**Def** A tracial vNa $M$ is **existentially closed** (e.c.) if:

whenever $M \leq N$, any quantifier-free formula $\varphi(x,y)$, any $a \in M$, we have

$$\inf_{b \in M} \varphi(a,b)^M = \inf_{c \in N} \varphi(a,c)^N.$$ 

**Lemma** $M$ is e.c. iff: whenever $M \leq N$, then there is $N \leftrightarrow M^n$ restricting to the diagonal embedding $M \rightarrow M^n$.

**Facts**

- Ec tracial vNas exist.
- If $M$ is a tracial vNa, there is an
e.c. tracial vNa \( N \cong M \).
If \( M \) is sep., can take \( N \) sep.

- E.c. tracial vNas are II_1 factors. Extra properties: McDuff, all autos approx inner, ...

**Def.** Call a tracial vNa embeddable if it embeds in \( \mathbb{M}^n \), i.e. if it's a model of \( \text{Th}_{\text{v}}(\mathbf{M}) \).

**Prop.** \( \mathbf{R} \) is an e.c. embeddable factor.

**Pf 1:** \( \mathbf{R} \subseteq M \hookrightarrow \mathbb{M}^n \)

\[ \text{may not be the diagonal embedding} \]

**Fact:** any embedding of \( \mathbf{R} \hookrightarrow \mathbb{M}^n \)
is unitarily conjugate to the diagonal embedding, i.e. \( \exists U \in \mathbb{U}(\mathbb{R}^n) \) s.t. \( u \varphi(x) u^* = x \forall x \in \mathbb{R} \).

**Digression** Kenley Jung: If \( N \to \mathbb{R}^n \) is s.t. any two embeddings are unit. conj, then \( N \cong \mathbb{R} \).

Atkinson-Elayavalli: If \( N \) is emb. and any two embeddings \( N \to \mathbb{R}^n \) are unit conj, then \( N \cong \mathbb{R} \).

AE-G.: If \( N \) is emb. and any two embeddings \( N \to \mathbb{R}^n \) are conj. by an auto, then \( N \cong \mathbb{R} \). (Model theory!)
Def. A tracial $vN\alpha^\wedge$ is \underline{locally universal} if for any $N$, $N \rightarrow M^\mu$.

CEP: $R$ is locally universal.

\underline{Lemma} An e.c. tracial $vN\alpha^\wedge$ is locally universal.

\textbf{Lt}: Given any $N$, want $N \rightarrow M^\mu$.

\[ N \leq N \otimes M \rightarrow M^\mu \]
\[ U \overset{\alpha}{\rightarrow} \text{blc} \]
\[ M \overset{\text{blc}}{\rightarrow} M \text{ is e.c.} \]

\underline{Cor} CEP $\iff R$ is an e.c. factor.

\textbf{Lt} (\(\Rightarrow\)) Know $R$ is an e.c. emb. factor.
CEP $\Rightarrow$ everything is emb.
\[(\equiv)\text{A e.c. } \Rightarrow \text{R loc univ } \Leftrightarrow \text{CEP}\]

Back to games
- 2 players
- Play finite sets of conditions of the form \(\exists (C) \leq r, \text{satisfiable}\)
- Extend each other's play.
- At the end, built a separable tracial \(\text{vN}\), called the compiled algebra.

Said a property \(P\) of tracial \(\text{vN}\) as \textbf{enforceable} if \(\exists\) (player II) has a strategy that forces the compiled algebra to have property \(P\).
Last time: being a l1 factor is enforceable.

**Proof** Being e.c. is enforceable.

**Pf:** Enough to show: given any g.t. 
(\(\mathcal{L}(x,y)\)) constants c, and rational \(r > 0\), either the compiled structure has no extension with \(\inf y \mathcal{L}(c,y) < r\) or else there are \(c'\) s.t. \(\mathcal{L}(c,c') < r\).

Player A opens the game with \(P_0\).

If there is a trivial \(\nu Na\) satisfying \(P_0\) and \(\inf y \mathcal{L}(c,y) < r\), then there are constants \(c'\) s.t. \(P_0 \land \mathcal{L}(c,c') < r\)

is a condition and \(E\) plays it.

Otherwise, every model of \(P_0\) thinks \(\inf y \mathcal{L}(c,y) \geq r\) and so the compiled
0 structure thinks that as well.

**Def** A separable tracial rN\(\mathcal{A}\) is **enforceable** if the property of being \(\cong \mathcal{M}\) is an enforceable property.

Q: Is there an enforceable II\(_1\) factor?

**Examples**

- The random graph is the enforceable graph.
- The enforceable field of char \(p\) is \(\mathbb{F}_p\).
Note: If $T$ is $VF$ and has JEP, then $M$ is enforceable iff it is e.c and embeds in all other e.c models (e-atomic).

**Thm:** There is no enforceable group.

**Pf:** If $G$ is the enforceable group, then $G$ embeds in every e.c group.

By a theorem of Macintyre, every f.g. subgroup of $G$ has solvable word problem. But every e.c. group has a f.g. subgroup w/ unsolv. word problem.

**Example:** There is an enf. Banach space, Gurarij Banach space.
**Thm** $R$ is the enforceable embeddable factor.

**Pr** Model theory reason: If $P$ is a $\forall \forall \exists$-axiomatizable property

$$\forall x \exists y \exists z \exists w \forall u \forall v \forall w \forall x \exists y \exists z \exists w \forall u \forall v \forall w \forall x$$

and there is existential

a locally universal object with property $P$, then $P$ is enforceable.

Us: $P =$ hyperfinite
Player $A$ opens with $p$ satisfiable
in some emb. $M$.

Since $M \models R^n$, $p$ is satisfied in $M$.
I can respond with approximate $p \cup \{e_{ij}\}$ are matrix units and $C_n$ is
Close to some $l_1$ comb $e_{ij}$'s $\exists$ Being hyperfinite is enf.

But being a $l_1$ factor is enf.

Use $\mathcal{R} \alpha$ is the unique sep. hyperfinite $l_1$ factor (Murray - von Neumann).

**Cor** TFAE:

1. CEP
2. $\mathcal{R} \alpha$ is the enf. $l_1$ factor
3. Being embeddable is enforceable.

Proof: 1 $\Rightarrow$ 2

2 $\Rightarrow$ 3 obvious.

3 $\Rightarrow$ 1 By 3, there is an $e.c.$ factor that is embeddable. This is $e.c.$ factor is locally universal, so CEP is true. $\square$
Fact If $\sigma$ is a sentence, then there is a unique $\sigma EIR$ s.t. $\sigma|=\sigma$ is enforceable.

If $\sigma E = \sigma^R$ for every universal $\sigma$, then being embeddable would be enforceable.

:: Exist universal $\sigma$ s.t. $\sigma E \neq \sigma^R$ and $\sigma|=\sigma E$ is enforceable, i.e. one can enforce the compiled algebra $M$ to satisfy $\sigma M \neq \sigma^R$, so $M \neq \mathbb{M}$.

Open Question Does the enforceable 11 factor $E$ s.t.?

\[ \text{yes} / \text{No} \]

\[ \text{REP.} \]
$E$ vs. $R$

Improvement of $E$ every ec $ll_1$ factor

$R$ every $ll_1$ factor

$R \cong \mathbb{R} \circ \mathbb{R}$

$E \neq E \circ E$

$E \neq 3 \circ 3$

Atkinson: every emb $R \rightarrow \mathbb{R}^n$ has a lift $\psi_0: R \rightarrow R$ that induce $R \rightarrow \mathbb{R}^n$.

Atkinson proved that characterizes $R$ amongst emb. factors.

$G. \ E$ has that property.

$R \cong L(C^1)$ for any ctbl, infinite, ICC amenable group
Is being a group via an enforceable property?