Embedding problems, games, and square roots

Isaac Goldbring

University of California, Irvine

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Embedding problems

- CEP: Every separable II_1 factor embeds into $\mathcal{R}^{\mathcal{U}}$.
- KEP: Every separable C^*-algebra embeds into $\mathcal{O}_2^{\mathcal{U}}$.
- MFP (resp. QDP): Every separable stably finite (resp. stably finite nuclear) C^*-algebra embeds into $\mathcal{Q}^{\mathcal{U}}$.
- $\mathcal{Z}$*EP*: Every separable stably projectionless (nuclear) C^*-algebra embeds into $\mathcal{Z}^{\mathcal{U}}$.

Definition

A separable element A of a class $\mathcal{C}$ is called *locally universal* (with respect to $\mathcal{C}$) if every separable element of $\mathcal{C}$ embeds into some (equiv. any) ultrapower of A.

Thus, all of the embedding problems ask whether or not certain canonical models of some given class are locally universal.

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JEP and locally universal models

Definition

A class $\mathcal{C}$ has the *joint embedding property* (JEP) if any two elements of $\mathcal{C}$ embed into a common element.

Example

- The theories of II_1 factors and C^*-algebras have JEP.
- It is unknown whether the other aforementioned classes have JEP.

Lemma

If an inductive axiomatizable class has JEP, then it has a locally universal element.

So there are locally universal II_1 factors and C^*-algebras. But the other classes may not even have a locally universal object.

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Existentially closed models

Definition

- Given $A \subseteq B$, we say that A is *existentially closed (e.c.) in B* if there is an embedding $B \hookrightarrow A^{\mathcal{U}}$ that restricts to the diagonal embedding $A \hookrightarrow A^{\mathcal{U}}$.
- If A belongs to some class $\mathcal{C}$, we say that A is e.c. for $\mathcal{C}$ if A is e.c. in B for every $B \in \mathcal{C}$ with $A \subseteq B$.

- In an inductive class, every element is contained in an e.c. element of the class. Often, the e.c. element can be taken to be separable if the original element is separable.
- What are *natural* examples of e.c. objects in operator algebras?

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- What are *natural* examples of e.c. objects in operator algebras?

E.c. vs. locally universal

- Suppose that $A \in \mathcal{C}$ is separable and e.c.
- Suppose that $B \in \mathcal{C}$ is separable and that there is $D \in \mathcal{C}$ into which both A and B embed (e.g. when $\mathcal{C}$ has JEP).
- Then since A is e.c. we have $B \hookrightarrow D \hookrightarrow A^\omega$.
- It follows that A is locally universal.
- If $\mathcal{C}$ is actually axiomatizable, then A being e.c. and locally universal implies that $\mathcal{C}$ has JEP.

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Embedding problems and e.c. models

Theorem

All of the embedding problems are equivalent to the corresponding canonical object being e.c for the corresponding class.

Proof for QDP.

First suppose that the QDP has a positive solution and that $\mathcal{Q} \subseteq A$ with A separable, s.f. nuclear. By assumption, $A \hookrightarrow \mathcal{Q}^{\mathcal{U}}$. Since $\mathcal{Q}$ is *strongly self-absorbing*, the composition $\mathcal{Q} \subseteq A \hookrightarrow \mathcal{Q}^{\mathcal{U}}$ is unitarily conjugate to the diagonal embedding.

Now suppose that $\mathcal{Q}$ is e.c. for s.f. nuclear. Given A separable s.f. nuclear, we have that $A \otimes \mathcal{Q}$ is also s.f. nuclear. Thus, by the previous slide, $\mathcal{Q}$ is locally universal for the class. □

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KEP

- In the rest of this talk, let us focus on KEP.
- The case of CEP is fairly similar but the other cases pose some additional difficulties.

Theorem (G. and Sinclair)

KEP is equivalent to the existence of an e.c. C^*-algebra that is also nuclear.

Proof.

If KEP holds, then $\mathcal{O}_2$ is e.c. Conversely, suppose that A is an e.c. C^*-algebra that is nuclear. Then by a deep theorem of Kirchberg, $A \otimes \mathcal{O}_2 \cong \mathcal{O}_2$. (One also needs to know that A is simple, which follows from being e.c.) However, another consequence of being e.c. is that $A \otimes \mathcal{O}_2 \cong A$. It follows that $A \cong \mathcal{O}_2$ and thus KEP holds. □

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Introducing the game

- The method of building e.c. models we are going to discuss goes under many names, e.g. Henkin constructions, model-theoretic forcing. We will discuss this concept using a certain kind of game.
- Let us fix a countably infinite set C of distinct symbols (*witnesses*) that are to represent generators of a separable C*-algebra that two players (traditionally named $\forall$ and $\exists$) are going to build together (albeit adversarially).
- The two players take turns playing finite sets of expressions of the form $|\|p(c)\| - r| < \epsilon$, where c is a tuple of variables, $p(c)$ is a $*$-polynomial, and each player's move is required to extend the previous player's move. These sets are called (open) *conditions*.
- Moreover, these conditions are required to be *satisfiable*, meaning that there should be some C*-algebra A and some tuple a from A such that $|\|p(a)\| - r| < \epsilon$ for each such expression in the condition.

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Introducing the game (cont'd)

- We play this game for ω many steps.
- At the end of this game, we have enumerated some countable, satisfiable set of expressions.
- Provided that the players behave, they can ensure that the play is *definitive*, meaning that the final set of expressions yields complete information about all $*$-polynomials over the variables C (that is, for each $*$-polynomial $p(c)$, there should be a unique r such that the play of the game implies that $\|p(c)\| = r$) and that this data describes a countable, dense $*$-subalgebra of a unique C*-algebra, which is often called the *compiled structure*.

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Enforceable properties

Definition

Given a property P of C^*-algebras, we say that P is an *enforceable* property is there a strategy for $\exists$ so that, regardless of player $\forall$'s moves, if $\exists$ follows the strategy, then the compiled structure will have that property.

Conjunction Lemma

If $(P_i : i \in \omega)$ are all enforceable properties, so is $\bigwedge_i P_i$.

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An example of enforceability

Example

$\mathcal{O}_2$-stability is an enforceable property of C^*-algebras.

Proof.

- We use the fact that a separable C^*-algebra A is $\mathcal{O}_2$-stable if and only if, for every finite set $F \subseteq A$, there are $u, v \in A$ that are "almost generators of $\mathcal{O}_2$" that almost commute with F.
- Here's the strategy: suppose that $\forall$ played the open condition p that only mentions witnesses amongst $C_0 \subseteq C$ (finite).
- $\exists$ can respond by taking $c, d \in C \setminus C_0$ and saying that c and d are almost generators of $\mathcal{O}_2$ that almost commute with C_0.
- This is indeed a condition: if p were satisfied in A, then this new set of expressions is satisfiable in $A \otimes \mathcal{O}_2$.

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A crucial fact and a crucial definition

Theorem

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In particular, this shows that any property possessed by an e.c. C^*-algebra (simple, purely infinite, $\mathcal{O}_2$-stable, trivial K-theory,...) is an enforceable property.

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If A is the unique countable e.c. model of some theory, then A is enforceable. For example, the random graph is the enforceable graph.

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KEP and enforceable models

Theorem

The following are equivalent:

1. *KEP has a positive solution.*
2. *Nuclearity is enforceable.*
3. $\mathcal{O}_2$ *is enforceable.*
4. $\mathcal{O}_2^{\mathcal{U}}$*-embeddability is enforceable.*

The only implication that we do not immediately know right now is (1)$\Rightarrow$(2).

KEP and enforceable models

Theorem

The following are equivalent:

1. *KEP has a positive solution.*
2. *Nuclearity is enforceable.*
3. $\mathcal{O}_2$ *is enforceable.*
4. $\mathcal{O}_2^{\mathcal{U}}$*-embeddability is enforceable.*

The only implication that we do not immediately know right now is (1)$\Rightarrow$(2).

Nuclearity is an $\forall\bigvee\exists$-property

Fact (Farah, et. al.; see also Bradd's tutorial)

There are existential formulae $\varphi_{mn}(x_1, \ldots, x_n)$ such that a C^*-algebra A is nuclear if and only if, for all n, we have

$$(\sup_x \inf_m \varphi_{mn}(x))^A = 0.$$

- Suppose that KEP holds. We show that $\exists$ can enforce nuclearity.
- Here is the strategy: suppose $\forall$ plays p.
- By KEP, p is realized in $\mathcal{O}_2^{\mathcal{U}}$ and thus in $\mathcal{O}_2$.
- If $c_1, \ldots, c_n$ are the witnesses named in p, we can find m such that $\varphi_{mn}(c) < \epsilon$.
- We can then respond with $p \cup \{\psi_{mn}(c, d) < \epsilon\}$, where $\varphi_{mn} = \inf_y \psi_{mn}(x, y)$ and d is a tuple of fresh witnesses.

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The dichotomy theorem

Theorem

Exactly one of the two possibilities occurs:

- *There is an enforceable* C^**-algebra; or*
- *Chaos: for every enforceable property P of* C^**-algebras, there are* $2^{\aleph_0}$ *many pairwise nonisomorphic* C^**-algebras with property P.*

Question

Suppose that we know that there is an enforceable C^*-algebra A. Must it be the case that $A \cong \mathcal{O}_2$?

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A digression on LLP and WEP

Definition

Let A be a C^*-algebra. Then A has *WEP* (resp. *LLP*) if and only if $A \otimes \mathrm{C}^*(\mathbb{F}) = A \otimes_{\max} \mathrm{C}^*(\mathbb{F})$ (resp. $A \otimes \mathcal{B}(H) = A \otimes_{\max} \mathcal{B}(H)$).

Theorem

1. *(Kirchberg) CEP is equivalent to the statement "LLP implies WEP."*
2. *(Junge-Pisier) WEP does not imply LLP.*

Theorem (G.-Sinclair)

1. *If A is e.c., then A has WEP.*
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A digression on LLP and WEP (cont'd)

- Let σ_{LLP} be the $L_{\omega_1,\omega}$-sentence axiomatizing LLP, i.e. $\sigma_{LLP}^A = 0$ if and only if A has LLP.
- Fact: There is $r \in [0, 1]$ such that the property that $\sigma_{LLP} = r$ is enforceable.
- Question: What is r?
- Answer: I don't know! But:
  - If $r > 0$, then we can enforce an e.c. C^*-algebra without LLP, which is then a possibly new example of a C^*-algebra with WEP but not LLP.
  - Suppose $r = 0$, so LLP is enforceable.
    - If nuclearity is enforceable, then KEP holds.
    - Otherwise, non-nuclearity is enforceable, whence, by the conjunction lemma, we can enforce an e.c. algebra that has LLP but is non-nuclear. This algebra would thus be non-nuclear but both WEP and LLP, which would be the first example of an algebra of this kind. ("Weak CEP")

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Square roots and KEP

Definition

A C^*-algebra A is a *tensor square* or *has a tensor square root* if there is a C^*-algebra B such that $A \cong B \otimes B$.

Clearly s.s.a. algebras are tensor squares.

Theorem (G.; G.-Sinclair)

KEP holds if and only if the property of being a tensor square is enforceable.

- We already know one direction: if KEP holds, then $\mathcal{O}_2$ is enforceable.
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E.c. models and approximately inner automorphisms

- Suppose that A is an e.c. C^*-algebra and $\alpha \in \mathrm{Aut}(A)$.
- Then we have $A \subseteq A \rtimes_\alpha \mathbb{Z}_2 \hookrightarrow A^{\mathcal{U}}$.
- Then the unitary in $A \rtimes_\alpha \mathbb{Z}_2$ that implements α gives us a sequence of unitaries in A that almost implement α.
- In other words, every automorphism of A is approximately inner.
- If in addition $A \cong B \otimes B$, then the half-flip $a \otimes b \mapsto b \otimes a$ is approximtaely inner, whence B has approximately inner half-flip, and thus so does A.
- This implies that A is nuclear, so KEP holds.

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