performs the shift on p^{Γ} by $\delta^{-1}\gamma$.

We observe that the following statements hold for all $\delta, \gamma \in \Gamma$:

(1) Each map $g^{\omega}_{\delta,\gamma}$ and $\bar{g}^{\omega}_{\delta,\gamma}$ pushes $\omega(\gamma)$ forward to $\omega(\delta)$. This is clear from the definitions together with clause (1) of Lemma 7.7.
(2) Both compositions $\bar{g}^{\omega}_{\gamma,\delta} \circ g^{\omega}_{\delta,\gamma}$ and $g^{\omega}_{\gamma,\delta} \circ \bar{g}^{\omega}_{\delta,\gamma}$ are equal to the identity $\omega(\gamma)$-almost-everywhere. This follows from the definitions and clause (2) of Lemma 7.7.
(3) The pairs $(g^{\omega}_{\delta,\gamma}, \omega(\gamma)), (\bar{g}^{\omega}_{\delta,\gamma}, \omega(\gamma)) \in M$ are continuous functions of ω. Indeed each pair can be expressed as the result of a composition $M_2 \to M$:

$$(g^{\omega}_{\delta,\gamma}, \omega(\gamma)) = (\psi(\omega(\delta), (\delta^{-1}\gamma)^s_*\omega(\gamma)), (\delta^{-1}\gamma)^s_*\omega(\gamma)) \cdot ((\delta^{-1}\gamma)^s, \omega(\gamma))$$

$$(\bar{g}^{\omega}_{\delta,\gamma}, \omega(\gamma)) = ((\delta^{-1}\gamma)^s, (\gamma^{-1}\delta)^s_*\omega(\delta)) \cdot (\psi((\gamma^{-1}\delta)^s_*\omega(\delta), \omega(\gamma)), \omega(\gamma)).$$

Continuity is a consequence of Lemma 7.5, Lemma 7.7.(3), and the continuity of the push-forward maps $\gamma^s_* : \mathrm{Prob}(p^{\Gamma}) \to \mathrm{Prob}(p^{\Gamma})$ for $\gamma \in \Gamma$.
(4) When $\delta^s_*\omega(\delta) = \gamma^s_*\omega(\gamma)$ both $g^{\omega}_{\delta,\gamma}$ and $\bar{g}^{\omega}_{\delta,\gamma}$ are equal to $(\delta^{-1}\gamma)^s$.
(5) $g^{\beta^s\cdot\omega}_{\beta\delta,\beta\gamma} = g^{\omega}_{\delta,\gamma}$ and $\bar{g}^{\beta^s\cdot\omega}_{\beta\delta,\beta\gamma} = \bar{g}^{\omega}_{\delta,\gamma}$ for all $\beta \in \Gamma$. This is immediate from the definitions together with the facts that $(\beta^s \cdot \omega)(\beta\delta) = \omega(\delta)$, $(\beta^s \cdot \omega)(\beta\gamma) = \omega(\gamma)$, and $(\beta\delta)^{-1}(\beta\gamma) = \delta^{-1}\gamma$.

Now for $\omega \in P^{(\Gamma)}$ and $F \in \mathcal{F}(\Gamma)$ we define a function σ^{ω}_F assigning to each pair $(\delta, \gamma) \in E_F$ a Borel measurable function $\sigma^{\omega}_F(\delta, \gamma) : p^{\Gamma} \to p^{\Gamma}$ as follows. First, when $\delta = \gamma$ we set $\sigma^{\omega}_F(\delta, \gamma)$ equal to the identity function. Next, if $(\delta, \gamma) \in F \cup \bar{F}$ then we set

$$\sigma^{\omega}_F(\delta, \gamma) = \begin{cases} g^{\omega}_{\delta,\gamma} & \text{if } (\delta, \gamma) \in F \\ \bar{g}^{\omega}_{\delta,\gamma} & \text{if } (\delta, \gamma) \in \bar{F}. \end{cases}$$

In general for $(\delta, \gamma) \in E_F$, consider the unique sequence of vertices $\gamma_0, \gamma_1, \ldots, \gamma_n$ forming a path in $F \cup \bar{F}$ from γ to δ, specifically $\gamma_0 = \gamma$, $\gamma_n = \delta$, and $(\gamma_{i+1}, \gamma_i) \in F \cup \bar{F}$ for all $0 \leq i < n$, and set

$$\sigma^{\omega}_F(\delta, \gamma) = \sigma^{\omega}_F(\gamma_n, \gamma_{n-1}) \circ \cdots \circ \sigma^{\omega}_F(\gamma_1, \gamma_0).$$

The following properties hold for all $\omega \in P^{(\Gamma)}$, $F \in \mathcal{F}(\Gamma)$, and $(\delta, \gamma) \in E_F$.

(6) $\sigma^\omega_F(\delta,\gamma)$ pushes $\omega(\gamma)$ forward to $\omega(\delta)$. This is an immediate consequence of (1) above.
(7) If β lies on the path from γ to δ then $\sigma^\omega_F(\delta,\gamma) = \sigma^\omega_F(\delta,\beta)\circ\sigma^\omega_F(\beta,\gamma)$. This is immediate from the definition.
(8) $\sigma^\omega_F(\gamma,\delta)\circ\sigma^\omega_F(\delta,\gamma)$ is equal to the identity $\omega(\gamma)$-almost-everywhere. When $(\delta,\gamma)\in F\cup\bar{F}$ this fact follows immediately from (2) above. The general case follows by induction on the length of the path from γ to δ. Specifically, if β lies on the path from γ to δ then by (7)

$$\sigma^\omega_F(\gamma,\delta)\circ\sigma^\omega_F(\delta,\gamma) = \sigma^\omega_F(\gamma,\beta)\circ\sigma^\omega_F(\beta,\delta)\circ\sigma^\omega_F(\delta,\beta)\circ\sigma^\omega_F(\beta,\gamma).$$

By induction we can assume $\sigma^\omega_F(\beta,\delta)\circ\sigma^\omega_F(\delta,\beta)$ is equal to the identity $\omega(\beta)$-almost-everywhere. Since $\sigma^\omega_F(\beta,\gamma)$ pushes $\omega(\gamma)$ forward to $\omega(\beta)$ by (6), it follows that $\omega(\gamma)$-almost-everywhere $\sigma^\omega_F(\gamma,\delta)\circ\sigma^\omega_F(\delta,\gamma)$ is equal to $\sigma^\omega_F(\gamma,\beta)\circ\sigma^\omega_F(\beta,\gamma)$, and by induction this latter function is equal to the identity $\omega(\gamma)$-almost-everywhere.
(9) Whenever β is in the same connected component as δ and γ, $\sigma^\omega_F(\delta,\gamma)$ is $\omega(\gamma)$-almost-everywhere equal to $\sigma^\omega_F(\delta,\beta)\circ\sigma^\omega_F(\beta,\gamma)$. To see this, let β' be the unique point belonging to all three of the paths between the points δ, γ, and β. Two applications of (7) yield the two equations:

$$\sigma^\omega_F(\delta,\beta)\circ\sigma^\omega_F(\beta,\gamma) = \sigma^\omega_F(\delta,\beta')\circ\sigma^\omega_F(\beta',\beta)\circ\sigma^\omega_F(\beta,\beta')\circ\sigma^\omega_F(\beta',\gamma)$$
$$\sigma^\omega_F(\delta,\gamma) = \sigma^\omega_F(\delta,\beta')\circ\sigma^\omega_F(\beta',\gamma).$$

The right-sides of these two equations are $\omega(\gamma)$-almost-everywhere equal since $\sigma^\omega_F(\beta',\gamma)$ pushes $\omega(\gamma)$ forward to $\omega(\beta')$ by (6) and $\sigma^\omega_F(\beta',\beta)\circ\sigma^\omega_F(\beta,\beta')$ is $\omega(\beta')$-almost-everywhere equal to the identity by (8). This establishes the claim.
(10) If $\gamma^s_*\omega(\gamma)=\omega(e)$ for all $\gamma\in\Gamma$ then $\sigma^\omega_F(\delta,\gamma)=(\delta^{-1}\gamma)^s$ for all $(\delta,\gamma)\in E_F$. This follows from (4).
(11) $\sigma^{\beta^s\cdot\omega}_{\beta^d\cdot F}(\beta\delta,\beta\gamma)=\sigma^\omega_F(\delta,\gamma)$ for all $\beta\in\Gamma$. This is a consequence of (5) together with the fact that if $\gamma_0,\dots,\gamma_n$ is the sequence of vertices forming a path in F from γ to δ then $\beta\gamma_0,\dots,\beta\gamma_n$ is the sequence of vertices for the path from $\beta\gamma$ to $\beta\delta$ in $\beta^d\cdot F$.

Additionally, we observe the following facts:

(11) The function $(\omega,F)\in P^{(\Gamma)}\times U_{\delta,\gamma}\mapsto(\sigma^\omega_F(\delta,\gamma),\omega(\gamma))\in M$ is continuous, where $U_{\delta,\gamma}$ is the open set $\{F\in\mathcal{F}(\Gamma):(\delta,\gamma)\in E_F\}$. Specifically, we can partition $U_{\delta,\gamma}$ into a collection $\mathcal{U}$ of clopen sets so that for every $U\in\mathcal{U}$ the path from δ to γ in F is constant over all $F\in U$. It's enough to observe that for each $U\in\mathcal{U}$ the map $(\omega,F)\mapsto(\sigma^\omega_F(\delta,\gamma),\omega(\gamma))$ is continuous on $P^{(\Gamma)}\times U$. Say $\gamma=\gamma_0,\gamma_1,\dots,\gamma_n=\delta$ is the path from γ to δ in F for all $F\in U$. Then from the definition of σ^ω_F and by (6) we see that $(\sigma^\omega_F(\delta,\gamma),\omega(\gamma))\in$

M is the composition of the elements $(\sigma^\omega_F(\gamma_{i+1}, \gamma_i), \omega(\gamma_i)) \in M$ when $F \in U$. It then follows from (3) and Lemma 7.5 that $(\sigma^\omega_F(\delta, \gamma), \omega(\gamma))$ is a continuous function of $(\omega, F) \in P^{(\Gamma)} \times U$.

(12) If $C \subseteq \Gamma$ is nonempty, $\gamma \in \Gamma$, and $C \cup \{\gamma\}$ is contained in a single E_F-class for all F in a set $H \subseteq \mathcal{F}(\Gamma)$, then the function $(\omega, F) \in P^{(\Gamma)} \times H \mapsto ((\sigma^\omega_F(\delta, \gamma))_{\delta \in C}, \omega(\gamma)) \in M^{(C)}$ is continuous. Indeed, for every $\delta \in C$ the map $(\omega, F) \in P^{(\Gamma)} \times H \mapsto (\sigma^\omega_F(\delta, \gamma), \omega(\gamma)) \in M$ is continuous by (11) since $H \subseteq U_{\delta,\gamma}$.

Let $C \in \Gamma/E_F$ be a connected component and fix any $\gamma_C \in C$. From ω we define a Borel probability measure on $(p^\Gamma)^C$ by the formula

$$\left(\prod_{\delta \in C} \sigma^\omega_F(\delta, \gamma_C)\right)_* \omega(\gamma_C).$$

Notice that this measure depends continuously on ω by (12) and Lemma 7.6. Additionally, the measure we obtain does not depend upon the choice of $\gamma_C \in C$ since for any other element $\gamma'_C \in C$ properties (6) and (9) imply

$$\begin{aligned}\left(\prod_{\delta \in C} \sigma^\omega_F(\delta, \gamma_C)\right)_* \omega(\gamma_C) &= \left(\prod_{\delta \in C} \sigma^\omega_F(\delta, \gamma_C)\right)_* \sigma^\omega_F(\gamma_C, \gamma'_C)_* \omega(\gamma'_C) \\ &= \left(\prod_{\delta \in C} \sigma^\omega_F(\delta, \gamma_C) \circ \sigma^\omega_F(\gamma_C, \gamma'_C)\right)_* \omega(\gamma'_C) \\ &= \left(\prod_{\delta \in C} \sigma^\omega_F(\delta, \gamma'_C)\right)_* \omega(\gamma'_C).\end{aligned}$$

For each $C \in \Gamma/E_F$ pick some $\gamma_C \in C$. Also let $f : (p^\Gamma)^\Gamma \to p^\Gamma$ be the flattening map given by the formula $f(z)(\gamma) = z(\gamma)(e)$. We define the Borel probability measure $\theta(\omega, F) \in \mathrm{Prob}(p^\Gamma)$ by

$$\theta(\omega, F) = f_* \prod_{C \in \Gamma/E_F} \left(\prod_{\delta \in C} \sigma^\omega_F(\delta, \gamma_C)\right)_* \omega(\gamma_C).$$

Formally the expression above to the right of f_* is a product measure on a product space indexed by the set Γ/E_F, however since the measure associated to the coordinate $C \in \Gamma/E_F$ is a measure on $(p^\Gamma)^C$, we identify the expression to the right of f_* as a Borel probability measure on $(p^\Gamma)^\Gamma$ (and therefore the application of f_* is defined).

We first check (b). Denote by $\pi_C : p^\Gamma \to p^C$ the projection map for $C \subseteq \Gamma$, and recall that

$$\phi(\omega(e), E_F) = \prod_{C \in \Gamma/E_F} (\pi_C)_* \omega(e).$$

Let us also write f for the map from $(p^\Gamma)^C$ to p^C given by the same formula as before: $f(z)(\gamma) = z(\gamma)(e)$ for $\gamma \in C$. Since $\prod_{\gamma \in C}(\gamma^{-1})^s$ maps p^Γ to $(p^\Gamma)^C$, it is easily checked that $\pi_C = f \circ \prod_{\gamma \in C}(\gamma^{-1})^s$. If $\gamma^s_* \omega(\gamma) = \omega(e)$ for all $\gamma \in \Gamma$ then using (10) we obtain

$$\begin{aligned}
\prod_{C \in \Gamma/E_F} (\pi_C)_* \omega(e) &= \prod_{C \in \Gamma/E_F} f_* \left(\prod_{\delta \in C} (\delta^{-1})^s \right)_* \omega(e) \\
&= \prod_{C \in \Gamma/E_F} f_* \left(\prod_{\delta \in C} (\delta^{-1})^s \right)_* (\gamma_C)^s_* \omega(\gamma_C) \\
&= \prod_{C \in \Gamma/E_F} f_* \left(\prod_{\delta \in C} (\delta^{-1}\gamma_C)^s \right)_* \omega(\gamma_C) \\
&= f_* \prod_{C \in \Gamma/E_F} \left(\prod_{\delta \in C} \sigma^\omega_F(\delta, \gamma_C) \right)_* \omega(\gamma_C) = \theta(\omega, F),
\end{aligned}$$

confirming (b).

Next we check (a). Let $\Delta \subseteq \mathcal{F}(\Gamma) \times \mathcal{E}(\Gamma)$ be the Borel set $\Delta = \{(F, E_F) : F \in \mathcal{F}(\Gamma)\}$. Since the map $F \in \mathcal{F}(\Gamma) \mapsto (F, E_F) \in \Delta$ is Borel, it suffices to show that $\theta(F, \omega)$ is a continuous function of $(\omega, F, E_F) \in P^{(\Gamma)} \times \Delta$. In fact it is enough to show that $\theta(\omega, F)(A)$ is a continuous function of (ω, F, E_F) when A is a cylinder set. Say $A = \prod_{\gamma \in \Gamma} B_\gamma$ where $\varnothing \neq B_\gamma \subseteq p$ for all $\gamma \in \Gamma$ and $B_\gamma = p$ for all $\gamma \in \Gamma \setminus D$ where $D \subseteq \Gamma$ is finite. Partition Δ into a collection $\mathcal{V}$ of clopen sets so that for each $V \in \mathcal{V}$ the restriction of E_F to D is constant over all $(F, E_F) \in V$. It will be enough to check that for each $V \in \mathcal{V}$ the map $(\omega, F, E_F) \mapsto \theta(\omega, F)(A)$ is continuous on $P^{(\Gamma)} \times V$. Say every $(F, E_F) \in V$ partitions D into the family of sets $\mathcal{D}$. When $(F, E_F) \in V$ and $C \in \Gamma/E_F$ we have that

$$\left(f_* \left(\prod_{\delta \in C} \sigma^\omega_F(\delta, \gamma_C) \right)_* \omega(\gamma_C) \right) \left(\prod_{\gamma \in C} B_\gamma \right)$$

is equal to 1 if $C \cap D = \varnothing$ and is equal to

$$\left(f_* \left(\prod_{\delta \in C'} \sigma^\omega_F(\delta, \gamma_{C'}) \right)_* \omega(\gamma_{C'}) \right) \left(\prod_{\gamma \in C'} B_\gamma \right)$$

when $C \cap D = C' \in \mathcal{D}$ and $\gamma_{C'}$ is any element of C' (changing γ_C to $\gamma_{C'}$ does not change the resulting value, as explained three paragraphs prior). Now notice that the map

$$(\omega, F, E_F) \in P^{(\Gamma)} \times V \mapsto f_* \left(\prod_{\delta \in C'} \sigma_F^{\omega}(\delta, \gamma_{C'}) \right)_* \omega(\gamma_{C'}) \in \mathrm{Prob}(p^{C'})$$

is continuous by (12), Lemma 7.6, and the fact that f_* is continuous. Therefore when $(\omega, F, E_F) \in P^{(\Gamma)} \times V$ we have that

$$\theta(\omega, F)(A) = \prod_{C' \in \mathcal{D}} \left(f_* \left(\prod_{\delta \in C'} \sigma_F^{\omega}(\delta, \gamma_{C'}) \right)_* \omega(\gamma_{C'}) \right) \left(\prod_{\gamma \in C'} B_\gamma \right)$$

is a continuous function of (ω, F, E_F) as claimed.

Lastly, to check (c) consider any $\beta \in \Gamma$. Since left-multiplication by β provides a bijection from the E_F-classes to the $E_{\beta^d \cdot F}$-classes, for each $C \in \Gamma/E_F$ we can use the point $\widehat{\gamma}_{\beta C} = \beta\gamma_C$ as the representative for βC in computing $\theta(\beta^s \cdot \omega, \beta^d \cdot F)$. Then by (11) and the fact that the map f is Γ^s-equivariant we have

$$\begin{aligned}
\beta^s_* \theta(\omega, F) &= \beta^s_* f_* \prod_{C \in \Gamma/E_F} \left(\prod_{\delta \in C} \sigma_F^{\omega}(\delta, \gamma_C) \right)_* \omega(\gamma_C) \\
&= f_* \prod_{\beta C \in \Gamma/E_{\beta^d \cdot F}} \left(\prod_{\beta\delta \in \beta C} \sigma_F^{\omega}(\delta, \gamma_C) \right)_* \omega(\gamma_C) \\
&= f_* \prod_{\beta C \in \Gamma/E_{\beta^d \cdot F}} \left(\prod_{\beta\delta \in \beta C} \sigma_{\beta^d \cdot F}^{\beta^s \cdot \omega}(\beta\delta, \beta\gamma_C) \right)_* \omega(\gamma_C) \\
&= f_* \prod_{\beta C \in \Gamma/E_{\beta^d \cdot F}} \left(\prod_{\delta \in \beta C} \sigma_{\beta^d \cdot F}^{\beta^s \cdot \omega}(\delta, \hat{\gamma}_{\beta C}) \right)_* \omega(\beta^{-1}\hat{\gamma}_{\beta C}) \\
&= f_* \prod_{C \in \Gamma/E_{\beta^d \cdot F}} \left(\prod_{\delta \in C} \sigma_{\beta^d \cdot F}^{\beta^s \cdot \omega}(\delta, \hat{\gamma}_C) \right)_* (\beta^s \cdot \omega)(\hat{\gamma}_C) = \theta(\beta^s \cdot \omega, \beta^d \cdot F). \quad \square
\end{aligned}$$

7.5. **Existence of the model companion.** We are now ready to prove that T_Γ^* exists when Γ is approximately treeable:

Proof of Theorem 7.4. We verify the criterion from Theorem 5.26. Fix $p, q \in \mathbb{N}$, nonempty open $\Lambda \subseteq \mathrm{Prob}_\Gamma(q^\Gamma \times p^\Gamma)$, and $\lambda \in \Lambda$. Let $\nu \in \mathrm{Prob}_\Gamma(q^\Gamma)$ be the pushforward of λ under the canonical projection map, and let $y \in q^\Gamma \mapsto \lambda_y \in$

$\mathrm{Prob}(p^\Gamma)$ be the disintegration of λ over ν, so that $\lambda = \int \delta_y \times \lambda_y \, d\nu$ where δ_y is the point-mass measure at y.

Let $\phi : \mathrm{Prob}(p^\Gamma) \times \mathcal{E}(\Gamma) \to \mathrm{Prob}(p^\Gamma)$ be the map from Lemma 7.8. Since ϕ is continuous, a routine calculation shows that the map $E \in \mathcal{E}(\Gamma) \mapsto \int \delta_y \times \phi(\lambda_y, E) \, d\nu \in \mathrm{Prob}(q^\Gamma \times p^\Gamma)$ is continuous. It follows from this that the function $\zeta : \mathrm{Prob}(\mathcal{E}(\Gamma)) \to \mathrm{Prob}(q^\Gamma \times p^\Gamma)$ defined by the formula

$$\zeta(\mu) = \int\int \delta_y \times \phi(\lambda_y, E) \, d\nu(y) \, d\mu(E)$$

is continuous. Moreover, if μ is an invariant Borel probability measure on $\mathcal{F}(\Gamma)$ and $\bar{\mu}$ is the pushforward of μ with respect to the map $F \in \mathcal{F}(\Gamma) \mapsto E_F \in \mathcal{E}(\Gamma)$, then $\zeta(\bar{\mu})$ is also an invariant measure since ϕ as well as each of the maps $y \mapsto \delta_y$, $y \mapsto \lambda_y$, and $F \mapsto E_F$ are Γ-equivariant. By Lemma 7.8 we have that $\zeta(\delta_{\Gamma\times\Gamma}) = \lambda$ belongs to the open set Λ, so by continuity and the fact that Γ is approximately treeable there is an invariant Borel probability measure $\mu \in \mathrm{Prob}(\mathcal{F}(\Gamma))$ satisfying $\zeta(\bar{\mu}) \in \Lambda$.

Recall the sets $P^* \subseteq \mathrm{Prob}(p^\Gamma) \times \mathrm{Prob}(p^\Gamma)$ and $P^{(\Gamma)} \subseteq \mathrm{Prob}(p^\Gamma)^\Gamma$ defined earlier. Define $\kappa_\infty : q^\Gamma \to \mathrm{Prob}(p^\Gamma)$ by $\kappa_\infty(y) = \lambda_y$. Pick a sequence of continuous functions κ_n, $n \in \mathbb{N}$, mapping q^Γ to the space of non-atomic Borel probability measures on p^Γ such that (κ_n) converges ν-almost-everywhere to κ_∞. For $n \in \mathbb{N} \cup \{\infty\}$ and $y \in q^\Gamma$ define $\omega_{n,y} \in \mathrm{Prob}(p^\Gamma)^\Gamma$ by

$$\omega_{n,y}(\gamma) = \kappa_n((\gamma^{-1})^s \cdot y).$$

Observe that for $\beta \in \Gamma$ we have $\beta^s \cdot \omega_{n,y} = \omega_{n,\beta^s\cdot y}$ since for all $\gamma \in \Gamma$

$$(\beta^s \cdot \omega_{n,y})(\gamma) = \omega_{n,y}(\beta^{-1}\gamma) = \kappa_n((\gamma^{-1}\beta)^s \cdot y) = \omega_{n,\beta^s\cdot y}(\gamma).$$

Our definitions ensure that $\omega_{n,y} \in P^{(\Gamma)}$ for all $n \in \mathbb{N}$ and all $y \in q^\Gamma$, and that $\omega_{\infty,y} \in P^{(\Gamma)}$ for ν-almost-every $y \in q^\Gamma$. This fact allows us to define for each $n \in \mathbb{N} \cup \{\infty\}$ a Borel probability measure on $q^\Gamma \times p^\Gamma$ by the formula

$$\int\int \delta_y \times \theta(\omega_{n,y}, F) \, d\nu(y) \, d\mu(F).$$

Notice that the above measure is in fact Γ-invariant since the measures ν and μ are Γ-invariant and the maps $y \mapsto \omega_{n,y}$, $y \mapsto \delta_y$, and θ are Γ-equivariant. Now observe that for every $\gamma \in \Gamma$

$$\gamma^s_* \omega_{\infty,y}(\gamma) = \gamma^s_* \kappa_\infty((\gamma^{-1})^s \cdot y) = \gamma^s_* \lambda_{(\gamma^{-1})^s\cdot y}$$

is equal to

$$\gamma^s_* (\gamma^{-1})^s_* \lambda_y = \lambda_y = \omega_{\infty,y}(e)$$

for ν-almost-every $y \in q^\Gamma$. Additionally, $\omega_{n,y}$ converges to $\omega_{\infty,y}$ as $n \to \infty$ for ν-almost-every $y \in q^\Gamma$. Combining these two facts with Lemma 7.9 yields

$$\begin{aligned}
&\lim_{n\to\infty} \int\int \delta_y \times \theta(\omega_{n,y}, F)\, d\nu(y)\, d\mu(F)\\
&= \int\int \delta_y \times \theta(\omega_{\infty,y}, F)\, d\nu(y)\, d\mu(F)\\
&= \int\int \delta_y \times \phi(\lambda_y, E_F)\, d\nu(y)\, d\mu(F) = \zeta(\bar{\mu}) \in \Lambda.
\end{aligned}$$

We fix from this point forward an $n \in \mathbb{N}$ satisfying $\int\int \delta_y \times \theta(\omega_{n,y}, F)\, d\nu(y)\, d\mu(F) \in \Lambda$.

Since $\omega_{n,y} \in P^{(\Gamma)}$ for all $y \in q^\Gamma$, we may define a function ξ taking invariant Borel probability measures on q^Γ to invariant Borel probability measures on $q^\Gamma \times p^\Gamma$ by the formula

$$\xi(\nu') = \int\int \delta_y \times \theta(\omega_{n,y}, F)\, d\nu'(y)\, d\mu(F).$$

Of course, by construction we have $\xi(\nu) \in \Lambda$. On the other hand, we chose κ_n, and hence the map $y \mapsto \omega_{n,y}$, to be continuous. Consequently the map $y \in q^\Gamma \mapsto \int \delta_y \times \theta(\omega_{n,y}, F)\, d\mu(F)$ is continuous as well. It follows that ξ is continuous and therefore $\xi^{-1}(\Lambda)$ is an open set of invariant Borel probability measures containing ν. Every $\nu' \in \xi^{-1}(\Lambda)$ is immediately seen from the definition to be the pushforward of $\xi(\nu')$ with respect to the projection map to q^Γ. We conclude that the pushforward of Λ with respect to the projection map is open and thus T_Γ^* exists. □

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