Partition regularity of nonlinear Diophantine equations

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Applications of Ultrafilters and Nonstandard Methods, IV

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Example

Trivially, for every $n \in \mathbb{N}$, the polynomial $x - n$ is PR.

Rado's theorem

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is PR on $\mathbb{N}$ if and only if the following condition is satisfied:

- *"There exists a nonempty set $J \subseteq \{1, \ldots, n\}$ such that $\sum_{j \in J} c_j = 0$."*

Nonlinear results/1

Theorem (Multiplicative Rado)

A nonlinear Diophantine equation $\prod_{i=1}^{n} x_i^{c_i} = 1$ *is PR on* $\mathbb{N}$ *if and only if the following condition is satisfied:*

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Let $P(z) \in \mathbb{Z}[z]$ be such that $P(0) = 0$. Then the equation $x - y = P(z)$ is PR.

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In 2010, by using algebraic properties of ultrafilters in $\beta\mathbb{N}$, Bergelson solved the problem in the positive.

Nonlinear results/3

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Theorem (Luperi Baglini)

Let $n, m > 0$. *For every choice of sets* $F_i \subseteq \{1, \ldots, m\}$, *the equation* $\sum_{i=1}^{n} c_i\, x_i (\prod_{j \in F_i} y_j) = 0$ *is partition regular whenever* $\sum_{i \in J} c_j = 0$ *for some nonempty* $J \subseteq \{1, \ldots, m\}$. *(It is agreed that* $\prod_{j \in \emptyset} y_j = 1$.*)*

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Theorem (Luperi Baglini)

Let $n, m > 0$. For every choice of sets $F_i \subseteq \{1, \ldots, m\}$, the equation $\sum_{i=1}^{n} c_i x_i (\prod_{j \in F_i} y_j) = 0$ is partition regular whenever $\sum_{i \in J} c_j = 0$ for some nonempty $J \subseteq \{1, \ldots, m\}$. (It is agreed that $\prod_{j \in \emptyset} y_j = 1$.)

Idea: use the existence of a multiplicatively idempotent ultrafilter $\mathcal{U}$ with good linear properties; study the ultrafilter using nonstandard analysis.

Nonlinear results/4

Theorem (Di Nasso, Riggio)

Let $k, n, m \in \mathbb{N}$ be such that $k \notin \{n, m\}$. Then the equation $x^m + y^n = z^k$ is not PR.

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Theorem (Moreira)

Let $\sum_{i=1}^{n} c_i = 0$. *Then* $\sum_{i=1}^{n} c_i x_i^2 = y$ *is PR.*

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Theorem (Moreira)

Let $\sum_{i=1}^{n} c_i = 0$. Then $\sum_{i=1}^{n} c_i x_i^2 = y$ is PR.

Idea: use ergodic methods involving the set of affinities $\{x \rightarrow ax + b\}$; alternatively, use an embeddability property of piecewise syndetic sets w.r.t. arithmetic progressions.

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Proposition

A Diophantine equation $P(x_1, \ldots, x_n) = 0$ *is PR if and only if there exists* $\mathcal{U} \in \beta\mathbb{N}$ *such that for every* $A \in \mathcal{U}$ *there exists* $a_1, \ldots, a_n \in A$ *with* $P(a_1, \ldots, a_n) = 0$.

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In this case, we say that $\mathcal{U}$ witnesses the PR of the equation (notation: $\mathcal{U} \models P(a_1, \ldots, a_n) = 0$).

Banach density and IP-sets

Definition

Let $A \subseteq \mathbb{N}$. The upper Banach density of A is
$BD(A) = \lim_{n\to+\infty} \sup_{m\in\mathbb{N}} \frac{|A\cap[m,m+n]|}{n+1}$.

We let $\Delta = \{\mathcal{U} \in \beta\mathbb{N} \mid BD(A) > 0\ \forall A \in \mathcal{U}\}$.

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Definition

Let $G = (g_i)_{i\in\mathbb{N}}$ be an increasing sequence of natural numbers. The IP-set generated by G is the set of finite sums

$$FS(G) = FS(g_i)_{i\in\mathbb{N}} = \left\{ \sum_{j=1}^{k} g_{i_j} \,\middle|\, i_1 < i_2 < \cdots < i_k \right\}.$$

A set $A \subseteq \mathbb{N}$ is called IP-large if it contains an IP-set. Multiplicative IP-sets and multiplicative IP-large sets are defined similarly.

Special ultrafilters

Various classes of ultrafilters are important in this field, the "best" being combinatorially rich ultrafilters:

Definition

$\mathcal{U}$ is combinatorially rich if $\mathcal{U} \in \mathbb{M} \cap \Delta \cap K(\odot)$ and $\mathcal{U} \odot \mathcal{U} = \mathcal{U}$, where $\mathbb{M} = \{\mathcal{V} \in \beta\mathbb{N} \mid \forall A \in \mathcal{V}\ A$ is central in $(\mathbb{N}, +)\}$.

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Notice that if $\mathcal{U}$ is combinatorially rich and $A \in \mathcal{U}$ then:

- A is central in $(\mathbb{N}, +)$, in particular it is IP;
- A is also multiplicatively IP;
- A contains solutions to all homogeneous PR equations (we will show this later), in particular to all linear equations;
- $BD(A) > 0$.

A surprisingly simple key Lemma

Lemma

Let $\mathcal{U}$ *be a common witness of the equations* $P_1(x_1, \ldots, x_n) = 0$ *and* $P_2(y_1, \ldots, y_m) = 0$.

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$$\begin{cases} P_1(x_1, \ldots, x_n) = 0; \\ P_2(y_1, \ldots, y_m) = 0; \\ x_1 = y_1. \end{cases}$$

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Proof.

Let $A\in\mathcal{U}$ be fixed. Let

$$\begin{aligned} \Lambda_1 &= \{\, a\in A \mid \exists\, a_2,\ldots,a_n\in A \text{ s.t. } P_1(a,a_2,\ldots,a_n)=0\,\}, \\ \Lambda_2 &= \{\, b\in A \mid \exists\, b_2,\ldots,b_m\in A \text{ s.t. } P_2(b,b_2,\ldots,b_m)=0\,\}. \end{aligned}$$

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Notice that $\Lambda_1, \Lambda_2 \in \mathcal{U}$, as otherwise $\neg(\mathcal{U} \models P_i = 0)$. Take $\Lambda_1 \cap \Lambda_2$. □

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It is readily seen that this is equivalent to the PR of the configuration $\{x, y, z, y + x^2, z + y^2\}$ (which had already been proven by ergodic methods).

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(Nonstandard) Let $\mathcal{U} \in \mathfrak{W}_P$. Let $\alpha_1, \ldots, \alpha_n \in \mu(\mathcal{U})$ be such that ${}^*P(\alpha_1, \ldots, \alpha_n) = 0$.

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(Nonstandard) Let $\mathcal{U} \in \mathfrak{W}_P$. Let $\alpha_1, \ldots, \alpha_n \in \mu(\mathcal{U})$ be such that ${}^*P(\alpha_1, \ldots, \alpha_n) = 0$. Let $\beta \in \mu(\mathcal{V})$. Then $\alpha_1 \cdot {}^*\beta, \ldots, \alpha_n \cdot {}^*\beta \in \mu(\mathcal{U} \odot \mathcal{V})$, and $P(\alpha_1 \cdot {}^*\beta, \ldots, \alpha_n \cdot {}^*\beta) = 0$. Hence $\mathcal{U} \odot \mathcal{V} \in \mathfrak{W}_P$. □

Corollary

Let $P(x_1, \ldots, x_n)$ be a homogeneous PR polynomial. Then $\mathcal{U} \models P(x_1, \ldots, x_n) = 0$ for every $\mathcal{U} \in \overline{K(\beta\mathbb{N}, \odot)}$.

The first generalization result

Theorem

Let $c\,(x_1 - x_2) = P(y_1, \ldots, y_k)$ *be a Diophantine equation where the polynomial* P *has no constant term and* $c \neq 0$. *If the set* $A \subseteq \mathbb{N}$ *is IP-large and has positive Banach density then there exist* $\xi_1, \xi_2 \in A$ *and mutually distinct* $\eta_1, \ldots, \eta_k \in A$ *such that* $c\,(\xi_1 - \xi_2) = P(\eta_1, \ldots, \eta_k)$. *Moreover, if* $k = 1$ *then one can take* $\xi_1 \neq \xi_2$.

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Definition

A polynomial with integer coefficients is called a Rado polynomial if it can be written in the form

$$c_1 x_1 + \cdots + c_n x_n + P(y_1, \dots, y_k)$$

where $n \geqslant 2$*,* P *has no constant term, and there exists a nonempty subset* $J \subseteq \{1, \dots, n\}$ *such that* $\sum_{j \in J} c_j = 0$*.*

Generalized Rado

Theorem

Let

$$R(x_1, \ldots, x_n, y_1, \ldots, y_k) = c_1x_1 + \ldots + c_nx_n + P(y_1, \ldots, y_k)$$

be a Rado polynomial. Then every ultrafilter $\mathcal{U} \in \overline{K(\odot)} \cap \overline{\mathbb{I}(\oplus)} \cap \Delta$ *is a PR-witness of* $R(x_1, \ldots, x_n, y_1, \ldots, y_k) = 0$.

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Proof.

Consider the following system:

$$\begin{cases} c_1 z + c_2 x_2 + \ldots + c_n x_n = 0; \\ c_1(w - x_1) = P(y_1, \ldots, y_k); \\ z = w. \end{cases}$$

□

Main positive result/1

Theorem

Let $\mathfrak{F}$ *be the family of polynomials whose PR on* $\mathbb{N}$ *is witnessed by at least an ultrafilter* $\mathcal{U} \in \mathbb{I}(\odot) \cap \overline{K(\odot)} \cap \overline{\mathbb{I}(\oplus)} \cap \Delta$. *Then* $\mathfrak{F}$ *includes:*

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(i) *If* $P(z, y_1, \ldots, y_k) \in \mathfrak{F}$ *and* $z - g(x_1, \ldots, x_n) \in \mathfrak{F}$, *then* $P(g(x_1, \ldots, x_n), y_1, \ldots, y_k) \in \mathfrak{F}$;

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(ii) *if* $P(x_1, \ldots, x_n) \in \mathfrak{F}$ *is homogeneous, then* $P\left(\frac{1}{x_1}, \ldots, \frac{1}{x_n}\right) \in \mathfrak{F}$.

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Let $n, m \in \mathbb{N}$ and assume that, for every $i \leqslant n$, $j \leqslant m$, the equations

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All these equations are PR and homogeneous and therefore, by the closure property (i), also

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Example

$P(x_1, x_2, x_3) = x_1 x_2 - 2x_3$ is PR but it does not belong to $\mathfrak{F}$.

u-equivalence and partition regularity

Definition

Two hypernatural numbers $\xi, \xi' \in {}^*\mathbb{N}$ *are* ***u-equivalent*** *if they cannot be distinguished by any hyper-extension, i.e. if for every* $A \subseteq \mathbb{N}$ *one has either* $\xi, \xi' \in {}^*A$ *or* $\xi, \xi' \notin {}^*A$.

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Proposition

A Diophantine equation $P(x_1, \ldots, x_n) = 0$ is PR if and only if there exist u-equivalent hypernatural numbers $\xi_1, \ldots, \xi_n$ with ${}^*P(\xi_1, \ldots, \xi_n) = 0$.

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- A polynomial $P(\mathbf{x}) = \sum_\alpha c_\alpha \mathbf{x}^\alpha$ is homogeneous if $\operatorname{supp}(P)$ is a homogeneous set of indexes.

Minimal and maximal indeces

Definition

Let $P(\mathbf{x}) = \sum_\alpha c_\alpha \mathbf{x}^\alpha \in \mathbb{Z}[x_1, \ldots, x_n]$. *We say that a multi-index* $\alpha \in supp(P)$ *is* ***minimal*** *if there are no* $\beta \in supp(P)$ *with* $\beta < \alpha$. *The notion of* ***maximal*** *multi-index is defined similarly.*

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For linear polynomials, every nonempty $J \subseteq \mathrm{Supp}(P) = \{\alpha(1), \dots, \alpha(n)\}$ is a Rado set of both minimal and maximal indexes.

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Example

In $c_{(2,1,1,0)} x_1^2 x_2 x_3 + c_{(1,2,7,0)} x_1 x_2^2 x_3^7 + c_{(2,2,2,1)} x_1^2 x_2^2 x_3^2 x_4$, the set $J = \{(2,1,1,0), (1,2,7,0)\}$ is a Rado set of minimal (but not maximal) indeces: just let $\Lambda = \{1, 2\} \subseteq \{1, 2, 3, 4\}$.

General necessary condition

Theorem

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Examples

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Let $P(x_1, x_2, x_3) = x_1^2 x_2 - 2x_3$. Pick any prime number p with $p \equiv 3$ or $p \equiv 5 \mod 8$, so that 2 is not a quadratic residue modulo p.

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Corollary

Let $P(\mathbf{x}) = \sum_\alpha c_\alpha \mathbf{x}^\alpha \in \mathbb{Z}[x_1, \ldots, x_n]$ be an homogeneous polynomial. If for every nonempty $J \subseteq supp(P)$ one has $\sum_{\alpha \in J} c_\alpha \neq 0$, then $P(\mathbf{x})$ is not PR.

Necessary condition for sums of polynomials in one variable/1

Theorem

For every $i = 1, \ldots, n$ let $P_i(x_i) = \sum_{s=1}^{d_i} c_{i,s} x_i^s$ be a polynomial of degree d_i in the variable x_i with no constant term.

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- *"There exists a nonempty set $J \subseteq \{1, \ldots, n\}$ such that $d_i = d_j$ for every $i, j \in J$, and $\sum_{j \in J} c_{j,d_j} = 0$."*

Necessary condition for sums of polynomials in one variable/2

Proof.

For every i, let $\Lambda(i) = \{s \mid c_{i,s} \neq 0\}$ be the support of $P_i(x_i)$, and for every s, let $\Gamma(s) = \{i \mid c_{i,s} \neq 0\}$. If we denote by

$$P(\mathbf{x}) \;=\; \sum_{i=1}^{n} P_i(x_i) \;=\; \sum_{i=1}^{n} \sum_{s \in \Lambda(i)} c_{i,s} x_i^s,$$

by the nonstandard characterization of non-trivial PR, we can pick infinite $\xi_1 \underset{u}{\sim} \ldots \underset{u}{\sim} \xi_n$ such that $P(\boldsymbol{\xi}) = 0$. Now fix any finite number $p \geqslant 2$, and write the numbers ξ_i in base p:

$$\xi_i \;=\; \sum_{t=0}^{\tau_i} a_{i,t}\, p^{\tau_i - t}$$

where $0 \leqslant a_{i,t} \leqslant p-1$ and $a_{i,0} \neq 0$.

Necessary condition for sums of polynomials in one variable/3

Proof.

Let $s_* \tau_* = \max\{s\,\tau_i \mid i \in \Gamma(s)\}$. It is not difficult to show that $d_i = s_*$ for every $i \in \Gamma(s_*)$, by the maximality of $s_* \tau_*$.

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Now let $I_* = \{i \in \Gamma(s_*) \mid \tau_i = \tau_*\}$, and decompose $P(\boldsymbol{\xi}) = \Theta + \Psi + \Phi$, where:

- $\Theta = \sum_{i \in I_*} c_{i,s_*}\, \xi_i^{s_*}$;
- $\Psi = \sum_{i \in \Gamma(s_*) \setminus I_*} c_{i,s_*}\, \xi_i^{s_*}$;
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Lemma

1. $\Theta = \left(\sum_{i \in I_*} c_{i,s_*}\right) \zeta + \Theta'$ *for suitable* $\zeta \geqslant p^{s_* \tau_*}$ *and* $|\Theta'| \ll p^{s_* \tau_*}$.

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With this decomposition, the lemma can be proven by showing that ζ_i^a and $\vartheta_{i,a}$ "does not mix in computations".

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Example

The polynomials $x^n + y^m = z^k$ are not PR for $k \notin \{n, m\}$.

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Open Problem 2. Are there simple decidable conditions under which a given (non-homogeneous) Diophantine equation with no constant term is PR on $\mathbb{N}$ if and only if it is PR on $\mathbb{Z}$ if and only if it is PR on $\mathbb{Q}$?

Open Problems/2

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Open Problem 5. Is there a characterization of PR infinite systems of Diophantine equations in terms of u-equivalence? (Or, equivalently, by means of ultrafilters?)

Thank You!

email: lorenzo.luperi.baglini@univie.ac.at