Last time:

CEP: Every tracial vNa embeds into an ultrapower $\mathcal{R}^{\mathcal{U}}$ of the hyperfinite II_1 factor $\mathcal{R}$.

Goal: CEP $\Rightarrow$ Kirchberg's ('90s) QWEP Problem

Recall: A concrete C^*-algebra is a $*$-subalgebra of $\mathcal{B}(\mathcal{H})$ closed in the operator norm topology.

Def A C^*-norm on a $*$-alg $\mathcal{A}$ is a norm $\|\cdot\|$ on $\mathcal{A}$ satisfying:

- $\|xy\| \leq \|x\| \cdot \|y\|$
- $\|x^*\| = \|x\|$
- (**C^*-equality**) $\|x^*x\| = \|x\|^2$

An abstract C^*-algebra is a $*$-algebra equipped with a complete C^*-norm.

Exercise: Concrete C^*-algebras are abstract C^*-alg.

Def A representation of a $*$-alg A is a $*$-alg hom $\pi: A \to \mathcal{B}(\mathcal{H})$.
It is faithful if π is injective.

Thm (Gelfand-Naimark) If A is an abstract C^*-alg, then A admits a faithful rep. $\pi: A \to \mathcal{B}(H)$.
(Then $A \cong \pi(A) \subseteq \mathcal{B}(\mathcal{H})$)
concrete C^*-alg.

Convention All C^*-algebras are unital. $\|1\| = 1$.

example If X is a compact Hausdorff space, $C(X) = \{\text{cont functions } X \to \mathbb{C}\}$, then $C(X)$ is an abelian C^*-alg:

$$f^*(x) := \overline{f(x)}$$

$$\|f\| := \sup_{x \in X} |f(x)|$$

Gelfand Duality: All abelian C^*-alg. are of this form.

Specral elements

- self-adjoint $x = x^*$ ($C(X)$: $f: X \to \mathbb{R}$)
- positive $x = y^* y$ for some y ($C(X)$: $f: X \to \mathbb{R}^{\geq 0}$)
- projection: self-adj & $x^2 = x$
  ($\mathcal{B}(\mathcal{H})$: orth projection onto closed subspace
  $C(X)$: $f: X \to \{0,1\}$)
- unitary: $x^* x = x x^* = 1$
  ($\mathcal{B}(\mathcal{H})$: unitary operators

$C(X)$: $f: X \xrightarrow{\text{"}} \text{"}S^1\text{"})$

Def A linear map $\phi: \mathcal{A} \to \mathcal{B}$ between C^*-alg is **positive** if it sends positive elements to positive elements.

e.g. $*$-hom. are positive maps.

Given any $n \geq 1$, view $M_n(\mathcal{A})$ = $n \times n$ matrices with entries from $\mathcal{A}$ as a C^*-subalg of $\mathcal{B}(\underbrace{\mathcal{H} \oplus \cdots \oplus \mathcal{H}}_{n\text{-times}})$. $(\mathcal{A} \subseteq \mathcal{B}(\mathcal{H}))$

If $\phi: \mathcal{A} \to \mathcal{B}$ is linear, get

$\phi^{(n)}: M_n(\mathcal{A}) \to M_n(\mathcal{B})$.

If each $\phi^{(n)}$ is positive, we say ϕ is **completely positive** (cp).

UCP: unital $\phi(1) = 1$ and cp.

$*$-homomorphisms are ucp.
Converse is false but not too badly:

If $\pi: A \to \mathcal{B}(K)$ is a $*$-hom. and
$V: \mathcal{H} \to K$ is an isometry, i.e.
$\langle V\xi, V\eta \rangle = \langle \xi, \eta \rangle$ for all $\xi, \eta \in \mathcal{H}$,
get $\phi: A \to \mathcal{B}(\mathcal{H})$ by
$$\phi(a)(\xi) := V^* \pi(a)(V\xi).$$
"Compression of π"
Check: ϕ is ucp

Stinespring's Thm: All ucp maps
$\phi: A \to \mathcal{B}(\mathcal{H})$ arise this way.

Examples coming from groups

G countable (discrete) group.

Last time: Left-regular representation

$\lambda : G \to \mathcal{B}(\ell^2(G))$ unitary rep.

$(\lambda(g)(f))(h) := f(g^{-1}h)$

Alternatively: If δ_h is the Dirac mass at $h \in G$, then $\lambda(g)(\delta_h) = \delta_{gh}$.

$L(G)$ = von Neumann algebra generated by $\lambda(G)$ group <u>von Neumann alg</u>

$C^*_r(G)$ = C^*-alg. generated by $\lambda(G)$

= <u>reduced</u> group <u>C^*-alg</u> $\subseteq L(G)$

$\pi : G \to \mathcal{U}(A)$ unitary rep. of G

Σ group of unitaries in a C^*-alg A.

Extends to $*$-alg map

$\pi : \mathbb{C}[G] \to A$

$\mathbb{C}[G]$ = group ring

= formal linear comb $\sum_{g \in G} c_g \cdot g$ w/ only fin many nonzero coeff.

$$\left(\sum c_g \cdot g\right)^* := \sum \overline{c_g} \cdot g^{-1}$$

Define $\| \sum c_g \cdot g \| = \sup \{ \| \sum_{g \in G} c_g \pi(g) \| :$
$\pi : G \to \mathcal{U}(A)$ unitary rep. $\}$.

Check: $\| \sum c_g \cdot g \| < \infty$

$\| \cdot \|$ is a C^*-norm on $\mathbb{C}[G]$.

Def $C^*(G)$ = completion of $\mathbb{C}[G]$ w.r.t. this norm

universal group C^*-alg. of G.

Universal property: If $\pi: G \to \mathcal{U}(A)$ is a unitary rep, then π extends uniquely to a $*$-alg hom $\pi: C^*(G) \to A$.

Cor For any separable C^*-alg A, there is a surjective $*$-hom $C^*(\mathbb{F}_\infty) \to A$.

Pf: A sep $\Rightarrow$ there exists unitaries $(u_i)_{i\in\mathbb{N}}$ that generate A.

$\mathbb{F}_\infty \to \mathcal{U}(A)$ group hom.

$e_i \mapsto u_i$

Extends to $*$-alg $C^*(\mathbb{F}_\infty) \to A$. $\square$

Get $C^*(G) \to C^*_r(G)$ surjective $*$-alg.
Usually not iso.
In fact: Isomorphism iff G amenable.

Functorial: $G \to H$ group hom
$\rightsquigarrow$ $C^*(G) \to C^*(H)$ $*$-alg.

If $H \leq G$, then $C^*(H) \subseteq C^*(G)$.

$C^*(G * H) \cong C^*(G) * C^*(H)$.

Tensor products of C^*-algebras

Vector spaces V, W $\rightsquigarrow$ tensor product $V \otimes W$.

$S: V_1 \to V_2$ lineer maps
$T: W_1 \to W_2$

$S \odot T : V_1 \odot W_1 \to V_2 \odot W_2$
$(S \odot T)(v \otimes w) = Sv \otimes Tw.$

If $\mathcal{H}, \mathcal{K}$ are Hilb spaces, then we have an inner product on $\mathcal{H} \odot \mathcal{K}$: $\langle \xi_1 \otimes \eta_1, \xi_2 \otimes \eta_2 \rangle$
$= \langle \xi_1, \xi_2 \rangle \cdot \langle \eta_1, \eta_2 \rangle.$

Complete: Hilbert space $\mathcal{H} \otimes \mathcal{K}.$
$S: \mathcal{H}_1 \to \mathcal{H}_2$ bounded linear
$T: \mathcal{K}_1 \to \mathcal{K}_2$
$S \otimes T: \mathcal{H}_1 \otimes \mathcal{K}_1 \to \mathcal{H}_2 \otimes \mathcal{K}_2.$

Main issue: $\mathcal{A}, \mathcal{B}$ C^*-alg.
Want "their" tensor product.
$\mathcal{A} \odot \mathcal{B}$ is a $*$-algebra again:

$$(x_1 \otimes y_1) \cdot (x_2 \otimes y_2) = (x_1 x_2) \otimes (y_1 y_2)$$
$$(x \otimes y)^* = x^* \otimes y^*.$$
unit $1 \otimes 1$

Want: C^*-norm on $\mathcal{A} \odot \mathcal{B}$.
The completion will be a C^*-alg. that could be called $\mathcal{A} \otimes \mathcal{B}$.

In general, no canonical C^* norm on $\mathcal{A} \odot \mathcal{B}$.

Two "natural" & "extreme" choices:

① $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$

$\mathcal{B} \subseteq \mathcal{B}(K)$

$\mathcal{A} \odot \mathcal{B} \subseteq \mathcal{B}(\mathcal{H} \otimes K)$

The operator norm on $\mathcal{B}(\mathcal{H} \otimes K)$ induces a C^*-norm on $\mathcal{A} \odot \mathcal{B}$ called $\|\cdot\|_{min}$.

Check: Independent of representation.

Thm (Takesaki) $\|\cdot\|_{min} \leq \|\cdot\|_\alpha$ for any C^*-norm $\|\cdot\|_\alpha$ on $\mathcal{A} \odot \mathcal{B}$.

Completion of $\mathcal{A} \odot \mathcal{B}$ w.r.t. $\|\cdot\|_{min}$ is denoted $\mathcal{A} \otimes_{min} \mathcal{B}$ or $\mathcal{A} \otimes \mathcal{B}$.

minimal tensor product

Properties

ⓐ If $\pi_{\mathcal{A}}: \mathcal{A}_1 \to \mathcal{A}_2$, $\pi_{\mathcal{B}}: \mathcal{B}_1 \to \mathcal{B}_2$ are $*$-homs, then $\pi_{\mathcal{A}} \odot \pi_{\mathcal{B}}$ extends uniquely to
$\pi_{\mathcal{A}} \otimes \pi_{\mathcal{B}}: \mathcal{A}_1 \otimes_{min} \mathcal{B}_1 \to \mathcal{A}_2 \otimes_{min} \mathcal{B}_2$.
"Ditto" for ucp maps.

ⓑ $C^*_r(G \times H) \cong C^*_r(G) \otimes_{min} C^*_r(H)$

② Define **maximal norm** $\|\cdot\|_{max}$ on $\mathcal{A} \odot \mathcal{B}$ by

$$\|x\|_{max} = \sup\{\|\pi(x)\|: \pi: \mathcal{A} \odot \mathcal{B} \to \mathcal{B}(\mathcal{H}) \text{ } *\text{-hom}\}$$

Check: $\|x\|_{max} < \infty$
$\|\cdot\|_{max}$ is a C^*-norm on $\mathcal{A} \odot \mathcal{B}$.

Fact: $\|\cdot\|_\alpha \le \|\cdot\|_{max}$ for any C^*-norm on $\mathcal{A}\odot\mathcal{B}$.

Completion: $\mathcal{A}\otimes_{max}\mathcal{B}$.

$\pi: \mathcal{A}\odot\mathcal{B} \to \mathcal{B}(\mathcal{H})$

Properties

ⓐ If $\pi_{\mathcal{A}}: \mathcal{A} \to \mathcal{B}(\mathcal{H})$, $\pi_{\mathcal{B}}: \mathcal{B} \to \mathcal{B}(\mathcal{H})$ are $*$-homs with commuting ranges, get $*$-hom $\pi_{\mathcal{A}}\otimes\pi_{\mathcal{B}}: \mathcal{A}\otimes_{max}\mathcal{B} \to \mathcal{B}(\mathcal{H})$.
"Ditto" for ucp maps.

ⓑ $C^*(G\times H) \cong C^*(G)\otimes_{max} C^*(H)$.

Fact: In general, $\mathcal{A}\otimes_{min}\mathcal{B} \not\cong \mathcal{A}\otimes_{max}\mathcal{B}$.
Always, $\mathcal{A}\otimes_{max}\mathcal{B} \twoheadrightarrow \mathcal{A}\otimes_{min}\mathcal{B}$.
When this map is an isomorphism, call $(\mathcal{A},\mathcal{B})$ a nuclear pair.

A is **nuclear** if $(A, \mathcal{B})$ is a nuclear pair for all $\mathcal{B}$.

examples

① $(C^*_r(\mathbb{F}_2), C^*_r(\mathbb{F}_2))$ not nuclear pair.

$(C^*(\mathbb{F}_2), C^*(\mathbb{F}_2))$ is a nuclear pair iff CEP! (Next time!)

② $(\mathcal{B}(H), \mathcal{B}(H))$ not nuclear pair. (Junge–Pisier)

Thm (Kirchberg) $\left(\overset{\text{surj. univ.}}{C^*(\mathbb{F}_\infty)}, \overset{\text{inj. univ.}}{\mathcal{B}(H)}\right)$ is a nuclear pair.

$C^*(G)$ nuclear iff $C^*_r(G)$
iff G amenable (80

$C^*(G) = C^*_r(G))$