Partition regularity of nonlinear Diophantine equations

Lorenzo Luperi Baglini

University of Vienna

Applications of Ultrafilters and Nonstandard Methods, IV

University of Hawaii at Manoa
Partition regularity of polynomials/equations

In this talk, we will repeatedly talk about the following property:

Definition
Let \( P_{x_1, \ldots, x_n} \) be a polynomial in \( \mathbb{Z}_r \). We say that the equation \( P_{x_1, \ldots, x_n} = 0 \) is (weakly) partition regular (PR) on \( \mathbb{N} \) if it has a monochromatic solution in every infinite coloring of \( \mathbb{N} \), i.e. for every coloring \( \mathbb{N} = \bigcup_{i=1}^k A_i \) with \( \mathbb{N} \cap A_i \) nonempty, there exist \( x_1, \ldots, x_n \in A_i \) such that \( P_{x_1, \ldots, x_n} = 0 \).

Example
Trivially, for every \( n \in \mathbb{N} \), the polynomial \( x^n \) is PR.
Partition regularity of polynomials/equations

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Let $P(x_1, \ldots, x_n) \in \mathbb{Z}[x_1, \ldots, x_n]$. 
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$$\forall k \in \mathbb{N}, \forall \mathbb{N} = A_1 \cup \cdots \cup A_k \exists i \leq k \exists x_1, \ldots, x_n \in A_i \text{ s.t. } P(x_1, \ldots, x_n) = 0.$$
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Trivially, for every $n \in \mathbb{N}$, the polynomial $x - n$ is PR.
Rado’s theorem

**Theorem (Schur)**

The polynomial $x + y - z$ is PR.
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- “There exists a nonempty set $J \subseteq \{1, \ldots, n\}$ such that $\sum_{j \in J} c_j = 0$.”
Theorem (Multiplicative Rado)

A nonlinear Diophantine equation $\prod_{i=1}^{n} x_i^{c_i} = 1$ is PR on $\mathbb{N}$ if and only if the following condition is satisfied:
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Idea: $ n \rightarrow 2^n $. 

Lorenzo Luperi Baglini
University of Vienna
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Nonlinear results/1

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Theorem (Lefmann)

Let $k \in \mathbb{N}$. A Diophantine equation of the form $c_1 x_1^{1/k} + \cdots + c_n x_n^{1/k} = 0$ is PR on $\mathbb{N}$ if and only if the following condition is satisfied:
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Nonlinear results/2

Theorem (Bergelson, Furstenberg, McCutcheon)

Let $P(z) \in \mathbb{Z}[z]$ be such that $P(0) = 0$. Then the equation $x - y = P(z)$ is PR.
Nonlinear results/2

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In 2010, by using algebraic properties of ultrafilters in \( \beta \mathbb{N} \), Bergelson solved the problem in the positive.
Nonlinear results/3

Independently, Hindman proved a more general version of Bergelson’s result:

\[
\begin{align*}
\text{Theorem (Hindman)} \\
\text{All Diophantine equations of the form } & x_1^n \pm m_1 y_1 = 0 \\
\text{are PR.} \\
\text{Idea: use the algebra of } & \beta^N, \text{in particular the existence of a ultralter } U \text{ such that every set } A^P U \text{is additively and multiplicatively IP.}
\end{align*}
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\begin{align*}
\text{Theorem (Luperi Baglini)} \\
\text{Let } n, m \neq 0. \text{ For every choice of sets } F_1, \ldots, F_m, \text{ the equation } & x_1^n + \cdots + x_n^{c_1} \pm j \in F_j y_1^{q_1} = 0 \text{ is partition regular whenever } c_j \neq 0 \text{ for some nonempty } J \neq 1, \ldots, m. \text{ (It is agreed that } \pm j \in F_j). \\
\text{Idea: use the existence of a multiplicatively idempotent ultralter } U \text{ with good linear properties; study the ultralter using nonstandard analysis.}
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Independently, Hindman proved a more general version of Bergelson’s result:

**Theorem (Hindman)**

All Diophantine equations of the form \( \sum_{i=1}^{n} x_i = \prod_{i=1}^{m} y_i \) are PR.

Idea: use the algebra of \( \beta \mathbb{N} \), in particular the existence of a ultrafilter \( U \) such that every set \( A \) is additively and multiplicatively \( IP \).

**Theorem (Luperi Baglini)**

Let \( n, m \geq 0 \). For every choice of sets \( F_i \) for \( i = 1, \ldots, m \), the equation \( \sum_{i=1}^{n} c_i x_i \pm \prod_{i=1}^{m} y_i = 0 \) is partition regular whenever \( \sum_{i=1}^{n} c_i \) for some nonempty \( J \) of \( i = 1, \ldots, m \). (It is agreed that \( \pm \) always holds.)

Idea: use the existence of a multiplicatively idempotent ultrafilter \( U \) with good linear properties; study the ultrafilter using nonstandard analysis.
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**Theorem (Luperi Baglini)**

Let $n, m > 0$. For every choice of sets $F_i \subseteq \{1, \ldots, m\}$, the equation $\sum_{i=1}^{n} c_i x_i (\prod_{j \in F_i} y_j) = 0$ is partition regular whenever $\sum_{i \in J} c_i = 0$ for some nonempty $J \subseteq \{1, \ldots, m\}$. (It is agreed that $\prod_{j \in \emptyset} y_j = 1$.)
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Let \( n, m > 0 \). For every choice of sets \( F_i \subseteq \{1, \ldots, m\} \), the equation \( \sum_{i=1}^{n} c_i x_i (\prod_{j \in F_i} y_j) = 0 \) is partition regular whenever \( \sum_{i \in J} c_j = 0 \) for some nonempty \( J \subseteq \{1, \ldots, m\} \). (It is agreed that \( \prod_{j \in \emptyset} y_j = 1 \).)

Idea: use the existence of a multiplicatively idempotent ultrafilter \( \mathcal{U} \) with good linear properties; study the ultrafilter using nonstandard analysis.
Theorem (Di Nasso, Riggio)

Let $k, n, m \in \mathbb{N}$ be such that $k \notin \{n, m\}$. Then the equation

$$x^m + y^n = z^k$$

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Idea: use nonstandard analysis, write numbers in base \( p \) for a sufficiently large prime number \( p \).
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Theorem (Moreira)

Let $\sum_{i=1}^{n} c_i = 0$. Then $\sum_{i=1}^{n} c_i x_i^2 = y$ is PR.
Nonlinear results/4

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Theorem (Moreira)

Let $\sum_{i=1}^{n} c_i = 0$. Then $\sum_{i=1}^{n} c_i x_i^2 = y$ is PR.

Idea: use ergodic methods involving the set of affinities $\{x \to ax + b\}$; alternatively, use an embeddability property of piecewise syndetic sets w.r.t. arithmetic progressions.
Partition regularity as a ultrafilters problem

$\beta\mathbb{N}$ turns out to be a natural setting where to study PR problems because of the following characterization (which is given here for equations, but holds in a way more general fashion):

**Proposition**

A Diophantine equation

$P(x_1,\ldots,x_n,q_0)$ is PR if and only if there exists $U_P^{\beta\mathbb{N}}$ such that for every $A_P \subseteq U_P$ there exists $a_1,\ldots,a_n \in A$ with $P(a_1,\ldots,a_n,q_0)$.

In this case, we say that $U_P$ witnesses the PR of the equation (notation: $U_P | \mathfrak{u}$).

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**Proposition**

A Diophantine equation \( P(x_1, \ldots, x_n) = 0 \) is PR if and only if there exists \( U \in \beta \mathbb{N} \) such that for every \( A \in U \) there exists \( a_1, \ldots, a_n \in A \) with \( P(a_1, \ldots, a_n) = 0 \).

Lorenzo Luperi Baglini
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**Proposition**

A Diophantine equation $P(x_1, \ldots, x_n) = 0$ is PR if and only if there exists $\mathcal{U} \in \beta\mathbb{N}$ such that for every $A \in \mathcal{U}$ there exists $a_1, \ldots, a_n \in A$ with $P(a_1, \ldots, a_n) = 0$.

In this case, we say that $\mathcal{U}$ witnesses the PR of the equation (notation: $\mathcal{U} \models P(a_1, \ldots, a_n) = 0$).
Banach density and IP-sets

**Definition**

Let $A \subseteq \mathbb{N}$. The upper *Banach density* of $A$ is

$$BD(A) = \lim_{n \to +\infty} \sup_{m \in \mathbb{N}} \frac{|A \cap [m, m+n]|}{n+1}.$$  

We let $\Delta = \{ \mathcal{U} \in \beta \mathbb{N} \mid BD(A) > 0 \ \forall A \in \mathcal{U} \}$. 

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**Definition**

Let $G = (g_i)_{i \in \mathbb{N}}$ be an increasing sequence of natural numbers. The IP-set generated by $G$ is the set of finite sums

$$FS(G) = FS(g_i)_{i \in \mathbb{N}} = \left\{ \sum_{j=1}^{k} g_{i_j} \mid i_1 < i_2 < \cdots < i_k \right\}.$$

A set $A \subseteq \mathbb{N}$ is called IP-large if it contains an IP-set. Multiplicative IP-sets and multiplicative IP-large sets are defined similarly.
Special ultrafilters

Various classes of ultrafilters are important in this field, the "best" being combinatorially rich ultrafilters:

**Definition**

\( \mathcal{U} \) is combinatorially rich if \( \mathcal{U} \in M \cap \triangle \cap K(\otimes) \) and \( \mathcal{U} \otimes \mathcal{U} = \mathcal{U} \), where

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M = \{ \mathcal{V} \in \beta\mathbb{N} \mid \forall A \in \mathcal{V} \ A \text{ is central in } (\mathbb{N}, +) \}.
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Notice that if \( \mathcal{U} \) is combinatorially rich and \( A \in \mathcal{U} \) then:

- \( A \) is central in \( (\mathbb{N}, +) \), in particular it is IP;
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Notice that if $\mathcal{U}$ is combinatorially rich and $A \in \mathcal{U}$ then:

- $A$ is central in $(\mathbb{N}, +)$, in particular it is IP;
- $A$ is also multiplicatively IP;
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Notice that if $\mathcal{U}$ is combinatorially rich and $A \in \mathcal{U}$ then:

- $A$ is central in $(\mathbb{N}, +)$, in particular it is IP;
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- $A$ contains solutions to all homogeneous PR equations (we will show this later), in particular to all linear equations;
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Notice that if $\mathcal{U}$ is combinatorially rich and $A \in \mathcal{U}$ then:

- $A$ is central in $(\mathbb{N}, +)$, in particular it is IP;
- $A$ is also multiplicatively IP;
- $A$ contains solutions to all homogeneous PR equations (we will show this later), in particular to all linear equations;
- $BD(A) > 0$. 

A surprisingly simple key Lemma

Lemma

Let \( U \) be a common witness of the equations \( P_1(x_1, \ldots, x_n) = 0 \) and \( P_2(y_1, \ldots, y_m) = 0 \).
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**Lemma**

Let \( U \) be a common witness of the equations \( P_1 (x_1, \ldots, x_n) = 0 \) and \( P_2 (y_1, \ldots, y_m) = 0 \). Then \( U \) is also a PR-witness of the system:

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\begin{align*}
P_1 (x_1, \ldots, x_n) &= 0; \\
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x_1 &= y_1.
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Let $\mathcal{U}$ be a common witness of the equations $P_1(x_1, \ldots, x_n) = 0$ and $P_2(y_1, \ldots, y_m) = 0$. Then $\mathcal{U}$ is also a PR-witness of the system:

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**Proof.**

Let $A \in \mathcal{U}$ be fixed. Let

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\begin{align*}
\Lambda_1 &= \{ a \in A \mid \exists a_2, \ldots, a_n \in A \text{ s.t. } P_1(a, a_2, \ldots, a_n) = 0 \}, \\
\Lambda_2 &= \{ b \in A \mid \exists b_2, \ldots, b_m \in A \text{ s.t. } P_2(b, b_2, \ldots, b_m) = 0 \}.
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Notice that $\Lambda_1, \Lambda_2 \in U$, as otherwise $\neg (U \models P_i = 0)$. 
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Let $U$ be a common witness of the equations $P_1(x_1, \ldots, x_n) = 0$ and $P_2(y_1, \ldots, y_m) = 0$. Then $U$ is also a PR-witness of the system:

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Notice that $\Lambda_1, \Lambda_2 \in U$, as otherwise $\neg (U \models P_i = 0)$. Take $\Lambda_1 \cap \Lambda_2$. □
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Take $U \models u - v = t^2$. 
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Take $\mathcal{U} \models u - v = t^2$.

Then $\mathcal{U}$ witnesses also of the PR of the system

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  u_1 - y &= x^2; \\
  u_2 - z &= t^2; \\
  y &= t.
\end{aligned}
\]

It is readily seen that this is equivalent to the PR of the configuration $\{x, y, z, y + x^2, z + y^2\}$ (which had already been proven by ergodic methods).
Homogeneous equations

Theorem

\[ P(x_1, \ldots, x_n) \text{ be a homogeneous PR polynomial.} \]
Homogeneous equations

Theorem

Let $P(x_1, \ldots, x_n)$ be a homogeneous PR polynomial. Then the set of its PR-witnesses

$$\mathcal{W}_P = \{ U \in \beta \mathbb{N} | U \models P(x_1, \ldots, x_n) = 0 \}$$

is a closed multiplicative two sided ideal.
# Homogeneous equations

## Theorem

Let \( P(x_1, \ldots, x_n) \) be a homogeneous PR polynomial. Then the set of its PR-witnesses

\[
\mathcal{W}_P = \{ U \in \beta N \mid U \models P(x_1, \ldots, x_n) = 0 \}
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is a closed multiplicative two sided ideal.

## Proof.

(Nonstandard) Let \( U \in \mathcal{W}_P \). Let \( \alpha_1, \ldots, \alpha_n \in \mu(U) \) be such that

\[
*P(\alpha_1, \ldots, \alpha_n) = 0.
\]
Homogeneous equations

**Theorem**

Let $P(x_1, \ldots, x_n)$ be a homogeneous PR polynomial. Then the set of its PR-witnesses

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is a closed multiplicative two sided ideal.

**Proof.**

(Nonstandard) Let $U \in \mathcal{W}_P$. Let $\alpha_1, \ldots, \alpha_n \in \mu(U)$ be such that

$$*P(\alpha_1, \ldots, \alpha_n) = 0.$$ Let $\beta \in \mu(V)$. Then $\alpha_1 \cdot *\beta, \ldots, \alpha_n \cdot *\beta \in \mu(U \odot V)$, and

$$P(\alpha_1 \cdot *\beta, \ldots, x_n \cdot *\beta) = 0.$$
Homogeneous equations

**Theorem**

Let $P(x_1,\ldots,x_n)$ be a homogeneous $PR$ polynomial. Then the set of its $PR$-witnesses

$$\mathcal{W}_P = \{U \in \beta\mathbb{N} \mid U \models P(x_1,\ldots,x_n) = 0\}$$

is a closed multiplicative two sided ideal.

**Proof.**

(Nonstandard) Let $U \in \mathcal{W}_P$. Let $\alpha_1,\ldots,\alpha_n \in \mu(U)$ be such that $*P(\alpha_1,\ldots,\alpha_n) = 0$. Let $\beta \in \mu(V)$. Then $\alpha_1 \cdot *\beta,\ldots,\alpha_n \cdot *\beta \in \mu(U \odot V)$, and $P(\alpha_1 \cdot *\beta,\ldots,\alpha_n \cdot *\beta) = 0$. Hence $U \odot V \in \mathcal{W}_P$.  \qed
Homogeneous equations

**Theorem**

Let $P(x_1, \ldots, x_n)$ be a homogeneous PR polynomial. Then the set of its PR-witnesses

$$\mathcal{W}_P = \{ \mathcal{U} \in \beta \mathbb{N} \mid \mathcal{U} \models P(x_1, \ldots, x_n) = 0 \}$$

is a closed multiplicative two sided ideal.

**Proof.**

(Nonstandard) Let $\mathcal{U} \in \mathcal{W}_P$. Let $\alpha_1, \ldots, \alpha_n \in \mu(\mathcal{U})$ be such that $\star P(\alpha_1, \ldots, \alpha_n) = 0$. Let $\beta \in \mu(\mathcal{V})$. Then $\alpha_1 \cdot \star \beta, \ldots, \alpha_n \cdot \star \beta \in \mu(\mathcal{U} \odot \mathcal{V})$, and $P(\alpha_1 \cdot \star \beta, \ldots, \alpha_n \cdot \star \beta) = 0$. Hence $\mathcal{U} \odot \mathcal{V} \in \mathcal{W}_P$. □

**Corollary**

Let $P(x_1, \ldots, x_n)$ be a homogeneous PR polynomial. Then $\mathcal{U} \models P(x_1, \ldots, x_n) = 0$ for every $\mathcal{U} \in K(\beta \mathbb{N}, \odot)$. 

Lorenzo Luperi Baglini
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The first generalization result

Theorem

Let \( c(x_1 - x_2) = P(y_1, \ldots, y_k) \) be a Diophantine equation where the polynomial \( P \) has no constant term and \( c \neq 0 \). If the set \( A \subseteq \mathbb{N} \) is IP-large and has positive Banach density then there exist \( \xi_1, \xi_2 \in A \) and mutually distinct \( \eta_1, \ldots, \eta_k \in A \) such that \( c(\xi_1 - \xi_2) = P(\eta_1, \ldots, \eta_k) \). Moreover, if \( k = 1 \) then one can take \( \xi_1 \neq \xi_2 \).
The first generalization result

**Theorem**

Let $c(x_1 - x_2) = P(y_1, \ldots, y_k)$ be a Diophantine equation where the polynomial $P$ has no constant term and $c \neq 0$. If the set $A \subseteq \mathbb{N}$ is $IP$-large and has positive Banach density then there exist $\xi_1, \xi_2 \in A$ and mutually distinct $\eta_1, \ldots, \eta_k \in A$ such that $c(\xi_1 - \xi_2) = P(\eta_1, \ldots, \eta_k)$. Moreover, if $k = 1$ then one can take $\xi_1 \neq \xi_2$.

**Definition**

A polynomial with integer coefficients is called a *Rado polynomial* if it can be written in the form

$$c_1x_1 + \cdots + c_nx_n + P(y_1, \ldots, y_k)$$

where $n \geq 2$, $P$ has no constant term, and there exists a nonempty subset $J \subseteq \{1, \ldots, n\}$ such that $\sum_{j \in J} c_j = 0$. 

Lorenzo Luperi Baglini  
University of Vienna  
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Generalized Rado

Theorem

Let

\[ R(x_1, \ldots, x_n, y_1, \ldots, y_k) = c_1 x_1 + \ldots + c_n x_n + P(y_1, \ldots, y_k) \]

be a Rado polynomial. Then every ultrafilter \( \mathcal{U} \in K(\otimes) \cap \mathbb{II}(\oplus) \cap \Delta \) is a \( PR \)-witness of \( R(x_1, \ldots, x_n, y_1, \ldots, y_k) = 0 \).
Generalized Rado

**Theorem**

Let

\[ R(x_1, \ldots, x_n, y_1, \ldots, y_k) = c_1 x_1 + \ldots + c_n x_n + P(y_1, \ldots, y_k) \]

be a Rado polynomial. Then every ultrafilter \( \mathcal{U} \in \overline{K(\odot)} \cap \overline{I(\oplus)} \cap \Delta \) is a PR-witness of \( R(x_1, \ldots, x_n, y_1, \ldots, y_k) = 0 \).

**Proof.**

Consider the following system:

\[
\begin{cases}
    c_1 z + c_2 x_2 + \ldots + c_n x_n = 0; \\
    c_1 (w - x_1) = P(y_1, \ldots, y_k); \\
    z = w.
\end{cases}
\]
Main positive result/1

Theorem

Let \( \mathcal{F} \) be the family of polynomials whose PR on \( \mathbb{N} \) is witnessed by at least an ultrafilter \( \mathcal{U} \in \mathcal{I}(\otimes) \cap \overline{K(\otimes)} \cap \mathcal{I}(\oplus) \cap \Delta \). Then \( \mathcal{F} \) includes:

- Every Radó polynomial;
- Every polynomial of the form
  \[ n \overset{1}{\ldots} i c_i x^i \overset{1}{\ldots} j P_f \] where \( n \overset{1}{\ldots} i c_i x^i \) is a Radó polynomial and sets \( F_i = \overset{1}{\ldots} m \); 
- Every polynomial
  \[ P_{px,y} x^k \overset{1}{\ldots} y^i \]
Main positive result/1

Theorem

Let \( \mathcal{F} \) be the family of polynomials whose PR on \( \mathbb{N} \) is witnessed by at least an ultrafilter \( \mathcal{U} \in \mathcal{I}(\otimes) \cap \overline{\mathcal{K}(\otimes)} \cap \mathcal{I}(\oplus) \cap \Delta \). Then \( \mathcal{F} \) includes:

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Theorem

Let $\mathcal{F}$ be the family of polynomials whose PR on $\mathbb{N}$ is witnessed by at least an ultrafilter $\mathcal{U} \in \mathbb{I}(\odot) \cap \overline{K(\odot)} \cap \mathbb{I}(\oplus) \cap \Delta$. Then $\mathcal{F}$ includes:

- Every Rado polynomial;
- Every polynomial of the form

$$\sum_{i=1}^{n} c_i x_i \left( \prod_{j \in F_i} y_j \right)$$

where $\sum_{i=1}^{n} c_i x_i$ is a Rado polynomial and sets $F_i \subseteq \{1, \ldots, m\}$;
Theorem

Let $\mathcal{F}$ be the family of polynomials whose PR on $\mathbb{N}$ is witnessed by at least an ultrafilter $\mathcal{U} \in \mathcal{I}(\emptyset) \cap \overline{\mathcal{K}(\emptyset)} \cap \mathcal{I}(\emptyset) \cap \Delta$. Then $\mathcal{F}$ includes:

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- Every polynomial of the form

$$
\sum_{i=1}^{n} c_i x_i \left( \prod_{j \in F_i} y_j \right)
$$

where $\sum_{i=1}^{n} c_i x_i$ is a Rado polynomial and sets $F_i \subseteq \{1, \ldots, m\}$;

- Every polynomial

$$
P(x, y_1, \ldots, y_k) = x - \prod_{i=1}^{k} y_i;
$$
Main positive result/1

Theorem

Let $\mathcal{F}$ be the family of polynomials whose PR on $\mathbb{N}$ is witnessed by at least an ultrafilter $\mathcal{U} \in \mathcal{I}(\bigodot) \cap \overline{K(\bigodot)} \cap \mathcal{I}(\bigoplus) \cap \Delta$. Then $\mathcal{F}$ includes:

- Every Rado polynomial;
- Every polynomial of the form
  \[
  \sum_{i=1}^{n} c_i x_i \left( \prod_{j \in F_i} y_j \right)
  \]
  where $\sum_{i=1}^{n} c_i x_i$ is a Rado polynomial and sets $F_i \subseteq \{1, \ldots, m\};$
- Every polynomial
  \[
  P(x, y_1, \ldots, y_k) = x - \prod_{i=1}^{k} y_i;
  \]
Theorem

- **Every polynomial**

\[ P(x, y_1, \ldots, y_k) = x - \prod_{i=1}^{k} y_i^{a_i}, \]

wherever the exponents \( a_i \in \mathbb{Z} \) satisfy \( \sum_{i=1}^{n} a_i = 1. \)
Main positive result/2

Theorem

- Every polynomial

\[ P(x, y_1, \ldots, y_k) = x - \prod_{i=1}^{k} y_i^{a_i}, \]

whenever the exponents \( a_i \in \mathbb{Z} \) satisfy \( \sum_{i=1}^{n} a_i = 1 \).

Moreover, the family \( \mathcal{F} \) satisfies the following closure properties:
Main positive result/2

Theorem

- Every polynomial

\[ P(x, y_1, \ldots, y_k) = x - \prod_{i=1}^{k} y_i^{a_i}, \]

whenever the exponents \( a_i \in \mathbb{Z} \) satisfy \( \sum_{i=1}^{n} a_i = 1 \).

Moreover, the family \( \mathcal{F} \) satisfies the following closure properties:

(i) If \( P(z, y_1, \ldots, y_k) \in \mathcal{F} \) and \( z - g(x_1, \ldots, x_n) \in \mathcal{F} \), then \( P(g(x_1, \ldots, x_n), y_1, \ldots, y_k) \in \mathcal{F} \);
Main positive result/2

Theorem

- Every polynomial

\[ P(x, y_1, \ldots, y_k) = x - \prod_{i=1}^{k} y_i^{a_i}, \]

whenever the exponents \( a_i \in \mathbb{Z} \) satisfy \( \sum_{i=1}^{n} a_i = 1 \).

Moreover, the family \( F \) satisfies the following closure properties:

(i) If \( P(z, y_1, \ldots, y_k) \in F \) and \( z - g(x_1, \ldots, x_n) \in F \), then
    \[ P(g(x_1, \ldots, x_n), y_1, \ldots, y_k) \in F; \]

(ii) if \( P(x_1, \ldots, x_n) \in F \) is homogeneous, then \( P \left( \frac{1}{x_1}, \ldots, \frac{1}{x_n} \right) \in F. \)
Some examples
Some examples

Example

Let $n, m \in \mathbb{N}$ and assume that, for every $i \leq n$, $j \leq m$, the equations

$$
\begin{align*}
    x_{i,1} &= \sum_{h=1}^{r_i} c_{i,h} x_{i,h}, \\
    y_{j,1} &= \sum_{k=1}^{s_j} d_{j,k} y_{j,k}
\end{align*}
$$

are PR.
Some examples

Example

Let \( n, m \in \mathbb{N} \) and assume that, for every \( i \leq n, j \leq m \), the equations

\[
    x_{i,1} = \sum_{h=1}^{r_i} c_{i,h} x_{i,h}, \quad y_{j,1} = \sum_{k=1}^{s_j} d_{j,k} y_{j,k}
\]

are PR.

Let \( a_1, \ldots, a_n, b_1, \ldots, b_m \) be such that \( \sum_{i=1}^{n} a_i = \sum_{j=1}^{m} b_j \) and consider the homogeneous PR equation

\[
    \prod_{i=1}^{n} t_i^{a_i} = \prod_{j=1}^{m} z_j^{b_j}.
\]
Some examples

Example

Let \( n, m \in \mathbb{N} \) and assume that, for every \( i \leq n, j \leq m \), the equations

\[
x_{i,1} = \sum_{h=1}^{r_i} c_{i,h} x_{i,h}, \quad y_{j,1} = \sum_{k=1}^{s_j} d_{j,k} y_{j,k}
\]

are PR.

Let \( a_1, \ldots, a_n, b_1, \ldots, b_m \) be such that \( \sum_{i=1}^{n} a_i = \sum_{j=1}^{m} b_j \) and consider the homogeneous PR equation \( \prod_{i=1}^{n} t_i^{a_i} = \prod_{j=1}^{m} z_j^{b_j} \).

All these equations are PR and homogeneous and therefore, by the closure property (i), also

\[
\prod_{i=1}^{n} \left( \sum_{h=1}^{r_i} c_{i,h} x_{i,h} \right)^{a_i} = \prod_{j=1}^{m} \left( \sum_{k=1}^{s_j} d_{j,k} y_{j,k} \right)^{b_j}
\]

is PR.
Some examples

Example

Let $n, m \in \mathbb{N}$ and assume that, for every $i \leq n$, $j \leq m$, the equations

$$x_{i,1} = \sum_{h=1}^{r_i} c_{i,h} x_{i,h}, \quad y_{j,1} = \sum_{k=1}^{s_j} d_{j,k} y_{j,k}$$

are PR.

Let $a_1, \ldots, a_n, b_1, \ldots, b_m$ be such that $\sum_{i=1}^{n} a_i = \sum_{j=1}^{m} b_j$ and consider the homogeneous PR equation $\prod_{i=1}^{n} t_{i}^{a_i} = \prod_{j=1}^{m} z_{j}^{b_j}$.

All these equations are PR and homogeneous and therefore, by the closure property (i), also

$$\prod_{i=1}^{n} \left( \sum_{h=1}^{r_i} c_{i,h} x_{i,h} \right)^{a_i} = \prod_{j=1}^{m} \left( \sum_{k=1}^{s_j} d_{j,k} y_{j,k} \right)^{b_j}$$

is PR.
Some examples

Example

For every $n \in \mathbb{N}$, the polynomial $u - v - z^n$ is in $\mathcal{F}$;
Some examples

Example
For every \( n \in \mathbb{N} \), the polynomial \( u - v - z^n \) is in \( \mathcal{F} \); moreover, for every \( k \geq 2 \) the function \( x = \prod_{j=1}^{k} x_j \) is in \( \mathcal{F} \).
Some examples

Example

For every $n \in \mathbb{N}$, the polynomial $u - v - z^n$ is in $\mathcal{F}$; moreover, for every $k \geq 2$ the function $x = \prod_{j=1}^{k} x_j$ is in $\mathcal{F}$. Therefore, for every $h, k \geq 2$ we can apply the closure property (i) of $\mathcal{F}$ to the system

$$
\begin{align*}
\begin{cases}
u - v = z^n; \\
x = \prod_{j=1}^{h} x_j; \\
y = \prod_{j=1}^{k} y_j; \\
x = u, y = v.
\end{cases}
\end{align*}
$$
Some examples

Example

For every $n \in \mathbb{N}$, the polynomial $u - v - z^n$ is in $\mathcal{F}$; moreover, for every $k \geq 2$ the function $x = \prod_{j=1}^{k} x_j$ is in $\mathcal{F}$. Therefore, for every $h, k \geq 2$ we can apply the closure property (i) of $\mathcal{F}$ to the system

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x = u, y = v.
\end{align*}
$$

Hence $\prod_{j=1}^{h} x_j - \prod_{j=1}^{k} y_j = z^n$ is in $\mathcal{F}$. In particular, $x_1 x_2 - y_1 y_2 = z^2$ is PR.
Some examples

Example

For every \( n \in \mathbb{N} \), the polynomial \( u - v - z^n \) is in \( \mathcal{F} \); moreover, for every \( k \geq 2 \) the function \( x = \prod_{j=1}^{k} x_j \) is in \( \mathcal{F} \). Therefore, for every \( h, k \geq 2 \) we can apply the closure property (i) of \( \mathcal{F} \) to the system

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    & u - v = z^n; \\
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\end{align*}
\]

Hence \( \prod_{j=1}^{h} x_j - \prod_{j=1}^{k} y_j = z^n \) is in \( \mathcal{F} \). In particular, \( x_1 x_2 - y_1 y_2 = z^2 \) is PR.

Example

\( P(x_1, x_2, x_3) = x_1 x_2 - 2x_3 \) is PR but it does not belong to \( \mathcal{F} \).
$u$-equivalence and partition regularity

**Definition**

Two hypernatural numbers $\xi, \xi' \in {}^*\mathbb{N}$ are *$u$-equivalent* if they cannot be distinguished by any hyper-extension, i.e. if for every $A \subseteq \mathbb{N}$ one has either $\xi, \xi' \in {}^*A$ or $\xi, \xi' \notin {}^*A$. 

Lorenzo Luperi Baglini

University of Vienna

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$u$-equivalence and partition regularity

**Definition**

Two hypernatural numbers $\xi, \xi' \in \mathbb{N}$ are $u$-equivalent if they cannot be distinguished by any hyper-extension, i.e. if for every $A \subseteq \mathbb{N}$ one has either $\xi, \xi' \in *A$ or $\xi, \xi' \notin *A$.

When the hyperextension has the $|\varphi(\mathbb{N})|^+\text{-enlarging}$ property, ultrafilters and hypernaturals can be identified:
u-equivalence and partition regularity

**Definition**

Two hypernatural numbers $\xi, \xi' \in \mathbb{N}^*$ are **u-equivalent** if they cannot be distinguished by any hyper-extension, i.e. if for every $A \subseteq \mathbb{N}$ one has either $\xi, \xi' \in *A$ or $\xi, \xi' \not\in *A$.

When the hyperextension has the $|\wp(\mathbb{N})|^+$-enlarging property, ultrafilters and hypernaturals can be identified:

- $\alpha \rightarrow U_\alpha = \{A \in \wp(\mathbb{N}) \mid \alpha \in *A\}$;
$u$-equivalence and partition regularity

Definition

Two hypernatural numbers $\xi, \xi' \in \ast\mathbb{N}$ are $u$-equivalent if they cannot be distinguished by any hyper-extension, i.e. if for every $A \subseteq \mathbb{N}$ one has either $\xi, \xi' \in \ast A$ or $\xi, \xi' \notin \ast A$.

When the hyperextension has the $|\wp(\mathbb{N})|^+\text{-enlarging}$ property, ultrafilters and hypernaturals can be identified:

- $\alpha \rightarrow \mathcal{U}_\alpha = \{A \in \wp(\mathbb{N}) \mid \alpha \in \ast A\}$;
- $\mathcal{U} \rightarrow \mu(\mathcal{U}) = \{\alpha \in \ast\mathbb{N} \mid \mathcal{U} = \mathcal{U}_\alpha\}$. 
u-equivalence and partition regularity

**Definition**

Two hypernatural numbers $\xi, \xi' \in *\mathbb{N}$ are **u-equivalent** if they cannot be distinguished by any hyper-extension, i.e. if for every $A \subseteq \mathbb{N}$ one has either $\xi, \xi' \in *A$ or $\xi, \xi' \notin *A$.

When the hyperextension has the $|\mathcal{P}(\mathbb{N})|^+$-enlarging property, ultrafilters and hypernaturals can be identified:

- $\alpha \to U_\alpha = \{A \in \mathcal{P}(\mathbb{N}) \mid \alpha \in *A\}$;
- $U \to \mu(U) = \{\alpha \in *\mathbb{N} \mid U = U_\alpha\}$.

**Proposition**

A Diophantine equation $P(x_1, \ldots, x_n) = 0$ is PR if and only if there exist u-equivalent hypernatural numbers $\xi_1, \ldots, \xi_n$ with $*P(\xi_1, \ldots, \xi_n) = 0$. 
Multi-index notations

- An $n$-dimensional multi-index is an $n$-tuple $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n$;
Multi-index notations

- An $n$-dimensional multi-index is an $n$-tuple $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n$;
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Multi-index notations

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Multi-index notations

- An $n$-dimensional multi-index is an $n$-tuple $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n$;
- $\alpha \preceq \beta$ means that $\alpha_i \leq \beta_i$ for all $i = 1, \ldots, n$;
- $\alpha < \beta$ means that $\alpha \preceq \beta$ and $\alpha \neq \beta$;
- If $\mathbf{x} = (x_1, \ldots, x_n)$ is a vector and $\alpha = (\alpha_1, \ldots, \alpha_n)$ is a multi-index, the product $\prod_{i=1}^n x_i^{\alpha_i}$ is denoted by $\mathbf{x}^\alpha$;
Multi-index notations

- An \( n \)-dimensional \textbf{multi-index} is an \( n \)-tuple \( \boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n \);
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- If \( \mathbf{x} = (x_1, \ldots, x_n) \) is vector and \( \alpha = (\alpha_1, \ldots, \alpha_n) \) is a multi-index, the product \( \prod_{i=1}^{n} x_i^{\alpha_i} \) is denoted by \( \mathbf{x}^\alpha \);
- The \textbf{length} of a multi-index \( \alpha = (\alpha_1, \ldots, \alpha_n) \) is \( |\alpha| = \sum_{i=1}^{n} \alpha_i \);
Multi-index notations

- An $n$-dimensional multi-index is an $n$-tuple $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n$;
- $\alpha \leq \beta$ means that $\alpha_i \leq \beta_i$ for all $i = 1, \ldots, n$;
- $\alpha < \beta$ means that $\alpha \leq \beta$ and $\alpha \neq \beta$;
- If $x = (x_1, \ldots, x_n)$ is a vector and $\alpha = (\alpha_1, \ldots, \alpha_n)$ is a multi-index, the product $\prod_{i=1}^{n} x_i^{\alpha_i}$ is denoted by $x^\alpha$;
- The length of a multi-index $\alpha = (\alpha_1, \ldots, \alpha_n)$ is $|\alpha| = \sum_{i=1}^{n} \alpha_i$;
- A set $I$ of $n$-dimensional multi-indexes having all the same length is called homogeneous;
Multi-index notations

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- Polynomials $P \in \mathbb{Z}[x_1, \ldots, x_n]$ are written in the form $P(\mathbf{x}) = \sum_{\alpha} c_{\alpha} \mathbf{x}^{\alpha}$ where $\alpha$ are multi-indexes;
Multi-index notations

- An $n$-dimensional **multi-index** is an $n$-tuple $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n$;
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- A set $I$ of $n$-dimensional multi-indexes having all the same length is called **homogeneous**;
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- The **support** of $P$ is the finite set $\text{supp}(P) = \{\alpha \mid c_{\alpha} \neq 0\}$;
Multi-index notations

- An \( n \)-dimensional \textbf{multi-index} is an \( n \)-tuple \( \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n \);
- \( \alpha \preceq \beta \) means that \( \alpha_i \leq \beta_i \) for all \( i = 1, \ldots, n \);
- \( \alpha < \beta \) means that \( \alpha \preceq \beta \) and \( \alpha \neq \beta \);
- If \( \mathbf{x} = (x_1, \ldots, x_n) \) is vector and \( \alpha = (\alpha_1, \ldots, \alpha_n) \) is a multi-index, the product \( \prod_{i=1}^n x_i^{\alpha_i} \) is denoted by \( \mathbf{x}^\alpha \);
- The \textbf{length} of a multi-index \( \alpha = (\alpha_1, \ldots, \alpha_n) \) is \( |\alpha| = \sum_{i=1}^n \alpha_i \);
- A set \( I \) of \( n \)-dimensional multi-indexes having all the same length is called \textit{homogeneous};
- Polynomials \( P \in \mathbb{Z}[x_1, \ldots, x_n] \) are written in the form \( P(\mathbf{x}) = \sum_\alpha c_\alpha \mathbf{x}^\alpha \) where \( \alpha \) are multi-indexes;
- The \textbf{support} of \( P \) is the finite set \( \text{supp}(P) = \{ \alpha \mid c_\alpha \neq 0 \} \);
- A polynomial \( P(\mathbf{x}) = \sum_\alpha c_\alpha \mathbf{x}^\alpha \) is \textbf{homogeneous} if \( \text{supp}(P) \) is a homogeneous set of indexes.
Minimal and maximal indeces

Definition

Let \( P(x) = \sum_{\alpha} c_{\alpha} x^\alpha \in \mathbb{Z}[x_1, \ldots, x_n] \). We say that a multi-index \( \alpha \in \text{supp}(P) \) is minimal if there are no \( \beta \in \text{supp}(P) \) with \( \beta < \alpha \). The notion of maximal multi-index is defined similarly.
Minimal and maximal indeces

Definition

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Lorenzo Luperi Baglini

University of Vienna

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Minimal and maximal indices

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For linear polynomials, every nonempty $J \subseteq \text{Supp}(P) = \{\alpha(1), \ldots, \alpha(n)\}$ is a Rado set of both minimal and maximal indexes.
Minimal and maximal indeces

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For linear polynomials, every nonempty $J \subseteq \text{Supp}(P) = \{\alpha(1), \ldots, \alpha(n)\}$ is a Rado set of both minimal and maximal indexes.

Example

In $c_{(2,1,1,0)} x_1^2 x_2 x_3 + c_{(1,2,7,0)} x_1 x_2^2 x_3^7 + c_{(2,2,2,1)} x_1^2 x_2^2 x_3^2 x_4$, the set $J = \{(2, 1, 1, 0), (1, 2, 7, 0)\}$ is a Rado set of minimal (but not maximal) indeces: just let $\Lambda = \{1, 2\} \subseteq \{1, 2, 3, 4\}$. 
General necessary condition

**Theorem**

Let \( P(x) = \sum_{\alpha} c_\alpha x^\alpha \in \mathbb{Z}[x_1, \ldots, x_n] \) be a polynomial with no constant term.
General necessary condition

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Let \( P(x) = \sum_{\alpha} c_\alpha x^\alpha \in \mathbb{Z}[x_1, \ldots, x_n] \) be a polynomial with no constant term. Suppose there exists a prime \( p \) such that:

1. \( z \mid \alpha \mod p \) has no solutions \( z \neq 0 \);
2. For every Rado set \( J \) of minimal indexes, \( \sum_{\alpha} c_\alpha z^\alpha \mod p \) has no solutions \( z \neq 0 \).

Then \( P(x) \) is not PR, except possibly for constant solutions \( x_1 \ldots x_n \).

**Proof.** Pick infinite \( \xi_1 \ldots \xi_n \) such that \( P(p^{\xi_1} \ldots p^{\xi_n}) = 0 \). Write \( \xi_i \) in base \( p \). Find the absurd playing with the exponents and the coefficients in this expansion.
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Proof.

Pick infinite \( \xi_1 u \ldots u \xi_n \) such that \( P(\xi)q^{\vert\alpha\vert} \equiv 0 \). Write \( \xi_i \) in base \( p \). Find the absurd playing with the exponents and the coefficients in this expansion.
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Proof.

Pick infinite $\xi_1 \sim \ldots \sim \xi_n$ such that $P(\xi) = \sum_{\alpha} c_{\alpha} \xi^{\alpha} = 0$. Write $\xi_i$ in base $p$. 
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Pick infinite $\xi_1 \sim_u \ldots \sim_u \xi_n$ such that $P(\xi) = \sum_{\alpha} c_{\alpha} \xi^{\alpha} = 0$. Write $\xi_i$ in base $p$. Find the absurd playing with the exponents and the coefficients in this expansion.
Examples

Let $P_{x_1, x_2, x_3}$. Pick any prime number $p$ with $p \equiv 3 \text{ or } 5 \mod 8$, so that $2$ is not a quadratic residue modulo $p$.

Then condition (1) is satisfied because $z_3 \equiv 2z \equiv 0$, and also condition (2) is easily verified.

Since it has no constant solutions $x_1$, $x_2$, $x_3$, we can conclude that $P_{x_1, x_2, x_3}$ is not PR.

Notice that, by Multiplicative Rado's Theorem, the seemingly similar equation $x_2 + x_2 + x_3$ is PR.

Corollary Let $P_{p^{x_1}, x_2, x_3}^\alpha c^\alpha x^{\alpha P}$ be a homogeneous polynomial. If for every nonempty $J = \text{supp} P$ one has $\alpha_j P J \equiv 0$, then $P_{p^{x_1}, x_2, x_3}$ is not PR.
Example

Let $P(x_1, x_2, x_3) = x_1^2 x_2 - 2x_3$. Pick any prime number $p$ with $p \equiv 3$ or $p \equiv 5 \mod 8$, so that $2$ is not a quadratic residue modulo $p$. 

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Notice that, by Multiplicative Rado’s Theorem, the seemingly similar equation \( x_1^2 x_2 = x_3 \) is PR.

Corollary

Let \( P(x) = \sum_{\alpha} c_{\alpha} x^\alpha \in \mathbb{Z}[x_1, \ldots, x_n] \) be an homogeneous polynomial. If for every nonempty \( J \subseteq \text{supp}(P) \) one has \( \sum_{\alpha \in J} c_{\alpha} \neq 0 \), then \( P(x) \) is not PR.
Theorem

For every $i = 1, \ldots, n$ let $P_i(x_i) = \sum_{s=1}^{d_i} c_i,s x_i^s$ be a polynomial of degree $d_i$ in the variable $x_i$ with no constant term.
**Theorem**

For every $i = 1, \ldots, n$ let $P_i(x_i) = \sum_{s=1}^{d_i} c_{i,s} x_i^s$ be a polynomial of degree $d_i$ in the variable $x_i$ with no constant term. If the Diophantine equation

$$\sum_{i=1}^{n} P_i(x_i) = 0$$

is PR then the following “Rado’s condition” is satisfied:
Theorem

For every $i = 1, \ldots, n$ let $P_i(x_i) = \sum_{s=1}^{d_i} c_{i,s} x_i^s$ be a polynomial of degree $d_i$ in the variable $x_i$ with no constant term. If the Diophantine equation

$$\sum_{i=1}^{n} P_i(x_i) = 0$$

is PR then the following “Rado’s condition” is satisfied:

- “There exists a nonempty set $J \subseteq \{1, \ldots, n\}$ such that $d_i = d_j$ for every $i, j \in J$, and $\sum_{j \in J} c_{j,d_j} = 0$.”
Necessary condition for sums of polynomials in one variable

Proof.

For every \( i \), let \( \Lambda(i) = \{ s \mid c_{i,s} \neq 0 \} \) be the support of \( P_i(x_i) \), and for every \( s \), let \( \Gamma(s) = \{ i \mid c_{i,s} \neq 0 \} \). If we denote by

\[
P(x) = \sum_{i=1}^{n} P_i(x_i) = \sum_{i=1}^{n} \sum_{s \in \Lambda(i)} c_{i,s} x_i^s,
\]

by the nonstandard characterization of non-trivial PR, we can pick infinite \( \xi_1 \sim \ldots \sim \xi_n \) such that \( P(\xi) = 0 \). Now fix any finite number \( p \geq 2 \), and write the numbers \( \xi_i \) in base \( p \):

\[
\xi_i = \sum_{t=0}^{\tau_i} a_{i,t} p^{\tau_i-t}
\]

where \( 0 \leq a_{i,t} \leq p - 1 \) and \( a_{i,0} \neq 0 \).
Necessary condition for sums of polynomials in one variable/3

Proof.

Let $s_\ast \tau_\ast = \max\{s \tau_i \mid i \in \Gamma(s)\}$. It is not difficult to show that $d_i = s_\ast$ for every $i \in \Gamma(s_\ast)$, by the maximality of $s_\ast \tau_\ast$. 
Necessary condition for sums of polynomials in one variable/3

Proof.

Let \( s_\ast \tau_\ast = \max\{s \tau_i \mid i \in \Gamma(s)\} \). It is not difficult to show that \( d_i = s_\ast \) for every \( i \in \Gamma(s_\ast) \), by the maximality of \( s_\ast \tau_\ast \).

Now let \( I_\ast = \{i \in \Gamma(s_\ast) \mid \tau_i = \tau_\ast\} \), and decompose \( P(\xi) = \Theta + \Psi + \Phi \), where:

- \( \Theta = \sum_{i \in I_\ast} c_{i,s_\ast} \xi_i^{s_\ast} \);
- \( \Psi = \sum_{i \in \Gamma(s_\ast) \setminus I_\ast} c_{i,s_\ast} \xi_i^{s_\ast} \);
- \( \Phi = \sum_{s \neq s_\ast} \sum_{i \in \Gamma(s)} c_{i,s} \xi_i^s \).
Necessary condition for sums of polynomials in one variable/3

Proof.

Let $s_* \tau_* = \max\{s \tau_i \mid i \in \Gamma(s)\}$. It is not difficult to show that $d_i = s_*$ for every $i \in \Gamma(s_*)$, by the maximality of $s_* \tau_*$. Now let $I_* = \{i \in \Gamma(s_*) \mid \tau_i = \tau_*\}$, and decompose $P(\xi) = \Theta + \Psi + \Phi$, where:

- $\Theta = \sum_{i \in I_*} c_{i,s_*} \xi^{s_*}_i$;
- $\Psi = \sum_{i \in \Gamma(s_*) \setminus I_*} c_{i,s_*} \xi^{s_*}_i$;
- $\Phi = \sum_{s \neq s_*} \sum_{i \in \Gamma(s)} c_{i,s} \xi^s_i$.

Lemma

1. $\Theta = \left(\sum_{i \in I_*} c_{i,s_*}\right) \zeta + \Theta'$ for suitable $\zeta \geq p^{s_* \tau_*}$ and $|\Theta'| \leq p^{s_* \tau_*}$. 

Lorenzo Luperi Baglini
University of Vienna
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Proof.

Let \( s_\tau = \max\{s \tau_i \mid i \in \Gamma(s)\} \). It is not difficult to show that \( d_i = s_\tau \) for every \( i \in \Gamma(s_\tau) \), by the maximality of \( s_\tau \).

Now let \( I_\tau = \{i \in \Gamma(s_\tau) \mid \tau_i = \tau_\tau\} \), and decompose \( P(\xi) = \Theta + \Psi + \Phi \), where:

- \( \Theta = \sum_{i \in I_\tau} c_{i,s_\tau} \xi^s \);
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Lemma

1. \( \Theta = \left(\sum_{i \in I_\tau} c_{i,s_\tau}\right) \zeta + \Theta' \) for suitable \( \zeta \geq p^{s_\tau \tau_\tau} \) and \( |\Theta'| \leq p^{s_\tau \tau_\tau} \).
2. \( |\Psi| \leq p^{s_\tau \tau_\tau} \).
Necessary condition for sums of polynomials in one variable/3

Proof.

Let $s_\ast \tau_\ast = \max\{s \tau_i \mid i \in \Gamma(s)\}$. It is not difficult to show that $d_i = s_\ast$ for every $i \in \Gamma(s_\ast)$, by the maximality of $s_\ast \tau_\ast$.

Now let $I_\ast = \{i \in \Gamma(s_\ast) \mid \tau_i = \tau_\ast\}$, and decompose $P(\xi) = \Theta + \Psi + \Phi$, where:

- $\Theta = \sum_{i \in I_\ast} c_{i,s_\ast} \xi_i^{s_\ast}$;
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Lemma

1. $\Theta = \left(\sum_{i \in I_\ast} c_{i,s_\ast}\right) \zeta + \Theta'$ for suitable $\zeta \geq p^{s_\ast \tau_\ast}$ and $\lvert \Theta' \rvert \leq p^{s_\ast \tau_\ast}$.

2. $\lvert \Psi \rvert \leq p^{s_\ast \tau_\ast}$.

3. $\lvert \Phi \rvert \leq p^{s_\ast \tau_\ast}$.
Necessary condition for sums of polynomials in one variable/3

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Let $s_{\ast} \tau_{\ast} = \max\{s \tau_i \mid i \in \Gamma(s)\}$. It is not difficult to show that $d_i = s_{\ast}$ for every $i \in \Gamma(s_{\ast})$, by the maximality of $s_{\ast} \tau_{\ast}$.

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Lemma

1. $\Theta = \left(\sum_{i \in I_{\ast}} c_{i,s_{\ast}}\right) \zeta + \Theta' \text{ for suitable } \zeta \geq p^{s_{\ast} \tau_{\ast}} \text{ and } |\Theta'| \leq p^{s_{\ast} \tau_{\ast}}.$
2. $|\Psi| \leq p^{s_{\ast} \tau_{\ast}}.$
3. $|\Phi| \leq p^{s_{\ast} \tau_{\ast}}.$
Necessary condition for sums of polynomials in one variable/3

Proof.

Since \( P(\xi) = \Theta + \Psi + \Phi = 0 \), the above inequalities imply that the sum of coefficients \( \sum_{i \in I_*} c_{i,s*} = 0 \).
Proof.

Since $P(\xi) = \Theta + \Psi + \Phi = 0$, the above inequalities imply that the sum of coefficients $\sum_{i \in I_*} c_{i,s_*} = 0$. We claim that $J = I_*$ is the desired set of indexes. In fact, $I_*$ is trivially nonempty; moreover, $d_i = d_j = s_*$ for all $i, j \in J$; and $\sum_{j \in J} c_{j,d_j} = \sum_{j \in J} c_{j,s_*} = 0$. 
necessary condition for sums of polynomials in one variable/3

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The most complicated part is the proof of the Lemma.
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The most complicated part is the proof of the Lemma. The idea is to let $\varphi : \mathbb{N} \to \mathbb{N}_0$ be the function s.t. $p^{\varphi(m)} \leq m < p^{\varphi(m)+1}$; and for every $t \in \mathbb{N}_0$, let $\psi_t(m) : \mathbb{N} \to \{0, 1, \ldots, p - 1\}$ be the function where $\psi_t(m)$ is the $(t + 1)$-th digit from the left when $m$ is written in base $p$. 
Necessary condition for sums of polynomials in one variable/3

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Since \( P(\xi) = \Theta + \Psi + \Phi = 0 \), the above inequalities imply that the sum of coefficients \( \sum_{i \in I_*} c_{i,s_*} = 0 \). We claim that \( J = I_* \) is the desired set of indexes. In fact, \( I_* \) is trivially nonempty; moreover, \( d_i = d_j = s_* \) for all \( i, j \in J \); and \( \sum_{j \in J} c_{j,d_j} = \sum_{j \in J} c_{j,s_*} = 0 \).

The most complicated part is the proof of the Lemma. The idea is to let \( \varphi : \mathbb{N} \to \mathbb{N}_0 \) be the function s.t. \( p^\varphi(m) \leq m < p^\varphi(m) + 1 \); and for every \( t \in \mathbb{N}_0 \), let \( \psi_t(m) : \mathbb{N} \to \{0, 1, \ldots, p - 1\} \) be the function where \( \psi_t(m) \) is the \((t + 1)\)-th digit from the left when \( m \) is written in base \( p \). Then the \( u \)-equivalences \( \xi_1 \sim_u \ldots \sim_u \xi_n \) imply, by overspill, that for every \( a \in \mathbb{N} \) one has \( \xi_i^a = \zeta_i^a + \vartheta_{i,a} \) where \( p^{a_{\tau_i}} \leq \zeta_i^a \leq \xi_i^a < p^{a_{\tau_i} + a} \) and \( \vartheta_{i,a} \leq p^{a_{\tau_i}} \).
Necessary condition for sums of polynomials in one variable/3

Proof.

Since \( P(\xi) = \Theta + \Psi + \Phi = 0 \), the above inequalities imply that the sum of coefficients \( \sum_{i \in I_*} c_{i,s_*} = 0 \). We claim that \( J = I_* \) is the desired set of indexes. In fact, \( I_* \) is trivially nonempty; moreover, \( d_i = d_j = s_* \) for all \( i, j \in J \); and \( \sum_{j \in J} c_{j,d_j} = \sum_{j \in J} c_{j,s_*} = 0 \).

The most complicated part is the proof of the Lemma. The idea is to let \( \phi : \mathbb{N} \to \mathbb{N}_0 \) be the function s.t. \( p^{\phi(m)} \leq m < p^{\phi(m)+1} \); and for every \( t \in \mathbb{N}_0 \), let \( \psi_t(m) : \mathbb{N} \to \{0, 1, \ldots, p - 1\} \) be the function where \( \psi_t(m) \) is the \((t + 1)\)-th digit from the left when \( m \) is written in base \( p \). Then the \( u \)-equivalences \( \xi_1 \sim_u \ldots \sim_u \xi_n \) imply, by overspill, that for every \( a \in \mathbb{N} \) one has \( \xi_i^a = \zeta_i^a + \vartheta_{i,a} \) where \( p^{a\tau_i} \leq \zeta_i^a \leq \xi_i^a < p^{a\tau_i+a} \) and \( \vartheta_{i,a} \leq p^{a\tau_i} \).

With this decomposition, the lemma can be proven by showing that \( \zeta_i^a \) and \( \vartheta_{i,a} \) "does not mix in computations".
Examples

A polynomial of the form

\[ P(x_1, x_2, \ldots, x_n) = c_1 x_1 + c_2 x_2 + \cdots + c_n x_n, \]

where \( P \) is a non-linear polynomial with no constant term, is PR if and only if it is a Rado polynomial.

Example

The polynomial

\[ P(x, y) = x^3 y^2 + x^2 y^3 \]

is not PR (even if it contains a partial sum of coefficients that equals zero).

Example

The polynomials

\[ x^n y^m z^k \]

are not PR for \( k < n, m \).

Lorenzo Luperi Baglini
University of Vienna
23 March 2019
Examples

Corollary

A polynomial of the form \( \sum_{i=1}^{n} c_i x_i + P(y) \), where \( P \) is a nonlinear polynomial with no constant term, is PR if and only if it is a Rado polynomial.
## Examples

### Corollary

A polynomial of the form \( \sum_{i=1}^{n} c_i x_i + P(y) \), where \( P \) is a nonlinear polynomial with no constant term, is PR if and only if it is a Rado polynomial.

### Example

The polynomial

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P(x, y) = x^3 + 2x + y^3 - 2y
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Example

The polynomials \( x^n + y^m = z^k \) are not PR for \( k \notin \{n, m\} \).
Op en Problems/1

Recently, Heule, Kullmann and Marek have proven the PR of the Pythagorean equation for 2-colorings.

Moreover, using number theoretical methods, Chow, Lindqvist and Prendiville have proven the following:

**Theorem**
For every \( k \in \mathbb{N} \) there exists \( s_0, p, q \) such that for every \( s \neq s_0 \), \( p, q \) and \( c_1, \ldots, c_s \in \mathbb{Z} \), \( z \not\equiv 0 \), the following equivalence holds:

The equation \( \sum_{i=1}^{s} c_i x_i \) is PR;

there exists a nonempty set \( J \) such that \( \sum_{j \in J} c_j \not\equiv 0 \).

\( s_0, p, q \) is of the order \( k \log k \).

Open Problem 2.
Are there simple decidable conditions under which a given (non-homogeneous) Diophantine equation with no constant term is PR on \( \mathbb{N} \) if and only if it is PR on \( \mathbb{Z} \) if and only if it is PR on \( \mathbb{Q} \)?
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Open Problem 1. Is $x^2 + y^2 = z^2$ PR?
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Theorem

For every $k \in \mathbb{N}$ there exists $s_0, p, q$ such that for every $s > s_0$ and $c_1, \ldots, c_s \in \mathbb{Z}$ such that $t_0 < t_1 < \ldots < t_s$ the following equivalence holds:

The equation $\sum_{i=1}^{s} c_i x^k_i$ is PR; there exists a nonempty set $J = \{t_1, \ldots, t_s\}$ such that $\sum_{j \in J} c_j = 0$.

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Op en Problems/2

Are equations of the form
\[ a_i^m b_j^2 \]
PR if and only if the linear or the quadratic part satisfy a Rado condition?

Op en Problem 4.
Are there simple Rado-like necessary and sufficient conditions under which a given Diophantine equation with no constant term is PR on sufficiently large finite fields \( F_p \)?

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Is there a characterization of PR in infinite systems of Diophantine equations in terms of \( u \)-equivalence? (Or, equivalently, by means of ultralimits?)
Open Problem 3. Are equations of the form \( \sum_{i=1}^{n} c_i x_i = \sum_{j=1}^{m} d_j y_j^2 \)
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Thank You!

email: lorenzo.luperi.baglini@univie.ac.at