Ultrafilters as nonstandard points:
some new applications in Ramsey Theory

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Introduction

In combinatorics of numbers one finds deep and fruitful interactions among diverse *non-elementary* methods, including ergodic theory, Fourier analysis, algebra in the space of ultrafilters, (discrete) topological dynamics.

In the last years, also the techniques of nonstandard analysis have been successfully applied to that area of research, starting from Jin's Theorem on sumsets of 2000.

There are two main areas of application:

- Additive combinatorics:
  Density-dependent results for sets of integers and their generalizations (e.g. *amenable groups*).

$$\overline{d}(A) = \limsup_{n\to\infty} \frac{|A \cap \{1,\ldots,n\}|}{n}$$

- Ramsey Theory:
  Properties that are preserved under finite partitions.

DN - Goldbring - Lupini, *Nonstandard Methods in Ramsey Theory and Combinatorial Number Theory*, Lecture Notes in Mathematics Series, vol. 2239, Springer, to appear.

Plan of the talk

1. Quick introduction to Ramsey Theory.
2. Hyper-quick presentation of nonstandard analysis.
3. The hypernatural numbers $^*\mathbb{N}$ of nonstandard analysis as ultrafilters on $\mathbb{N}$, and primary examples of applications (proofs of *Ramsey's Theorem* and *Hindman's Theorem*).
4. Ramsey Theory of Diophantine equations by nonstandard methods.
5. Topological dynamics in $^*\mathbb{N}$ and a proof of *van der Waerden's Theorem*.

Schur's Theorem

In the early years of XX century, Issai Schur was working on *Fermat's last theorem* over finite fields.

Theorem (Schur 1916)

For every m, the equation $x^m + y^m = z^m$ has non-trivial solutions in $\mathbb{Z}/p\mathbb{Z}$ for all sufficiently large primes p.

The crucial combinatorial lemma that he used in the proof was the finite version of the following property:

Theorem (Schur)

In every finite coloring $\mathbb{N} = C_1 \cup \ldots \cup C_r$ there is a monochromatic Schur triple $a, b, a+b \in C_i$ (where $a \neq b$.)

van der Waerden Theorem

A fundamental result in combinatorial number theory was proved by Bartel van der Waerden at the age of 23.

Theorem (van der Waerden 1926)

In every finite coloring $\mathbb{N} = C_1 \cup \ldots \cup C_r$ *there are arbitrarily long monochromatic arithmetic progressions.*

Ramsey's theorem

Ramsey gave significant contributions to mathematics, economics and philosophy (he was Wittgenstein's supervisor).
He died at the age of 26.

At the end of the 1920's years, Frank Ramsey proved a result in mathematical logic about the decidability of a relevant fragment of first-order order logic (namely, the class of formulas whose prenex normal form have an $\exists\forall$ quantifier prefix and do not contain any function symbols).

In the proof, he needed a combinatorial property (now known as Ramsey's Theorem) that eventually became the cornerstone of a whole field of research known as *Ramsey Theory*.

Ramsey Theory

"Complete disorder is impossible": structure can be found in any "sufficiently large" set.

Theorem (Ramsey 1928)

Let X be infinite. For every finite coloring of the k-tuples $[X]^k = C_1 \cup \ldots \cup C_r$, there exists an infinite homogeneous set H, i.e., all k-tuples $[H]^k \subseteq C_i$ are monochromatic.

When $k = 1$, this theorem simply says in any finite partition of $\mathbb{N}$ one of the pieces must be infinite.

As a straight application of Ramsey's Theorem, one can prove

Theorem (Schur – Infinite version)

In every finite coloring $\mathbb{N} = C_1 \cup \ldots \cup C_r$ there is a monochromatic Schur triple $a, b, a+b$.

Proof.

Color any pair $\{x < y\} \in [\mathbb{N}]^2$ with the same color as $y - x$. By *Ramsey's Theorem*, we can pick an infinite homogeneous set H. Then for every $x < y < z$ in H where $y - z \neq z - y$, the following numbers form a monochromatic Schur triple:

$$a = y - x, \quad b = z - y, \quad a + b = z - x$$

□

Improving on Schur's theorem

- In any finite coloring of $\mathbb{N}$, can one find 3 elements $a < b < c$ such that all sums $a, b, c, a+b, a+c, b+c, a+b+c$ are monochromatic? More generally, can one find k-many elements with that additive property?

Theorem (Folkman 1970)

For every k and for every finite coloring $\mathbb{N} = C_1 \cup \ldots \cup C_r$ there exists $|X| = k$ such that all sums $x_1 + \ldots + x_n$ of distinct elements in X are monochromatic.

Hindman's Theorem

A few years later, Neil Hindman succeeded in extending Folkman's theorem to infinite X.

Theorem (Hindman 1974)

For every finite coloring of the natural numbers $\mathbb{N} = C_1 \cup \ldots \cup C_r$ there exists an infinite X such that all sums $x_1 + \ldots + x_n$ of distinct elements from X are monochromatic.

Partition regularity

A typical result in Ramsey theory has the following form:

Ramsey property

A property that cannot be destroyed by a finite partition. That is:

- If X has the "structural property" P, and $X = C_1 \cup \ldots \cup C_r$ is a finite partition (coloring), then one of the colors C_i preserves the "structural property" P.

Definition

A family $\mathcal{F}$ of sets of integers is partition regular (P.R.) if

$$A_1 \cup \ldots \cup A_n = A \in \mathcal{F} \implies \exists\, i \text{ s.t. } A_i \in \mathcal{F}$$

Examples of P.R. families

- $\{A \subseteq \mathbb{N} \mid A \text{ is infinite}\}$ is P.R.
- $\left\{A \subseteq \mathbb{N} \,\middle|\, \overline{d}(A) = \limsup_{n\to\infty} \frac{|A\cap[1,n]|}{n} > 0\right\}$ is P.R.
- $\left\{A \subseteq \mathbb{N} \,\middle|\, \sum_{a\in A} \frac{1}{a} = \infty\right\}$ is P.R.
- $\{A \subseteq \mathbb{N} \mid A \text{ contains arbitrarily long arithmetic progressions}\}$ is P.R.
- $\{A \subseteq \mathbb{N} \mid A \text{ is piecewise syndetic}\}$ is P.R.
  Piecewise syndetic = it has bounded gaps on arbitrarily large intervals.

Nonstandard analysis

Basically, nonstandard analysis consists of two properties:

1. STAR MAP:
   Every mathematical object of interest X is extended to an object ${}^*X$, called hyper-extension or nonstandard extension. (When X is infinite, ${}^*X$ is a *proper* extension.)

2. TRANSFER:
   ${}^*X$ is a sort of weakly isomorphic copy of X, in the sense that it satisfies exactly the same elementary properties as X.

The notion of "elementary property" is made precise by using the formal language of 1st order logic.

Transfer principle

If $P(A_1, \ldots, A_n)$ is any elementary property of $A_1, \ldots, A_n$ then

$$P(A_1, \ldots, A_n) \iff P({}^*A_1, \ldots, {}^*A_n)$$

Examples:

- The hyperintegers ${}^*\mathbb{Z}$ are a *discretely ordered ring*.
- The hyperreal numbers ${}^*\mathbb{R}$ are an *ordered field* that properly extends the real line $\mathbb{R}$.

$\mathbb{Z}$ and ${}^*\mathbb{Z}$, and similarly $\mathbb{R}$ and ${}^*\mathbb{R}$, cannot be distinguished by any elementary property.

- A property of X is elementary if it talks about elements of X ("first-order" property).
  E.g., the properties of ordered field are elementary properties of $\mathbb{R}$.

- A property of X is not elementary if it talks about subsets or functions of X ("second-order" property).
  E.g., the well-ordering property of $\mathbb{N}$ and the completeness property of $\mathbb{R}$ are not elementary.

The hypernatural numbers

Usually, nonstandard analysis focuses on the hyperreal numbers ${}^*\mathbb{R}$. For applications in combinatorial number theory, one focuses on the discrete setting given by the hyperintegers ${}^*\mathbb{Z}$.

${}^*\mathbb{Z}$ is a *discretely ordered ring* whose positive part are the hypernatural numbers ${}^*\mathbb{N}$.

$${}^*\mathbb{N} \;=\; \{\underbrace{1, 2, \ldots, n, \ldots}_{\text{finite numbers}} \quad \underbrace{\ldots, N-2, N-1, N, N+1, N+2, \ldots}_{\text{infinite numbers}}\}$$

Why NSA in combinatorics?

- Arguments of elementary finite combinatorics can be used in a hyperfinite setting to prove results about infinite sets of integers, also in the case of null asymptotic density.
- Nonstandard proofs for density-depending results usually work also in more general settings, such as amenable groups or semigroups.
- The nonstandard integers (or hyperintegers) ${}^*\mathbb{Z}$ may serve as a “bridge” between the *discrete* and the *continuum*.

- Tools from *analysis* and *measure theory*, such as *Birkhoff Ergodic Theorem* and *Lebesgue Density Theorem*, can be used in ${}^*\mathbb{Z}$.

- Hypernatural numbers can play the role of ultrafilters on $\mathbb{N}$ and be used in *Ramsey Theory* problems (*e.g.*, partition regularity of Diophantine equations).

- Hypernatural numbers ${}^*\mathbb{N}$ have a natural compact topology and one can consider the (discrete) topological dynamics in the system $({}^*\mathbb{N}, S)$ where $S : \xi \mapsto \xi + 1$ is the shift operator.

- Model-theoretic tools are available, most notably saturation. *E.g.*, saturation is needed for the *Loeb measure construction*.

The u-equivalence relation

Definition

For $\xi, \zeta \in {}^*\mathbb{N}$, we say that $\xi \underset{u}{\sim} \zeta$ are u-equivalent if they are indistinguishable by any "standard property":

- For every $A \subseteq \mathbb{N}$ one has either $\xi, \zeta \in {}^*A$ or $\xi, \zeta \notin {}^*A$.

- Any hypernatural number $\xi \in {}^*\mathbb{N}$ generates an ultrafilter on $\mathbb{N}$:

$$\mathfrak{U}_\xi = \{A \subseteq \mathbb{N} \mid \xi \in {}^*A\}.$$

- $\xi \underset{u}{\sim} \eta \iff \mathfrak{U}_\xi = \mathfrak{U}_\eta$.
- By *saturation*, every ultrafilter on $\mathbb{N}$ is generated by some $\xi \in {}^*\mathbb{N}$ (actually, by at least $\mathfrak{c}^+$-many ξ).
- In some sense, in our nonstandard setting every ultrafilter $\mathcal{U} = \mathfrak{U}_\xi$ is a *principal* ultrafilter.

Independent and indiscernible pairs

Definition

$(\alpha, \beta) \in {}^*\mathbb{N} \times {}^*\mathbb{N}$ is an independent pair if for every $A \subseteq \mathbb{N} \times \mathbb{N}$:

- $(\alpha, \beta) \in {}^*A \Longrightarrow (n, \beta) \in {}^*A$ for some $n \in \mathbb{N}$.

An indiscernible pair is an independent pair (α, β) where $\alpha \underset{u}{\sim} \beta$.

Fact

(α, β) is an independent pair if and only if

$$\mathfrak{U}_{(\alpha,\beta)} = \{A \subseteq \mathbb{N} \times \mathbb{N} \mid (\alpha, \beta) \in {}^*A\} = \mathfrak{U}_\alpha \otimes \mathfrak{U}_\beta.$$

By considering *iterated hyper-extensions*, one easily obtains indiscernible pairs $(\alpha, {}^*\alpha) \in {}^{**}\mathbb{N} \times {}^{**}\mathbb{N}$.

Ramsey's Theorem: a nonstandard proof

Theorem (Ramsey 1928 – Infinite version)

Let $[X]^k = C_1 \cup \ldots \cup C_r$ *be a finite coloring of the k-tuples of an infinite set* X*. Then there exists an infinite* homogeneous *set* H*, i.e. all k-tuples from* H *are monochromatic:* $[H]^k \subseteq C_i$.

Nonstandard proof ($k = 2$)

- Pick an indiscernible pair (α, β) and let $C = C_i$ be the color such that $\{\alpha, \beta\} \in {}^*C$.
- Then there exists $h_1 \in \mathbb{N}$ such that $\{h_1, \beta\} \in {}^*C$, and hence $\{h_1, \alpha\} \in {}^*C$.
- Let $\Gamma_1 := \{\{n, m\} \in C \mid n > h_1 \text{ and } \{h_1, m\} \in C\}$. Since $\{\alpha, \beta\} \in {}^*\Gamma_1$, we have $\{h_2, \beta\} \in {}^*\Gamma_1$ for some $h_2 \in \mathbb{N}$. So, $\{h_2, \beta\} \in {}^*C$, and hence $\{h_2, \alpha\} \in {}^*C$; $h_2 > h_1$; $\{h_1, h_2\} \in C$.
- Let $\Gamma_2 := \{\{n, m\} \in C \mid n > h_2 \text{ and } \{h_1, m\}, \{h_2, m\} \in C\}$. $\{\alpha, \beta\} \in {}^*\Gamma_1 \Rightarrow \{h_3, \beta\} \in {}^*\Gamma_1$ for some $h_3 \in \mathbb{N}$. So, $\{h_3, \beta\} \in {}^*C$, and hence $\{h_3, \alpha\} \in {}^*C$; $h_3 > h_2$; $\{h_1, h_3\}, \{h_2, h_3\} \in C$.
- Iterate the construction to define the infinite homogeneous set $H = \{h_i\}_{i \in \mathbb{N}}$ such that $[H]^2 \subseteq C$.

Ramsey Theory of Diophantine equations

Definition

An equation $F(X_1, \ldots, X_n) = 0$ is partition regular (PR) on $\mathbb{N}$ if for every finite coloring of $\mathbb{N}$ there exist a monochromatic solution, *i.e.* monochromatic elements $x_1, \ldots, x_n$ such that $F(x_1, \ldots, x_n) = 0$.

- Schur's Theorem: *In every finite coloring of $\mathbb{N}$ one finds monochromatic triples $a, b, a + b$.*
  So, the equation $X + Y = Z$ is PR.
- van der Waerden's Theorem: *In every finite coloring of $\mathbb{N}$ one finds arbitrarily long arithmetic progressions.*
  So, the equation $X + Y = 2Z$ is PR.
  (Solutions are the 3-term arithmetic progressions.)
- Not all linear equations are PR! *E.g.*, $X + Y = 3Z$ is *not* PR.

The problem of partition regularity of linear Diophantine equations was completely solved by Richard Rado.

Theorem (Rado 1933)

The Diophantine equation $c_1X_1 + \ldots + c_nX_n = 0$ *is PR if and only if* $\sum_{i \in I} c_i = 0$ *for some (nonempty)* $I \subseteq \{1, \ldots, n\}$.

Numerous PR results have been proved for linear equations (especially about infinite systems), but the study on the nonlinear case has been sporadic, until very recently.

Nonstandard characterization

Nonstandard characterization

An equation $F(X_1, \ldots, X_n) = 0$ is partition regular on $\mathbb{N}$ if there exist $\xi_1 \underset{u}{\sim} \ldots \underset{u}{\sim} \xi_n$ in ${}^*\mathbb{N}$ such that ${}^*F(\xi_1, \ldots, \xi_n) = 0$.

So, *Schur's Theorem* states the existence of $\xi \neq \zeta$ in ${}^*\mathbb{N}$ such that:

$$\xi \underset{u}{\sim} \zeta \underset{u}{\sim} \xi + \zeta$$

Idempotent elements

Definition

An element $\nu \in {}^*\mathbb{N}$ is idempotent if the generated filter $\mathfrak{U}_\nu = \mathfrak{U}_\nu \oplus \mathfrak{U}_\nu$ is idempotent, *i.e.*,

$$A \in \mathfrak{U}_\nu \Leftrightarrow \{n \mid A - n \in \mathfrak{U}_\nu\} \in \mathfrak{U}_\nu .$$

By using iterated hyper-extension, one obtains the following

Characterization of idempotents

An element $\nu \in {}^*\mathbb{N}$ is idempotent $\Longleftrightarrow \nu \underset{u}{\sim} \nu + {}^*\nu$.

Theorem (Bergelson-Hindman 1990)

Let $\mathcal{U}$ be an idempotent ultrafilter. Then every $A \in 2\mathcal{U} \oplus \mathcal{U}$ contains an arithmetic progression of length 3. In consequence, $X - 2Y + Z = 0$ is PR.

Nonstandard proof.

Let ν be the idempotent element such that $\mathcal{U} = \mathfrak{U}_\nu$. Then

- $\xi = 2\nu + \quad + {}^{**}\nu$
- $\zeta = 2\nu + {}^{*}\nu + {}^{**}\nu$
- $\vartheta = 2\nu + 2{}^{*}\nu + {}^{**}\nu$

are *u*-equivalent numbers of ${}^{***}\mathbb{N}$ that generate $2\mathcal{U} \oplus \mathcal{U}$.
For every $A \in 2\mathcal{U} \oplus \mathcal{U}$, the elements $\xi, \zeta, \vartheta \in {}^{***}A$ form a 3-term arithmetic progression and so, by *transfer*, there exists a 3-term arithmetic progression in A.

The previous nonstandard argument admits a strong generalization.

Theorem ("Idempotent Ultrafilter Rado" - DN 2015)

Let $c_1X_1 + \ldots + c_nX_n = 0$ be a Diophantine equation with $n \geq 3$. If $c_1 + \ldots + c_n = 0$ then there exist integers $a_1, \ldots, a_{n-1}$ such that for every idempotent ultrafilter $\mathcal{U}$, the ultrafilter

$$\mathcal{V} = a_1\mathcal{U} \oplus \ldots \oplus a_{n-1}\mathcal{U}$$

is an injective PR-witness, i.e. for every $A \in \mathcal{V}$ there exist distinct $x_1, \ldots, x_n \in A$ with $c_1x_1 + \ldots + c_kx_k = 0$.

Properties of u-equivalence

Theorem

(1) *If* $\xi \underset{u}{\sim} \zeta$ *and* $\xi \neq \zeta$ *then* $|\xi - \zeta|$ *is infinite.*

Let $f : \mathbb{N} \to \mathbb{N}$ *be any function. Then*

(2) *If* $\xi \underset{u}{\sim} \zeta$ *then* ${}^*f(\xi) \underset{u}{\sim} {}^*f(\zeta)$.

(3) *If* ${}^*f(\xi) \sim \xi$ *then* ${}^*f(\xi) = \xi$.

Remark

The last property corresponds to the following basic (non-trivial) fact about ultrafilters:

$$f(\mathcal{U}) = \mathcal{U} \implies \{n \mid f(n) = n\} \in \mathcal{U}$$

Examples of nonlinear equations

In the last few years, research on the partition regularity of nonlinear equations has made significant progress. In particular, the nonstandard framework of hypernatural numbers and u-equivalence relation, combined with the use of idempotent elements, revealed quite effective.

To introduce the method, let us see in details two simple examples.

$\longrightarrow$ Luperi Baglini's talk.

Theorem (Csikvári - Gyarmati - Sárkožy 2012)

The equation $X + Y = Z^2$ is not partition regular (except for $X = Y = Z = 2$).

Proof. By contradiction, let $\alpha \underset{u}{\sim} \beta \underset{u}{\sim} \gamma$ be such that $\alpha + \beta = \gamma^2$. Let $f : \mathbb{N} \to \mathbb{N}_0$ be the function defined by $2^{f(n)} \leq n < 2^{f(n)+1}$, and set $a := f(\alpha)$. Notice that a is infinite, as otherwise α would be finite and $\alpha \sim \beta \sim \gamma$ would imply $\alpha = \beta = \gamma = 2$.
Now assume WLOG $\alpha \geq \beta$. Then

$$2^a \leq \alpha < \alpha + \beta = \gamma^2 \leq 2\alpha < 2 \cdot 2^{a+1} \;\Rightarrow\; 2^{\frac{a}{2}} < \gamma < 2^{\frac{a}{2}+1}.$$

So, either $f(\gamma) = \lfloor \frac{a}{2} \rfloor$ or $f(\gamma) = \lfloor \frac{a}{2} \rfloor + 1$. Since $a = f(\alpha) \sim f(\gamma)$, we have either $a \sim \lfloor \frac{a}{2} \rfloor$ or $a \sim \lfloor \frac{a}{2} \rfloor + 1$. But then either $a = \lfloor \frac{a}{2} \rfloor$ (which cannot occur since a is positive), or $a = \lfloor \frac{a}{2} \rfloor + 1$ (and hence $a = 1$ or $a = 2$, contradicting a infinite).

Theorem (DN)

The equation $X^2 + Y^2 = Z$ is not partition regular.

Proof. By contradiction, let $\alpha \underset{u}{\sim} \beta \underset{u}{\sim} \gamma$ be such that $\alpha^2 + \beta^2 = \gamma$. α, β, γ are even numbers, since they cannot all be odd; then

$$\alpha = 2^a \alpha_1, \quad \beta = 2^b \beta_1, \quad \gamma = 2^c \gamma_1$$

where $a \underset{u}{\sim} b \underset{u}{\sim} c$ are positive and $\alpha_1 \underset{u}{\sim} \beta_1 \underset{u}{\sim} \gamma_1$ are odd.
Case 1: If $a < b$ then $2^{2a}(\alpha_1^2 + 2^{2b-2a}\beta_1^2) = 2^c \gamma_1$. Since $\alpha_1^2 + 2^{2b-2a}\beta_1^2$ and γ_1 are odd, it follows that $2a = c \underset{u}{\sim} a$. But then $2a = a$ and so $a = 0$, a contradiction. (Same proof if $b > a$.)
Case 2: If $a = b$ then $2^{2a}(\alpha_1^2 + \beta_1^2) = 2^c \gamma_1$. Since α_1, β_1 are odd, $\alpha_1^2 + \beta_1^2 \equiv 2 \mod 4$, and so $2^c \gamma_1 = 2^{2a+1}\alpha_2$ where α_2 is odd. But then $2a + 1 = c \underset{u}{\sim} a$ and so $2a + 1 = a$, a contradiction.

OPEN QUESTION

Is the Pythagorean equation partition regular?

$$X^2 + Y^2 = Z^2$$

- $X^2 + Y^2 = Z^2$ is PR for 2-colorings (computer-assisted proof by Heule - Kullmann - Marek 2016).
- $X + Y = Z$ is PR (Schur's Theorem).
- $X^2 + Y = Z$ is PR (corollary of Sarkozy - Fürstenberg 1978).
- $X + Y = Z^2$ is *not* PR (Csikvári - Gyarmati - Sárközy 2012), but it is PR for 2-colorings (Green-Lindqvist 2016).
- $X^2 + Y^2 = Z$ is *not* PR (DN 2016).
- $X^2 + Y = Z^2$ is PR (Moreira 2016)
- $X_1X_2 + Y^2 = Z_1Z_2$ is PR (DN - Luperi Baglini 2017).
- $X^n + Y^n = Z^k$ where $k \neq n$ are not PR (DN - Riggio 2016).
- $X^n + Y^n = Z^n$ where $n > 2$ has no solutions (Fermat's Theorem!).

${}^*\mathbb{N}$ as a topological space

In the following we will assume the property of $\mathfrak{c}^+$-*enlargement*.

There is a natural topology τ_S on ${}^*\mathbb{N}$, named the S-topology (standard topology) whose basic (cl)open sets are the hyper-extensions: $\{{}^*A \mid A \subseteq \mathbb{N}\}$.

While the Stone-Čech compactification $\beta\mathbb{N}$ is – in a precise sense – the largest possible Hausdorff compactification of the discrete space $\mathbb{N}$, the hypernatural numbers ${}^*\mathbb{N}$ are an even larger compactification that retains the nice algebraic properties of the natural numbers.

1. ${}^*\mathbb{N}$ is compact and completely regular (but not Hausdorff). [X is *completely regular* if for every closed C and $x \notin C$ there is a continuous $f : X \to \mathbb{R}$ with $f(x) = 0$ and $f \equiv 1$ on C.]

2. $\mathbb{N}$ is dense in ${}^*\mathbb{N}$.

3. Every $f : \mathbb{N} \to K$ where K is compact Hausdorff is naturally extended to a continuous $\overline{f} : {}^*\mathbb{N} \to K$. [$\overline{f}(\xi)$ be the unique $x \in K$ that is "near" to ${}^*f(\xi)$, in the sense that ${}^*f(\xi) \in {}^*U$ for all neighborhoods U of x.]

4. By means of hyper-extensions, every function $f : \mathbb{N}^k \to \mathbb{N}$ is extended to a continuous ${}^*f : {}^*\mathbb{N}^k \to {}^*\mathbb{N}$ that satisfies the same "elementary properties" as f. In particular, sum and product on $\mathbb{N}$ are extended to commutative operations on ${}^*\mathbb{N}$.

Let us remark that ${}^*\mathbb{N}$ is not Hausdorff, and in fact, not even T_0.

Notice that two elements $\xi, \zeta \in {}^*\mathbb{N}$ are *not* separated by the S-topology precisely when they generate the same ultrafilter $\mathfrak{U}_\xi = \mathfrak{U}_\zeta$.

In case of topological spaces X that are not T_0, one considers the Kolmogorov equivalence $\equiv_K$ between points that are "topologically indistinguishable", that is:

$x \equiv_K y$ if and only if x and y have exactly the same neighborhoods.

The quotient space $X/\equiv_K$, called the Kolmogorov quotient of X, is of course always a T_0-space.

In the case of ${}^*\mathbb{N}$, the Kolmogorov equivalence coincides with the u-equivalence: $\xi \underset{u}{\sim} \eta \Leftrightarrow \mathfrak{U}_\xi = \mathfrak{U}_\eta$, and the following holds:

- The Kolmogorov quotient ${}^*\mathbb{N}/\underset{u}{\sim}$ is (isomorphic to) the Stone-Čech compactification $\beta\mathbb{N}$.

OPEN QUESTION

Are there hypernatural numbers ${}^*\mathbb{N}$ where the S-topology is Hausdorff ? (That is, where $\xi \underset{u}{\sim} \zeta \Leftrightarrow \xi = \zeta$.)

The question above has an equivalent reformulation in terms of ultrafilters.

- Are there ultrafilters $\mathcal{U}$ on $\mathbb{N}$ with the following property?

$$f(\mathcal{U}) = g(\mathcal{U}) \implies \{n \mid f(n) = g(n)\} \in \mathcal{U}.$$

The property is consistent with ZFC; indeed, it is satisfied when $\mathcal{U}$ is selective (and selective ultrafilters are consistent). However, it is still unknown whether Hausdorff ultrafilters exist in ZFC.

Dynamics in ${}^*\mathbb{N}$: the shift operator

Let us consider the shift operator S on the compact space ${}^*\mathbb{N}$:

$$S : \nu \longmapsto \nu + 1.$$

- $({}^*\mathbb{N}, S)$ is a discrete topological dynamical system.
- The (positive) orbit of a point $\nu \in {}^*\mathbb{N}$ is the set

$$\mathrm{orb}(\nu) := \{(\underbrace{S \circ \ldots \circ S}_{k-\text{times}})(\nu) \mid k \in \mathbb{N}\} = \{\nu + k \mid k \in \mathbb{N}\}$$

- $\nu \in {}^*\mathbb{N}$ is recurrent if $\mathrm{orb}(\nu) \cap {}^*A \neq \emptyset$ for every neighbourhood ${}^*A$ of ν, that is:

$$\nu \in {}^*A \implies \nu + k \in {}^*A \text{ for some positive integer } k.$$

Self-recurrent points

Definition

A point $\nu \in {}^*\mathbb{N}$ is self-recurrent if:

$$\nu \in {}^*A \implies \nu + a \in {}^*A \text{ for some } a \in A.$$

- A point ν is self-recurrent if and only if it generates an idempotent ultrafilter $\mathfrak{U}_\nu = \mathfrak{U}_\nu \oplus \mathfrak{U}_\nu$.

By using a self-recurrent point, one can give a short proof of Hindman's Theorem.

Theorem (Hindman 1974)

For every finite coloring of $\mathbb{N}$ there exists an infinite sequence (x_i) such that all finite sums $FS(X) = \{x_F := \sum_{i \in F} x_i \mid F \subset \mathbb{N} \text{ finite}\}$ are monochromatic.

Hindman's Theorem via self-recurrence

- Pick a self-recurrent point $\nu \in {}^*\mathbb{N}$ and let C be the color such that $\nu \in {}^*C$.
- Pick $x_1 \in C$ with $\nu + x_1 \in {}^*C$.
- Inductively, assume that we found $x_1 < \ldots < x_n$ such that $x_F = \sum_{i \in F} x_i \in C$ and $\nu + x_F \in {}^*C$ for every $F \subseteq \{1, \ldots, n\}$.
- Denote $C - x_F = \{n \mid n + x_F \in C\}$.
- Since $\nu \in \bigcap_F {}^*(C - x_F) = {}^*\bigcap_F (C - x_F)$, we can pick $x_{n+1} \in \bigcap_F (C - x_F)$ such that $\nu + x_{n+1} \in \bigcap_F {}^*(C - x_F)$. Then, x_G and $\nu + x_G \in {}^*C$ for every $G \subseteq \{1, \ldots, n+1\}$.

Definition

A point $\nu \in {}^*\mathbb{N}$ is uniformly recurrent (almost periodic) if $\mathrm{orb}(\nu) \cap {}^*A$ is syndetic for every neighborhood ${}^*A$.

A set $B \subseteq \mathbb{N}$ is syndetic if one has a "bounded return time", that is, there exists N such that for every $n \in \mathbb{N}$ there exists $k \leq N$ with $n + k \in B$.

If the space X is compact Hausdorff, then by *Zorn's Lemma* it is proved that in every dynamical system (X, T) one finds uniformly recurrent points. The same proof also works in the non-Hausdorff case, provided the space is *regular*, and so:

- In the dynamical system $({}^*\mathbb{N}, S)$ there exist uniformly recurrent points.

van der Waerden's Theorem

Uniformly recurrent points in the dynamical system $({}^*\mathbb{N}, S)$ can be used to give a proof of the classic van der Waerden's Theorem.

Theorem (van der Waerden 1926)

In every finite coloring $\mathbb{N} = C_1 \cup \ldots \cup C_r$ there are arbitrarily long monochromatic arithmetic progressions.

We will prove the following:

Theorem

Let ν be a uniformly recurrent point of $({}^*\mathbb{N}, S)$. If $\nu \in {}^*A$ then orb$(\nu) \cap {}^*A$ contains arbitrarily long arithmetic progressions. So, by transfer, A contains arbitrarily long arithmetic progressions.

Given a finite coloring $\mathbb{N} = C_1 \cup \ldots \cup C_r$, pick the color $C = C_i$ such that $\nu \in {}^*C$. Then C contains arbitrarily long arithmetic progressions.

Nonstandard proof of vdW's Thm - I

By induction on k, we prove that if $\nu + m \in {}^*A$ for some $m \geq 0$, then there exists an arithmetic progression of length k in $\mathrm{orb}(\nu + m) \cap {}^*A$. The basis $k = 1$ is trivial ($A - m$ is syndetic).

Successor step $k + 1$: By the hypothesis, $\{n \in \mathbb{N} \mid \nu + n \in {}^*A\}$ is syndetic, and so there exists $N \in \mathbb{N}$ such that for every $x \in \mathrm{orb}(\nu)$ one has $x + t \in {}^*A$ for some $t \leq N$.

Pick $t_0 \leq N$ such that $\nu + t_0 \in {}^*A$. By the inductive hypothesis there exist ℓ_1 and y_1 such that $\nu + \ell_1 + t_0 + iy_1 \in {}^*A$ for $i = 1, \ldots, k$.

Pick $t_1 \leq N$ with $\nu + \ell_1 + t_1 \in {}^*A$. If $t_1 = t_0$, we have found the desired arithmetic progression of length $k + 1$.

Nonstandard proof of vdW's Thm - II

Otherwise, consider the set

$$A_1 = (A - t_1) \cap \bigcap_{i=1}^{k} (A - t_0 - iy_1).$$

Since $\nu + \ell_1 \in {}^*A_1$, by the inductive hypothesis there exist ℓ_2 and y_2 such that $\nu + \ell_1 + \ell_2 + iy_2 \in {}^*A_1$ for $i = 1, \ldots, k$. So,

$$\nu + \ell_1 + \ell_2 + t_1 + iy_2,\ \nu + \ell_1 + \ell_2 + t_0 + i(y_1 + y_2) \in {}^*A.$$

Pick $t_2 \leq N$ such that $\alpha + \ell_1 + \ell_2 + t_2 \in {}^*A$. Notice that if $t_2 = t_1$ or $t_2 = t_0$ then we have obtained the desired arithmetic progression of length $k + 1$.

Nonstandard proof of vdW's Thm - III

Otherwise, consider the set

$$A_2 = (A - t_2) \cap \bigcap_{i=1}^{k} (A - t_1 - iy_2) \cap \bigcap_{i=1}^{k} (A - t_0 - i(y_1 + y_2)).$$

Since $\nu + \ell_1 + \ell_2 \in {}^*A_2$, we can apply the inductive hypothesis and proceed as above.

Now $\{1, \ldots, N\}$ is finite, and so after finitely many steps we find $t_n = t_m$ where $n > m$, and we obtain the following arithmetic progression of length $k + 1$ in ${}^*A$:

$$\nu + \ell_1 + \ldots + \ell_n + t_n + i(y_{m+1} + \ldots + y_n) \quad i = 0, 1, \ldots, k.$$


The End

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