Spectral gap and definability

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Defining von Neumann algebras

- $\mathcal{H}$ a complex Hilbert space, $B(H)$ the set of bounded operators on H.
- The **weak operator topology** on $B(H)$ is induced by the family of semi-norms given by, for every $\zeta, \eta \in H$,

$$a \mapsto |\langle a\zeta, \eta\rangle|.$$

- $M \subseteq B(H)$ is a **von Neumann algebra** if it is a unital *-algebra closed in the weak operator topology.
- Equivalently, any unital *-algebra $M \subseteq B(H)$ which satisfies $M'' = M$ is a von Neumann algebra, where

$$M' = \{a \in B(H) : ab = ba \text{ for all } b \in M\}.$$

Traces and tracial von Neumann algebras

Definition

A linear functional τ on a von Neumann-algebra M is a (finite, normalized) **trace** if

- it is positive ($\tau(a^*a) \geq 0$ for all $a \in M$),
- $\tau(a^*a) = \tau(aa^*)$ for all $a \in M$, and
- $\tau(1) = 1$.

We say it is **faithful** if $\tau(a^*a) = 0$ implies $a = 0$.

A **tracial von Neumann algebra** is a pair (M,τ) consisting of a von Neumann algebra and a faithful trace τ on M. τ induces a norm on M

$$\|a\|_2 = \sqrt{\tau(a^*a)}.$$

Examples

- $M_n(\mathbb{C})$ with the normalized trace is a tracial vNa; $B(H)$ for infinite-dimensional H is not.
- Inductive limits of tracial von Neumann algebras are tracial von Neumann algebras. In particular, $\mathcal{R}$, **the** inductive limit of the $M_n(\mathbb{C})$'s, is a tracial von Neumann algebra called the hyperfinite II_1 factor.
- $L(G)$ - Suppose G is a group and $\mathcal{H}$ has an orthonormal generating set ζ_h for $h \in G$. Let u_g for $g \in G$ be the operator determined by
$$u_g(\zeta_h) = \zeta_{gh}.$$
$L(G)$ is the von Neumann algebra generated by the u_g's. It is tracial: for $a \in L(G)$, let $\tau(a) = \langle a(\zeta_e), \zeta_e \rangle$.

Tracial ultraproducts

Suppose M_i are von Neumann algebras with faithful traces τ_i for all $i \in I$ and $\mathcal{U}$ is an ultrafilter on I. The bounded product is

$$\prod^b M_i := \{\bar{a} \in \prod M_i : \lim_{i \to \mathcal{U}} \|a_i\| < \infty\}$$

and we have a two-sided ideal

$$c_{\mathcal{U}} = \{\bar{a} \in \prod^b M_i : \lim_{i \to \mathcal{U}} \tau_i(a_i^* a_i) = 0\}.$$

The ultraproduct, $\prod_{\mathcal{U}} M_i$, is defined as $\prod^b M_i / c_{\mathcal{U}}$. It is a tracial von Neumann algebra with the trace given by $\tau(\bar{x}) = \lim_{i \to \mathcal{U}} \tau_i(x_i)$.

The class of tracial von Neumann algebras forms an elementary class where the model-theoretic ultraproduct construction coincides with the tracial ultraproduct construction.

II_1 factors

- A von Neumann algebra whose center is $\mathbb{C}$ is called a **factor**.
- A tracial factor is type I if all its projections have rational trace and is type II_1 if the range of the trace on projections is [0,1].
- $\mathcal{R}$, $\mathcal{R}^{\mathcal{U}}$, $\prod_U M_n(\mathbb{C})$ and $L(\Gamma)$ (for Γ a discrete ICC group) are all II_1 factors.
- The class of II_1 factors is an elementary class.

Continuum many theories of II_1 factors

Theorem (BCI)

If $(M_\alpha)_{\alpha \in 2^\omega}$ is McDuff's family of pairwise nonisomorphic separable II_1 factors, then for any ultrafilters $\mathcal{U}, \mathcal{V}$ on any index sets and distinct α, β, we have that $M_\alpha^\mathcal{U} \not\cong M_\beta^\mathcal{V}$.

Corollary

For distinct α, β, we have that $M_\alpha \not\equiv M_\beta$.

Theorem (G.-Hart-Towsner)

There are concrete sentences distinguishing the McDuff factors. Moreover, if $d(\alpha, \beta) = 2^{-k}$, then the sentence distinguishing M_α and M_β has "complexity" $5k + 3$.

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A motivating conversation

During the 2017 NCGOA, Chifan and Hart had a conversation about the above results. Part of that conversation was the following quote of Chifan:

"It's all spectral gap."

Hart relayed this quote to me. My initial response:

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In some sense, the point of this talk is me understanding this comment model-theoretically.

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Definability in a structure

Definition

Suppose that **M** is a structure, $A \subseteq \mathbf{M}$, and (φ_n) a sequence of formulae with parameters from A. If $\varphi_n^{\mathbf{M}}$ converges uniformly, we call the sequence a **formula in M over** A.

Definition

Suppose that φ is a formula in **M** over A. We will say that $Z(\varphi^{\mathbf{M}})$ is φ**-definable** if for every $\epsilon > 0$, there is $\delta > 0$ such that, for all $a \in \mathbf{M}^{\vec{x}}$, if $\varphi(\vec{a})^{\mathbf{M}} < \delta$, then $d(\vec{a}, Z(\varphi^{\mathbf{M}})) \leq \epsilon$. $Z(\varphi)$ is **definable in M over** A if it is ψ-definable for some ψ.

Theorem

Suppose that φ is a formula in **M** *over A. Then $Z(\varphi)$ is φ-definable if and only if $Z(\varphi^{\mathbf{M}})^{\mathcal{U}} = Z(\varphi^{\mathbf{M}^{\mathcal{U}}})$ for every ultrafilter $\mathcal{U}$.*

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An application: definability of relative commutants

Proposition

Suppose that M is a tracial von Neumann algebra and N is a definable subalgebra. Then $N' \cap M$ is also definable.

Proof.

This follows from the fact that, for $x \in M$, we have

$$\|x - \mathbb{E}_{N' \cap M}(x)\|_2 \leq \sup_{y \in N_1} \|[x, y]\|_2.$$

The right hand side is a formula if N is definable. □

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Introducing spectral gap

Proposition

Let $\pi : G \to \mathrm{U}(\mathcal{H}_\pi)$ be a unitary representation. The following are equivalent:

1. There exists finite $F \subseteq G$ and $c > 0$ such that, for all $\zeta \in \mathcal{H}_\pi$, we have
$$\max_{g \in F} \|\pi(g)\zeta - \zeta\| \geq c\|\zeta\|.$$
2. For any nonprincipal ultrafilter $\mathcal{U}$, $\pi^{\mathcal{U}}$ is ergodic.
3. For all $\epsilon > 0$, there is a finite $F \subseteq G$ and $\delta > 0$ such that, for all $\zeta \in H_\pi$, we have
$$\max_{g \in F} \|\pi(g)\zeta - \zeta\| \leq \delta \Rightarrow \|\zeta\| \leq \epsilon.$$

Introducing spectral gap (cont'd)

Definition

A unitary representation π has **spectral gap** if $\pi|\text{Erg}(\pi)$ satisfies the equivalent properties above.

Fact (Hulanicki-Reiter)

λ_Γ has spectral gap if and only if Γ is non-amenable.

Lemma

π has spectral gap if and only if, for any nonprincipal ultrafilter $\mathcal{U}$, we have $\text{Fix}(\pi^{\mathcal{U}}) = \text{Fix}(\pi)^{\mathcal{U}}$.

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Spectral gap and definability

- Let T_Γ denote the theory of unitary representations of Γ.
- Let φ_Γ be the T_Γ-formula $\sum_m 2^{-m}\|\gamma_m \cdot x - x\|$.
- For any unitary representation π of Γ, we have $Z(\varphi_\Gamma^{\mathcal{H}_\pi}) = \mathrm{Fix}(\pi)$.

Theorem

For a given representation π of Γ, π has spectral gap if and only if $\mathrm{Fix}(\pi)$ is a φ_Γ-definable subset of $\mathcal{H}_\pi$.

Remark

One cannot replace "φ_Γ-definable" in the previous theorem with "definable." For example, suppose that Γ is an infinite amenable group. Then $\mathrm{Fix}(\lambda_\Gamma) = \{0\}$ (which is clearly a definable subset of $\ell^2\Gamma$) but λ_Γ does not have spectral gap.

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Property (T)

Definition

We say that Γ has **property (T)** if every unitary representation of Γ has spectral gap.

Lemma

*Γ has property (T) if and only if there is a finite $F \subseteq \Gamma$ and $\delta > 0$ such that: for every unitary representation π, if π has a (F, δ)-almost invariant vector, then $\text{Fix}(\pi) \neq \{0\}$. Such a pair (F, δ) is called a **Kazhdan pair** for Γ.*

Proposition

Suppose that (F, δ) is a Kazhdan pair for Γ. Then for any unitary representation π of Γ and any $\epsilon > 0$, if $\xi \in \mathcal{H}_\pi$ is $(F, \delta\epsilon)$-invariant, then there is $\eta \in \text{Fix}(\pi)$ such that $\|\xi - \eta\| < \epsilon\|\xi\|$.

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Property (T) and definability

Theorem

The following are equivalent:

1. Γ *has property (T).*
2. *The* T*-functor* Fix *is a* T_Γ*-definable set.*

In this case, a simple T_Γ*-formula witnesses the definability.*

Theorem

Let $T_{\Gamma \curvearrowright}$ *denote the theory of pmp actions of* Γ*. Then the following are equivalent:*

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The hard part uses the Connes-Shmidt-Weiss characterization of (T).

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Introducing spectral gap for subfactors

Throughout, M is a separable II_1 factor and N is a von Neumann subalgebra.

Definition

We say that N **has spectral gap in** M if the unitary representation $U(N) \to L^2(M)$ given by $u \mapsto uxu^*$ has spectral gap.

Example

If Γ has property (T), then $L(\Gamma)$ has spectral gap in any II_1 factor extension. This holds more generally for property (T) II_1 tracial vNas.

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w-spectral gap

Observation

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$$\|[x, u_i]\|_2 \leq \delta\|x\|_2 \Rightarrow \|x - \mathbb{E}_{N' \cap M}(x)\|_2 \leq \epsilon\|x\|_2.$$

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N has **weak spectral gap in** (or **w-spectral gap** in) M if for all $\epsilon > 0$, there are $u_1, \ldots, u_n \in U(M)$ and $\delta > 0$ such that, for all $x \in M_1$,

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Comparing the notions

Lemma

1. *N has spectral gap in M if and only if $N' \cap L^2(M)^{\mathcal{U}} = L^2(N' \cap M)^{\mathcal{U}}$.*
2. *N has w-spectral gap in M if and only if $N' \cap M^{\mathcal{U}} = (N' \cap M)^{\mathcal{U}}$.*

Fact (Connes)

Suppose that N is a II_1 factor. Then the following are equivalent:

1. N has spectral gap in N;
2. N has w-spectral gap in N (i.e. $N' \cap N^{\mathcal{U}} = \mathbb{C} \cdot 1$);
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Moreover, if these equivalent conditions hold, then N has spectral gap in $N \bar{\otimes} S$ for any tracial von Neumann algebra S.

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- Let $\{u_m\}$ be an enumeration of a countable dense subset U(N).
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- Note that $Z(\varphi_N) = N' \cap M$.

Theorem

N has w-spectral gap in M if and only if $N' \cap M$ is a φ_N-definable subset of M. In this case, $(N' \cap M)' \cap M$ is also definable.

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Once again we cannot replace "φ_N-definable" with "definable" in the previous theorem. For instance, if $N = M$, then $M' \cap M = \mathbb{C}$, which is a definable subset of M, but M has w-spectral in itself if and only if M does not have property Gamma

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Spectral gap subfactors of ec factors

Proposition

Suppose that M is an existentially closed II_1 factor and N is a subalgebra of M with w-spectral gap. Then N satisfies the bicommutant condition $(N' \cap M)' \cap M = N$.

Proof.

Suppose, towards a contradiction, that $b \in (N' \cap M)' \cap M$ but $b \notin N$. Let $Q := M *_N (N \overline{\otimes} L(\mathbb{Z}))$. Since $M \subseteq Q$ and M is e.c., there is $i : Q \to M^{\mathcal{U}}$ such that i is the diagonal embedding on M. Let $c \in Q$ be the canonical unitary in $L(\mathbb{Z})$. Then $i(c) \in N' \cap M^{\mathcal{U}} = (N' \cap M)^{\mathcal{U}}$, so we can write $i(c) = (c_n)^\bullet$ with each $c_n \in N' \cap M$. By choice of b, we have $[b, c_n] = 0$ for all n, whence $[i(b), i(c)] = 0$ and hence $[b, c] = 0$, contradicting the fact that $b \notin N$. □

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II_1 factors do not have a model companion

Corollary (G.-Hart-Sinclair)

There is an e.c. II_1 factor M and an elementary extension $\tilde{M}$ of M such that $\tilde{M}$ is not e.c.

Proof (G.).

Let N be a property (T) factor and let M be an e.c. factor containing N. We show that $M^{\mathcal{U}}$ is not e.c. This follows from the previous slide and the computation:

$$N^{\mathcal{U}} \subseteq ((N' \cap M)^{\mathcal{U}})' \cap M^{\mathcal{U}} = (N' \cap M^{\mathcal{U}})' \cap M^{\mathcal{U}},$$

whence it follows that $N \neq (N' \cap M^{\mathcal{U}})' \cap M^{\mathcal{U}}$ and thus $M^{\mathcal{U}}$ is not e.c. □

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$$N^{\mathcal{U}} \subseteq ((N' \cap M)^{\mathcal{U}})' \cap M^{\mathcal{U}} = (N' \cap M^{\mathcal{U}})' \cap M^{\mathcal{U}},$$

whence it follows that $N \neq (N' \cap M^{\mathcal{U}})' \cap M^{\mathcal{U}}$ and thus $M^{\mathcal{U}}$ is not e.c. □

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Question

Are any two e.c. II_1 factors elementarily equivalent?

Assuming CEP, $\mathcal{R}$ is e.c. It seems that if M is an e.c. II_1 factor containing a property (T) factor, then since that subfactor is definable, we should not have $\mathcal{R} \equiv M$. Similar reasoning applies as well to:

Question

If N is non-Gamma, then is it possible that $\mathcal{R} \equiv N\bar{\otimes}\mathcal{R}$?

II_1 factors of the form $N\bar{\otimes}\mathcal{R}$ for N non-Gamma are called **strongly McDuff.**

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Questions (cont'd)

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Can a strongly McDuff II_1 factor ever be e.c.?

Call a non-Gamma factor N **bc-good** if it has a proper subalgebra $\tilde{N}$ with w-spectral gap such that $(\tilde{N}' \cap N)' \cap N \neq \tilde{N}$.

Corollary

If N is bc-good, then $N \bar{\otimes} \mathcal{R}$ *is not e.c.*

If N is not bc-good, then every w-spectral gap subfactor is definable.

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Are all non-Gamma factors bc-good?

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Back to the beginning

- There are self-functors T_0 and T_1 on the category of countable groups.
- Iterating gives us functors T_α for any $\alpha \in 2^{\leq \omega}$.
- We set $M_\alpha(\Gamma) := L(T_\alpha(\Gamma))$.

Theorem (G.-Hart-Towsner)

For each nonamenable ICC group Γ, there is an integer $m(\Gamma)$ and a sequence $(c_n(\Gamma))$ of positive real numbers such that, for any $n, t \in \mathbb{N}$ with $t \geq 1$ and any $\alpha \in 2^n$, we have:

$$\begin{aligned} \theta_{m,n}^{M_\alpha(\Gamma)^{\overline{\otimes} t}} &= 0 \text{ for all } m \geq 1 && \text{if } \alpha(n-1) = 1; \\ \theta_{m(\Gamma),n}^{M_\alpha(\Gamma)^{\overline{\otimes} t}} &\geq c_n(\Gamma) && \text{if } \alpha(n-1) = 0. \end{aligned}$$

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The role of spectral gap

- The base case uses the spectral gap characterization of nonamenability, which gives $m(\Gamma)$ and $c_0(\Gamma)$.
- A **generalized McDuff ultraproduct for Γ and α** is one of the form $\prod_{\mathcal{U}} M_\alpha^{\overline{\otimes} t_s}$.
- One defines **pairs of good unitaries** (u, v) to be pairs of unitaries that generate a w-spectral gap subalgebra in a precise numerical way. Such pairs (in ultraproducts) are the zeroset of a formula.
- Given two such pairs (u_1, v_1) and (u_2, v_2) in $M_\alpha(\Gamma)^{\mathcal{U}}$ with $C(u_2, v_2) \subseteq C(u_1, v_1)$, one has that $C(u_2, v_2)' \cap C(u_1, v_1)$ is a generalized McDuff ultraproduct with respect to Γ and a string that is one digit shorter, hinting at an inductive procedure.
- One then needs to be able to *relativize* previously constructed sentences; this heavily uses the *uniform* definability of such relative commutants.

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