Asymptotic spherical means as Loeb integrals

Irfan Alam

Department of Mathematics
Louisiana State University
Baton Rouge, LA

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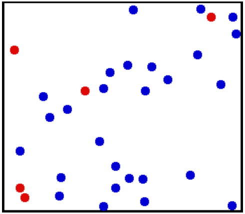
History



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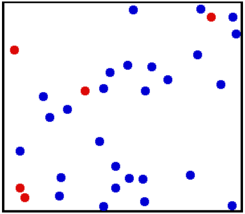

Kinetic Theory of Gases

$$\bar{E} \propto T$$

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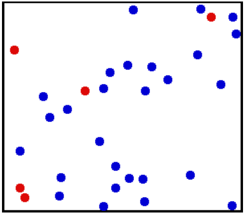

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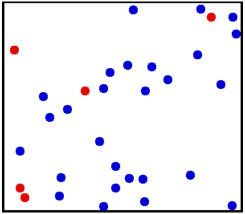

Kinetic Theory of Gases

$$\bar{E} \propto T \Rightarrow E = kTN.$$

$$\sum_{j=1}^{N} ||\vec{v_j}||^2 = \frac{2}{m}E = \frac{2}{m}kTN.$$

History

Maxwell(1860) and Boltzmann(1868)

J.C. Maxwell

L.E. Boltzmann

Along a particular direction, the probability that a given particle has velocity between x and "$x + dx$" is

$$\frac{1}{\alpha\sqrt{\pi}} e^{-\frac{x^2}{\alpha^2}} dx.$$

History

Poincaré’s *Calcul des probabilités* (1912)

$$\lim_{n\to\infty} \int_{S^{n-1}(\sqrt{n})} \mathbb{1}_{x_1\in(a,b)} d\bar{\sigma}_n(x_1,\ldots,x_n) = \int_a^b \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx.$$

J.H. Poincaré

History

A simple generalization

For any bounded measurable function $f : \mathbb{R}^k \to \mathbb{R}$, we have

$$\lim_{n\to\infty} \int_{S^{n-1}(\sqrt{n})} f(x_1, \ldots x_k) d\bar{\sigma}_n(x_1, \ldots, x_n) = \int_{\mathbb{R}^k} f d\mu.$$

J.H. Poincaré

History

Back to Maxwell-Boltzmann

The probability that a given particle has speed between v and "$v + dv$" is

$$\frac{4}{\alpha^3\sqrt{\pi}} v^2 e^{-\frac{v^2}{\alpha^2}} dv.$$

The mean velocity is

$$\frac{2\alpha}{\sqrt{\pi}}.$$

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The mean velocity is

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- This corresponds to integrating $\sqrt{{x_1}^2 + {x_2}^2 + {x_3}^2}$ in the sense of Poincaré.

What about a general μ-integrable $f : \mathbb{R}^k \to \mathbb{R}$?

$$\int_{S^{n-1}(\sqrt{n})} f d\bar{\sigma}_n = a_n b_n \int_{\mathbb{R}^k} \frac{1}{(\sqrt{2\pi})^k} \left(1 - \frac{||x||^2}{n}\right)^{\frac{n}{2}} \frac{\mathbb{1}_{B_k(\sqrt{n})}(x) f(x)}{\left(1 - \frac{||x||^2}{n}\right)^{\frac{k+2}{2}}} d\lambda(x),$$

$$\text{where } a_n = \frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n-k}{2}\right) \cdot \left(\frac{n-k}{2}\right)^{\frac{k}{2}}} \text{ and } b_n = \left(1 - \frac{k}{n}\right)^{\frac{k}{2}}.$$

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- Dominated convergence theorem does not work directly.
- Not even clear if f is integrable over $S^{n-1}(\sqrt{n})$.

What about a general μ-integrable $f : \mathbb{R}^k \to \mathbb{R}$?

Trying to work with truncations

- We know the result for $f_m := f\mathbb{1}_{|f|\leq m}$. Therefore,

$$\lim_{n\to\infty} \int_{S^{n-1}(\sqrt{n})} f_m d\bar{\sigma}_n = \int_{\mathbb{R}^k} f_m d\mu$$

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- We wanted $\lim_{n\to\infty} \lim_{m\to\infty} \int_{S^{n-1}(\sqrt{n})} f_m d\bar{\sigma}_n$ though.
- No good general theory of switching double limits, unfortunately.

Our results applicable to this setting

Theorem 1

If $(\Omega_n, \mathbb{P}_n)$ *and* $(\mathbb{R}^k, \mathbb{P})$ *are such that* $\lim_{n\to\infty} \mathbb{P}_n(B) = \mathbb{P}(B)$ *for all* $B \in \mathcal{B}(\mathbb{R}^k)$, *and* $f : \mathbb{R}^k \to \mathbb{R}$ *is measurable, then*

$$\lim_{m\to\infty} \lim_{n\to\infty} \int_{\Omega_n \cap \{|f|>m\}} |f| \, d\mathbb{P}_n = 0 \iff \lim_{n\to\infty} \int_{\Omega_n} f d\mathbb{P}_n = \int_{\mathbb{R}^k} f d\mathbb{P}.$$

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Theorem 2

If $f : \mathbb{R}^k \to \mathbb{R}$ *is* μ*-integrable, then*

$$\lim_{m\to\infty} \lim_{n\to\infty} \int_{S_n \cap \{|f|>m\}} |f| \, d\bar{\sigma}_n = 0.$$

Thus, $\lim_{n\to\infty} \int_{S_n} f d\bar{\sigma}_n = \int_{\mathbb{R}^k} f d\mu$ *for all* μ*-integrable* f.

More general spherical integrals

- Sengupta (and some of his PhD students) recently worked on a generalization where they fix an affine subspace of $\ell^2(\mathbb{R})$ and integrate over the corresponding sections of spheres.

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- More on this later.

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- If f is integrable on $(S_n, \bar{\sigma}_n)$ for large n with $u_n := \int_{S_n} f d\bar{\sigma}_n$, then transfer gives:

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- By transfer, $\bar{\sigma}_N : {}^*\mathcal{B}(S_N) \to {}^*[0,1]$ is a finitely additive function satisfying $\bar{\sigma}_N(S_N) = 1$, where ${}^*\mathcal{B}(S_N)$ is an algebra. The above is the value of the extension of the intregral operator at $((S_N, {}^*\mathcal{B}(S_N)), {}^*f)$.

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- **st**$(\bar{\sigma}_N)$ extends to a probability measure by Carathédory Extension Theorem. Its completion $(S_N, L({}^*\mathcal{B}(S_N)), L\bar{\sigma}_N)$ is called the Loeb space of $(S_N, {}^*\mathcal{B}(S_N), \bar{\sigma}_N)$.

S-integrability

- Suppose $(\Omega, \mathcal{F}, \mathbb{P})$ is an internal probability space and $F \in {}^*L^1(\Omega)$ is such that $L\mathbb{P}(F \in {}^*\mathbb{R}_{\text{fin}}) = 1$.

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- In our case, for $f : \mathbb{R}^k \to \mathbb{R}$, we have:

$$L\bar{\sigma}_N({}^*f \in {}^*\mathbb{R}_{\text{fin}}) = L\bar{\sigma}_N\left(\cup_{m\in\mathbb{N}}\{|{}^*f| < m\}\right)$$

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&= \lim_{m\to\infty} L\bar{\sigma}_N\left({}^*\{|f| < m\}\right) \\
&= \lim_{m\to\infty} \mu\left(|f| < m\right) = 1.
\end{aligned}
$$

Proof of the result for spherical integrals

Theorem 1

If $(\Omega_n, \mathbb{P}_n)$ and $(\mathbb{R}^k, \mathbb{P})$ are such that $\lim_{n\to\infty} \mathbb{P}_n(B) = \mathbb{P}(B)$ for all $B \in \mathcal{B}(\mathbb{R}^k)$, and $f : \mathbb{R}^k \to \mathbb{R}$ is measurable, then

$$\lim_{m\to\infty} \lim_{n\to\infty} \int_{\Omega_n \cap \{|f|>m\}} |f| \, d\mathbb{P}_n = 0 \iff \lim_{n\to\infty} \int_{\Omega_n} f d\mathbb{P}_n = \int_{\mathbb{R}^k} f d\mathbb{P}.$$

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Given $\epsilon \in \mathbb{R}_{>0}$, there exists $\ell_\epsilon \in \mathbb{N}$ such that the following holds:

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Given $\epsilon \in \mathbb{R}_{>0}$, there exists $\ell_\epsilon \in \mathbb{N}$ such that the following holds: for any $m \geq \ell_\epsilon$, there is an $n_{\epsilon,m} \in \mathbb{N}$ such that

$$\int_{\Omega_n \cap \{|f|\geq m\}} |f| \, d\nu_n < \epsilon \text{ for all } n \geq n_{\epsilon,m}.$$

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By transfer, for any $N > \mathbb{N}$,

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$$\mathbf{st}\left({}^*\int_{\Omega_N} {}^*f d\mathbb{P}_N\right) = \int_{\Omega_N} \mathbf{st}({}^*f) dL\mathbb{P}_N \overset{DCT}{=} \lim_{m\to\infty} \int_{\Omega_N} \mathbf{st}({}^*f_m) dL\mathbb{P}_n$$

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□

Proof of the result for spherical integrals(cont.)

Theorem 2

If $f : \mathbb{R}^k \to \mathbb{R}$ is μ-integrable, then

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Thus, $\lim_{n\to\infty} \int_{S_n} f d\bar{\sigma}_n = \int_{\mathbb{R}^k} f d\mu$ *for all μ-integrable f.*

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$$\textit{Thus, } \lim_{n\to\infty} \int_{S_n} f d\bar{\sigma}_n = \int_{\mathbb{R}^k} f d\mu \textit{ for all } \mu\textit{-integrable } f.$$

We use the following lemma:

Lemma 1

There is an $n' \in \mathbb{N}$ and a $C \in \mathbb{R}_{>0}$ such that

$$\int_{S_n} |g| \, d\bar{\sigma}_n \leq C \left(\mathbb{E}_\mu(|g|) + \sqrt{\mathbb{E}_\mu(g^2)} \right) \; \forall g \in L^2(\mathbb{R}^k, \mu), n \geq n'.$$

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With $g_j = \sqrt{|f| \cdot \mathbb{1}_{|f| \in (j, j+1]}}$, we have

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Proof of the result for spherical integrals(cont.)

Lemma 1

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&\leq 4C \cdot \mathbb{E}_\mu\left(|f| \cdot \mathbb{1}_{m < |f| \leq m'}\right).
\end{aligned}$$

Proof of the result for spherical integrals(cont.)

- For any $f \in L^1(\mathbb{R}^k, \mu)$, we get

$$\lim_{m\to\infty} \limsup_{n\to\infty} \int_{S_n \cap \{|f| \geq m\}} |f| \, d\bar{\sigma}_n \leq \lim_{m\to\infty} 4C \cdot \mathbb{E}_\mu(|f| \cdot \mathbb{1}_{|f|>m}) = 0.$$

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Idea of the proof of Lemma

$$\int\limits_{S^{n-1}(\sqrt{n})} g d\bar{\sigma}_n = a_n b_n \int\limits_{\mathbb{R}^k} \frac{1}{(\sqrt{2\pi})^k}\left(1 - \frac{||x||^2}{n}\right)^{\frac{n}{2}} \frac{\mathbb{1}_{B_k(\sqrt{n})}(x) g(x)}{\left(1 - \frac{||x||^2}{n}\right)^{\frac{k+2}{2}}} d\lambda(x),$$

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$$\lim_{n\to\infty} \int_{\{x\in\mathbb{R}^k : \frac{n}{2} < ||x||^2 < n\}} \left(1 - \frac{||x||^2}{n} \right)^{\frac{n}{4}} d\lambda(x) = 0.$$

Recent works on Gaussian Radon transforms

A series of three papers in Journal of Functional Analysis:

- Irina Holmes and Ambar Sengupta (2012)
  A Gaussian Radon transform for Banach spaces
- Ambar Sengupta (2016)
  The Gaussian Radon transform as a limit of spherical transforms
- Amy Peterson and Ambar Sengupta (2019)
  The Gaussian mean for high-dimensional spherical means

Recent works on Gaussian Radon transforms

- For a vector $x = (x_1, x_2, \ldots) \in \mathbb{R}^{\mathbb{N}}$ and $n \in \mathbb{N}$, we define

$$x_{(n)} := (x_1, \ldots, x_n).$$

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Spheres intersected by affine subspaces

Let $u^{(1)}, \ldots, u^{(\gamma)}$ be mutually orthonormal vectors in $\ell^2(\mathbb{R})$. For $p_1, p_2, \ldots, p_\gamma \in \mathbb{R}$ and $n \in \mathbb{N}$, define:

$$S_{A_n} := S^{n-1}(\sqrt{n}) \cap \{x \in \mathbb{R}^n : \langle x, (u^{(i)})_{(n)} \rangle = p_i \text{ for } i = 1, \ldots, \gamma\}.$$

Recent works on Gaussian Radon transforms

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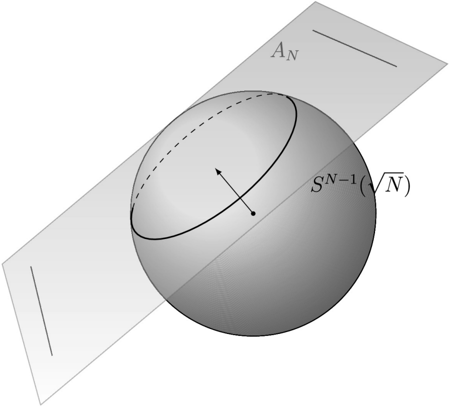

Recent works on Gaussian Radon transforms

The motivation behind this theory

- Originally (1917), for a two-dimensional function $f : \mathbb{R}^2 \to \mathbb{R}$, Radon defined a map $Rf : \{\text{straight lines on the plane}\} \to \mathbb{R}$ by $Rf(L) = \int_L f$.

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- There is an inverse transform, hence the Radon transform of a function completely characterizes that function.
- Higher dimensional versions were developed: the Radon transform is now an important tool in Integral Geometry.
- No straightforward generalization of the transform to functions on infinite dimensions.

Recent works on Gaussian Radon transforms

The motivation behind this theory (cont.)

- Sengupta defined the *Gaussian Radon transform* of an $f : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}$ to be the map that takes a hyperplane L in $\ell^2(\mathbb{R})$ to a Gaussian integral of f on L.

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- Since integrals on the spheres $S^{n-1}(\sqrt{n})$ are known to approximate Gaussian integrals, one expects the integrals on $S^{n-1}(\sqrt{n}) \cap L_n$ (where L_n is the "n^{th} approximation" to L) to converge to the Gaussian Radon transform (at L).

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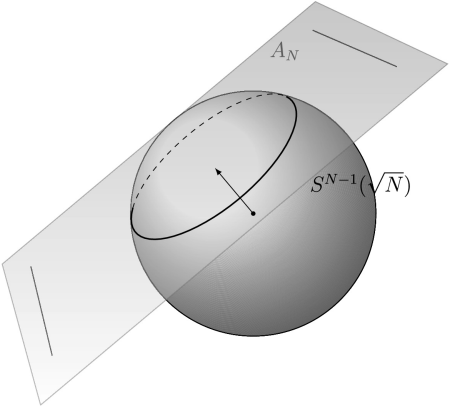
- Taking an affine subspace A of $\ell^2(\mathbb{R})$ instead of a hyperplane L is the next step of the generalization.

Recent works on Gaussian Radon transforms

$$S_{A_n} := S^{n-1}(\sqrt{n}) \cap \{x \in \mathbb{R}^n : \langle x, (u^{(i)})_{(n)} \rangle = p_i \text{ for } i = 1, \ldots, \gamma\}.$$



Recent works on Gaussian Radon transforms

Theorem 3

Let $f : \mathbb{R}^k \to \mathbb{R}$ be bounded and Borel measurable. Then

$$\lim_{n\to\infty} \int_{S_{A_n}} f d\bar{\sigma} = \int_{\mathbb{R}^k} f d\mu_{\bar{\eta}, u^{(1)}, \ldots, u^{(\gamma)}}.$$

Here $\mu_{\bar{\eta}, u^{(1)}, \ldots, u^{(\gamma)}}$ is the Gaussian measure on $\mathbb{R}^k$ with

- ***Mean***

$$\bar{\eta} = p_1 (u^{(1)})_{(k)} + \ldots p_\gamma (u^{(\gamma)})_{(k)}.$$

- ***Covariance***

$$I - \left|\left|(u^{(1)})_{(k)}\right|\right|^2 P_{(u^{(1)})_{(k)}} - \ldots - \left|\left|(u^{(\gamma)})_{(k)}\right|\right|^2 P_{(u^{(\gamma)})_{(k)}}.$$

Some notation

- With the fixed $u^{(1)}, \ldots, u^{(\gamma)}$, we define

$$S_{H_N} := S^{N-1}(\sqrt{N}) \cap (u^{(1)})^{\perp}_{(N)} \cap \ldots \cap (u^{(\gamma)})^{\perp}_{(N)}.$$

- We write $S_{n,u^{(1)},\ldots,u^{(\gamma)}}$ for the same set when the vectors are not clear from context.

Calculation in an easy case

Lemma 4

If $u^{(1)}, \ldots, u^{(\gamma)}$ are orthonormal in $\mathbb{R}^k$ then:

$$\lim_{n\to\infty} \int_{S_{n,u^{(1)},\ldots,u^{(\gamma)}}} f d\bar{\sigma}_n = \int_{\mathbb{R}^k} f d\mu_{0;u^{(1)},\ldots u^{(\gamma)}}$$

for all bounded Borel $f : \mathbb{R}^k \to \mathbb{R}$.

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Corollary 5

If $u^{(1)}, \ldots, u^{(\gamma)}$ are orthonormal in $\mathbb{R}^m$ for some $m \in \mathbb{N}$, then

$$\lim_{n\to\infty} \int_{S_{n,u^{(1)},\ldots,u^{(\gamma)}}} f d\bar{\sigma}(x_1, \ldots, x_n) = \int_{\mathbb{R}^k} f d\mu_{0;(u^{(1)}),\ldots,(u^{(\gamma)})}$$

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A hyperfinite approximation

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$$\theta_f(v^{(1)}, \ldots, v^{(\gamma)}) := \int_{\mathbb{R}^k} f d\mu_{0; v^{(1)}, \ldots, v^{(\gamma)}},$$

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$$\mathbb{N} \subseteq \left\{ m \in {}^*\mathbb{N} : \ m \leq N, \text{ and } \forall (v^{(1)}, \ldots, v^{(\gamma)}) \in ({}^*\mathbb{R}^m)^{[\gamma]} \right.$$
$$\left. \left(\left| {}^*a_f(v^{(1)}, \ldots, v^{(\gamma)}, N) - {}^*\theta_f(v^{(1)}, \ldots, v^{(\gamma)}) \right| < \frac{1}{m} \right) \right\}.$$

A hyperfinite approximation (cont.)

- By overflow, there is an $M > \mathbb{N}$ such that

$$^*a_f(v^{(1)}, \ldots, v^{(\gamma)}, N) \approx {}^*\theta_f(v^{(1)}, \ldots, v^{(\gamma)}),$$

for all $(v^{(1)}, \ldots, v^{(\gamma)}) \in ({}^*\mathbb{R}^M)^{[\gamma]}$.

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- Let $w^{(1)}, \ldots, w^{(\gamma)}$ be the orthonormalization of $(u^{(1)})_{(M)}, \ldots, (u^{(\gamma)})_{(M)}$. Also, let $z^{(1)}, \ldots, z^{(\gamma)}$ be the orthonormalization of $(u^{(1)})_{(N)}, \ldots, (u^{(\gamma)})_{(N)}$. Then

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A hyperfinite approximation (cont.)

Proof of $\diamond$ requires the following result:

Proposition 6

Let $f : \mathbb{R}^k \to \mathbb{R}$ be bounded and uniformnly continuous. Fix $N > \mathbb{N}$. For all $i \in \{1, \ldots, \gamma\}$ let $v^{(i)}, v'^{(i)} \in ({}^*\mathbb{R}_{fin})^N$ be such that:

(i) $||v^{(i)}||, ||v'^{(i)}|| \in {}^*\mathbb{R}_{fin}$.

(ii) $\mathbf{st}(||v^{(i)}||)\, \mathbf{st}(||v'^{(i)}||) > 0$.

(iii) $||v^{(i)}|| - ||v'^{(i)}|| \approx 0$.

Further, assume that $\{v^{(1)}, \ldots, v^{(\gamma)}\}$ and $\{v'^{(1)}, \ldots, v'^{(\gamma)}\}$ are both ${}^*\mathbb{R}$-independent sets. Then

$$\int_{S_{N,v^{(1)},\ldots,v^{(\gamma)}}} \mathbf{st}({}^*f(x))dL\bar{\sigma}(x) = \int_{S_{N,v'^{(1)},\ldots,v'^{(\gamma)}}} \mathbf{st}({}^*f(x))dL\bar{\sigma}(x).$$

Idea of the proof of Proposition 6

- Orthonormalize the two sets of vectors to get (say) $\{w^{(1)}, \ldots, w^{(\gamma)}\}$ and $\{z^{(1)}, \ldots, z^{(\gamma)}\}$ respectively. Define H and H' as the spaces orthogonal to the respective sets.

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- $\dim(H \cap H') \geq N - 2\gamma$. Find $(N - 2\gamma)$ orthonormal vectors in it; call them $w^{(i)} = z^{(i)}$ for $i \in \{\gamma + 1, \gamma + 2, \ldots, N - \gamma\}$.

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- Extend $\{w^{(1)}, \ldots, w^{N-\gamma}\}$ to a full ONB $\{w^{(1)}, \ldots, w^{(N)}\}$. It is actually possible to extend $\{z^{(1)}, \ldots, z^{(N-\gamma)}\}$ as well in a way that $\left|\left|w^{(i)} - z^{(i)}\right|\right| \approx 0$ for all $i \in \{1, \ldots, N\}$.

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- Define $R : {}^*\mathbb{R}^N \to {}^*\mathbb{R}^N$ by $R(w^{(i)}) = z^{(i)}$ for all i.

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- $\dim(H \cap H') \geq N - 2\gamma$. Find $(N - 2\gamma)$ orthonormal vectors in it; call them $w^{(i)} = z^{(i)}$ for $i \in \{\gamma + 1, \gamma + 2, \dots, N - \gamma\}$.
- Extend $\{w^{(1)}, \dots, w^{N-\gamma}\}$ to a full ONB $\{w^{(1)}, \dots, w^{(N)}\}$. It is actually possible to extend $\{z^{(1)}, \dots, z^{(N-\gamma)}\}$ as well in a way that $\left|\left|w^{(i)} - z^{(i)}\right|\right| \approx 0$ for all $i \in \{1, \dots, N\}$.
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An interesting result implicitly needed in the argument

Lemma 7

Let $u^{(1)}, \ldots, u^{(\gamma)}$ be $\mathbb{R}$-linearly independent in $\mathbb{R}^{\mathbb{N}}$. Then $(u^{(1)})_{(m)}, \ldots, (u^{(\gamma)})_{(m)}$ are linearly independent for all large m.

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- Let $b_i = \frac{a_i}{\max_{j \in \{1,\dots,\gamma\}} |a_j|}$. Then $\sum_{i=1}^{\gamma} \mathbf{st}(b_i) u^{(i)} = \mathbf{0} \in \mathbb{R}^{\mathbb{N}}$.

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- This contradicts the $\mathbb{R}$-independence of $u^{(1)}, \ldots, u^{(\gamma)}$.

□

So far...

- We want a result on the limiting behavior of integrals over domains that depend on two things:
  1. A finite number of vectors $u^{(1)}, \ldots, u^{(\gamma)}$ in $\ell^2(\mathbb{R})$.
  2. A function $f : \mathbb{R}^k \to \mathbb{R}$.
- And, we want the result for as many functions as possible.
- The result is easy for bounded functions if the vectors are eventually zero (as sequences).
- The n^{th} domain is the intersection of $S^{n-1}(\sqrt{n})$ with an affine subspace determined by the n^{th} truncation of $u^{(i)}$'s.
- For $N > \mathbb{N}$, Overflow approximates when the N^{th} sphere is intersected by the M^{th} affine space for some hyperfinite M.
- But this domain is separated from the N^{th} domain by an infinitesimal rotation! So we get the result for all bounded uniformly continuous f!

Going from hyperplanes to affine subspaces

Recall $S_{H_N} := S^{N-1}(\sqrt{N}) \cap (u^{(1)})^{\perp}_{(N)} \cap \ldots \cap (u^{(\gamma)})^{\perp}_{(N)}$, and $S_{A_N} := \{x \in S^{N-1}(\sqrt{N}) : \langle x, (u^{(i)})_{(N)} \rangle = p_i \text{ for all } i\}$.

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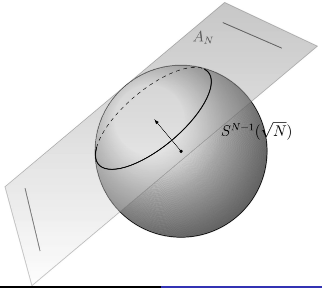

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- S is a translate of S_{H_N}, while S_{A_N} is "infinitesimally close" to S. So the result for bounded uniformly continuous functions on S_{H_N} generalizes to S_{A_N}.

Beyond bounded uniformly continuous functions

Theorem 8

Let $\{\mathbb{P}_n\}_{n\in\mathbb{N}}$ be Borel measures on $\mathbb{R}^k$ that are absolutely continuous with respect to Lebesgue measure. If their densities are uniformly bounded by some $B \in \mathbb{R}$ and $\mathbb{P}$ is Radon, then

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- Thus $\lim_{m\to\infty} L\mathbb{P}_N({}^*C_m) = L\mathbb{P}_N({}^*U)$. Then $\mathbb{P}(U) \geq \lim_{m\to\infty} \mathbb{P}(C_m) \geq L\mathbb{P}_N({}^*U)$.

□

Extending beyond bounded functions...

Theorem 1

If $(\Omega_n, \mathbb{P}_n)$ and $(\mathbb{R}^k, \mathbb{P})$ are such that $\lim_{n\to\infty} \mathbb{P}_n(B) = \mathbb{P}(B)$ for all $B \in \mathcal{B}(\mathbb{R}^k)$, and $f : \mathbb{R}^k \to \mathbb{R}$ is measurable, then

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Ongoing work on characterizing the functions satisfying the double limit condition:

- Trying to appropriately bound the n^{th} expectation of a Gaussian square integrable function, as in the lemma for full spheres.

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- Trying to appropriately bound the n^{th} expectation of a Gaussian square integrable function, as in the lemma for full spheres.
- Have some results for continuous functions, and decent progress to generalize to Gaussian integrable functions.

Possibilities of future work

- Methods potentially applicable to a lot of situations when the measure spaces are evolving over time.

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- Work on "truly infinite-dimensional Gaussian Radon transform." Also, spheres intersected with truncations of an infinite-codimensional affine space of $\ell^2(\mathbb{R})$, where nonstandard methods seem very appropriate.
- Physical interpretations of the results on sections of spheres (for instance, interpretations in the Kinetic Theory of Gases).

Possibilities of future work(cont.)

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Theorem 9

Let $\mathbb{P}$ be a Radon probability on E^k. Let (Ω_n, ν_n) be a sequence of probability spaces such that Ω_n eventually lives in spaces containing E^k. Let $f : E^k \to \mathbb{R}$ be $\mathbb{P}$-integrable. Given any $\epsilon, \delta, \theta \in \mathbb{R}_{>0}$ there exist an $n_0 \in \mathbb{N}$ and functions $g_n : E^k \to \mathbb{R}$ for all $n \in \mathbb{N}_{\geq n_0}$ such that the following hold:

(i) $|g_n|$ *is bounded by* n *for all* $n \in \mathbb{N}_{\geq n_0}$.

(ii) $\nu_n(|g_n - f| > \delta) < \epsilon$ *for all* $n \in \mathbb{N}_{\geq n_0}$.

(iii) $\left| \int_{\Omega_n} g_n d\nu_n - \int_{E^k} f d\mathbb{P} \right| < \theta$ *for all* $n \in \mathbb{N}_{\geq n_0}$.

Thank You!

- Thank you!
- An early preprint of the first paper "Limiting Probability Measures" available on arXiv.
- A preprint of the second paper (currently tentatively titled “The Gaussian Radon Transform as an Integral over an Infinite Sphere”) should be up in the coming weeks.
- Questions or comments?