Goals:

1. Finish $\text{CEP} \Rightarrow \text{QWEP}$
2. Start Complexity Theory

Last time:

**Theorem (Kirchberg)** $\left( C^*(1B_0), B(H) \right)$ is a **nuclear pair**:

\[ C^*(1B_0) \otimes \min B(H) = C^*(1B_0) \otimes \max B(H). \]

**Exercise**: If $A \leq B$, then

\[ A \otimes \min C \leq B \otimes \min C \text{ for any } C. \]

**Issue**: In general not true for $\otimes \max$,

\[ A \otimes \max C \rightarrow B \otimes \max C \text{ need not be isometric.} \]
One instance when it does hold: $A \subseteq A^{**}$ ?

Universal representation $\text{Tiu}: A \rightarrow B(\mathcal{H})$

$A^{**}$ = von Neumann subalgebra of $B(\mathcal{H})$

generated $\text{Tiu}(A)$.

$A \otimes \text{max } C \subseteq A^{**} \otimes \text{max } C$.

Fact: If $A \subseteq B$ and there is a ucp $B \rightarrow A^{**}$ that restricts to the identity on $A$, then $A \otimes \text{max } C \subseteq B \otimes \text{max } C$.

Thm For $A \subseteq B$, TFAE:

\begin{enumerate}
  \item \text{(A)} $A \otimes \text{max } C \subseteq B \otimes \text{max } C$
\end{enumerate}
There is a weak conditional map $B \to A^{**}$ weakly cp-complemented.

When $A$ is weakly cp-comp in $B(\mathcal{H})$, say $A$ has the weak expectation property (WEP).

Thm (Kirchberg) $A$ has WEP iff $(A, C^*(1\mathcal{F}_0))$ nuclear pair.

Q Does $C^*(1\mathcal{F}_0)$ have WEP?  
I.e. Is $(C^*(1\mathcal{F}_0), C^*(1\mathcal{F}_0))$ a nuclear pair?

Tautology: If $C^*(1\mathcal{F}_0)$ has WEP, then every separable $C^*$-algebra has QWEP, that is, is a quotient of something with WEP.
Converse is true: If $C^*(1F_0)$ has QWEP, then it actually has WEP.

Reason: $C^*(1F_0)$ has \textbf{lifting property}: 

\[ \exists \text{ucp} \rightarrow \mathcal{B} \]
\[ \downarrow \]
\[ C^*(1F_0) \xrightarrow{\text{hom}} \mathcal{B}/J \]

LLP + QWEP $\Rightarrow$ WEP

\[ \text{Lemma (Kirchberg) TFAE:} \]

1. $C^*(1F_0)$ has WEP
2. $(C^*(1F_0), C^*(1F_0))$ nuclear pair for some (equivalency) $k \in \{2, 3, \ldots, \infty\}$. 

Kirchberg's QWEP Problem
3. Every sep C*-alg has QWEP
4. LLP $\Rightarrow$ WEP.

**Now:** CEP $\Rightarrow$ QWEP

(i) $R$ has WEP ($R$ is "injective")
(ii) $\ell^\infty(R)$ has WEP
(iii) $R^\ast$ has QWEP: $R^\ast = \ell^\infty(\mathbb{N})/\mathbb{N}$
(iv) If $M$ is a tracial vNa and $M \subseteq R^\ast$, then $M$ is weakly cp-comp in $R^\ast$ and so $M$ has QWEP.

**Moral:** CEP $\Rightarrow$ all tracial vNas have QWEP.

Since $A$ is weakly cp-comp in the vNa $A^{\text{**}}$, to show $A$ has QWEP,
Enough to show $A^{**}$ has QWEP.

Issue: $A^{**}$ probably doesn't have a trace.

(v) For a general vNa $M$, there is a 1-parameter subgroup $\sigma_t$ of automorphisms of $M$ (modular group) so that $M\rtimes_\sigma R$ is semifinite. (Takesaki)

Finite vNas have QWEP $\implies$ semifinite ones.

(vi) $M$ is weakly cp-compact in $M\rtimes_\sigma R$.

Complexity Theory

Turing machine
If $M$ is a Turing machine, let $f^M : \{0,1\}^* \to \{0,1\}$ denote the partial function it computes.
$\{0,1\}^*$ = set of finite binary sequences

If $T : \mathbb{N} \to \mathbb{N}$ is a function, say $M$ runs in time $T(n)$ if: for every $z \in \{0,1\}^*$, upon input $z$, the machine halts in $\leq T(|z|)$ many steps.

$M$ runs in polynomial time if...

exponential time $\sim 2^{12}$

doubly exponential time

language $L \subseteq \{0,1\}^*$
"codes for problem instances"
Complexity class set of languages

Least complex \( P = \) set of languages \( L \) so that there is a poly time Turing machine that computes \( X_L \) = char. function of \( L \)

\( \text{EXP} = \) same but w/ exp time machine \( P \subseteq \text{EXP} \)

\( \subseteq \) Time Hierarchy Thm

\( \text{NP} = \) set of languages \( L \) for which there is a poly time machine \( M \) and poly. \( p(n) \) so that: "witness"

- if \( z \in L \), then there is \( w \in \Sigma^* \) such that \( M(z,w) = 1 \).
- if \( z \notin L \), then for every \( w \in \Sigma^* \), \( M(z,w) = 0 \)
Verifier & prover

Input: z

Prover: Trying to convince verifier z is her proof ("All powerful")

Verifier: Checks if z is a valid proof that z is L using M

**Example**

Graph isomorphism

$L = \{ (G_1, G_2) : G_1, G_2 \text{ finite graphs, } G_1 \cong G_2 \}$

Belongs to NP

$P \subseteq NP \subseteq EXP$

$\exists P = NP?$
Also $\text{NEXP, NEEXP...}$

$\text{NP \neq NEXP \neq NEEXP}$

$\text{PSPACE = set of languages } L \text{ that can be decided using machines that use a poly. amount of space.}$

$\text{P \subseteq PSPACE}$

$\text{NP \subseteq PSPACE \subseteq \text{Exp} \subseteq \text{NEXP}}$

$\text{PSPACE = NEXP?}$

$\text{BPP: Just like NP except for a randomly chosen } r \in \{0,1\}^P(\text{121}),$

$\text{Prob}(M(z,r) = \chi_L(z)) \geq \frac{2}{3}.$
Interactive proofs

\[
\begin{align*}
V(z) &= a_1 \\
P(z, a_1) &= a_2 \\
V(z, a_1, a_2) &= a_3 \\
P(z, a_1, a_2, a_3) &= a_4 \\
\vdots
\end{align*}
\]

Then \( V(z, a_1, \ldots, a_{2k}) = \text{yes or no} \)

This is just NP in disguise.

\[
\begin{align*}
V(z, r) &= a_1 \\
P(z, a_1) &= a_2 \\
V(z, r, a_1, a_2) &= a_3 \\
P(z, a_1, a_2, a_3) &= a_4 \\
\vdots
\end{align*}
\]
\[ V(z, r, a_1, \ldots, a_{2k}) = \text{yes or no} \]

\[ L \text{ belongs to IP if there is such } V \]

\[ \text{and } p \text{ so that:} \]

\[ \text{If } z \in L, \text{ then there is a function } P \]

\[ \text{so that } \Pr(\text{prob}(V(z, r, a_1, \ldots, a_{2k}) = 1)) \geq \frac{2}{3} \]

\[ \text{If } z \notin L, \text{ then for every such } P, \]

\[ \Pr(\text{prob}(V(z, r, a_1, \ldots, a_{2k}) = 1)) \leq \frac{1}{3} \]

**Example** Graph non-isom. is in IP.

Protocol: Input \((G_1, G_2)\).

Verifier flips a coin, getting \(i \in \{1, 2\}\).

Randomly picks a permutation \(\sigma\) of vertices of \(G_i\), obtaining \(H \cong G_i\)

Prover returns \(a \in \{1, 2\}\).
If $G_1 \not\cong G_2$, prover can always get it right.
If $G_1 \cong G_2$, prover can do no better than guessing.

Graph: $G = (V, E)$  
$V$ vertex set  
$E \subseteq V \times V$

$(v, w) \in E$  

$(v, v) \notin E$  

$(v, w) \in E$  

$(w, v) \notin E$