Ultrafilters as nonstandard points: some new applications in Ramsey Theory

Mauro Di Nasso – Università di Pisa – Italia

AMS Sectional Meeting - Special Session: “Applications of Ultrafilters and Nonstandard Methods” Honolulu, March 23, 2019
In combinatorics of numbers one finds deep and fruitful interactions among diverse non-elementary methods, including ergodic theory, Fourier analysis, algebra in the space of ultrafilters, (discrete) topological dynamics.

In the last years, also the techniques of nonstandard analysis have been successfully applied to that area of research, starting from Jin's Theorem on sumsets of 2000.
There are two main areas of application:

- **Additive combinatorics:**
  Density-dependent results for sets of integers and their generalizations (e.g. *amenable groups*).

  \[
  \overline{d}(A) = \limsup_{n \to \infty} \frac{|A \cap \{1, \ldots, n\}|}{n}
  \]

- **Ramsey Theory:**
  Properties that are preserved under finite partitions.

Plan of the talk

1. Quick introduction to Ramsey Theory.
2. Hyper-quick presentation of nonstandard analysis.
3. The hypernatural numbers $^*\mathbb{N}$ of nonstandard analysis as ultrafilters on $\mathbb{N}$, and primary examples of applications (proofs of Ramsey’s Theorem and Hindman’s Theorem).
5. Topological dynamics in $^*\mathbb{N}$ and a proof of van der Waerden’s Theorem.
Schur’s Theorem

In the early years of XX century, Issai Schur was working on Fermat’s last theorem over finite fields.

**Theorem (Schur 1916)**

For every $m$, the equation $x^m + y^m = z^m$ has non-trivial solutions in $\mathbb{Z}/p\mathbb{Z}$ for all sufficiently large primes $p$.

The crucial combinatorial lemma that he used in the proof was the finite version of the following property:

**Theorem (Schur)**

In every finite coloring $\mathbb{N} = C_1 \cup \ldots \cup C_r$ there is a monochromatic Schur triple $a, b, a + b \in C_i$ (where $a \neq b$.)
Part 1: Ramsey Theory

van der Waerden Theorem

A fundamental result in combinatorial number theory was proved by Bartel van der Waerden at the age of 23.

**Theorem (van der Waerden 1926)**

In every finite coloring \( \mathbb{N} = C_1 \cup \ldots \cup C_r \) there are arbitrarily long monochromatic arithmetic progressions.
Ramsey’s theorem

Ramsey gave significant contributions to mathematics, economics and philosophy (he was Wittgenstein’s supervisor). He died at the age of 26.

At the end of the 1920’s years, Frank Ramsey proved a result in mathematical logic about the decidability of a relevant fragment of first-order order logic (namely, the class of formulas whose prenex normal form have an $\exists \forall$ quantifier prefix and do not contain any function symbols).
In the proof, he needed a combinatorial property (now known as Ramsey’s Theorem) that eventually became the cornerstone of a whole field of research known as Ramsey Theory.

**Ramsey Theory**

“Complete disorder is impossible”: structure can be found in any “sufficiently large” set.

**Theorem (Ramsey 1928)**

Let $X$ be infinite. For every finite coloring of the $k$-tuples $[X]^k = C_1 \cup \ldots \cup C_r$, there exists an infinite homogeneous set $H$, i.e., all $k$-tuples $[H]^k \subseteq C_i$ are monochromatic.

When $k = 1$, this theorem simply says in any finite partition of $\mathbb{N}$ one of the pieces must be infinite.
As a straight application of Ramsey's Theorem, one can prove

**Theorem (Schur – Infinite version)**

In every finite coloring $\mathbb{N} = C_1 \cup \ldots \cup C_r$ there is a monochromatic Schur triple $a, b, a + b$.

**Proof.**

Color any pair $\{x < y\} \in [\mathbb{N}]^2$ with the same color as $y - x$. By *Ramsey’s Theorem*, we can pick an infinite homogeneous set $H$. Then for every $x < y < z$ in $H$ where $y - z \neq z - y$, the following numbers form a monochromatic Schur triple:

$$a = y - x, \quad b = z - y, \quad a + b = z - x$$
In any finite coloring of \( \mathbb{N} \), can one find 3 elements \( a < b < c \) such that all sums \( a, b, c, a + b, a + c, b + c, a + b + c \) are monochromatic? More generally, can one find \( k \)-many elements with that additive property?

**Theorem (Folkman 1970)**

For every \( k \) and for every finite coloring \( \mathbb{N} = C_1 \cup \ldots \cup C_r \) there exists \( |X| = k \) such that all sums \( x_1 + \ldots + x_n \) of distinct elements in \( X \) are monochromatic.
Hindman’s Theorem

A few years later, Neil Hindman succeeded in extending Folkman’s theorem to infinite $X$.

**Theorem (Hindman 1974)**

For every finite coloring of the natural numbers $\mathbb{N} = C_1 \cup \ldots \cup C_r$ there exists an infinite $X$ such that all sums $x_1 + \ldots + x_n$ of distinct elements from $X$ are monochromatic.
A typical result in Ramsey theory has the following form:

**Definition**
A family $\mathcal{F}$ of sets of integers is **partition regular** (P.R.) if

$$A_1 \cup \ldots \cup A_n = A \in \mathcal{F} \implies \exists i \text{ s.t. } A_i \in \mathcal{F}$$
Examples of P.R. families

- \( \{ A \subseteq \mathbb{N} \mid A \text{ is infinite} \} \) is P.R.
- \( \left\{ A \subseteq \mathbb{N} \mid \bar{d}(A) = \limsup_{n \to \infty} \frac{|A \cap [1,n]|}{n} > 0 \right\} \) is P.R.
- \( \left\{ A \subseteq \mathbb{N} \mid \sum_{a \in A} \frac{1}{a} = \infty \right\} \) is P.R.
- \( \{ A \subseteq \mathbb{N} \mid A \text{ contains arbitrarily long arithmetic progressions} \} \) is P.R.
- \( \{ A \subseteq \mathbb{N} \mid A \text{ is piecewise syndetic} \} \) is P.R.  
  \textbf{Piecewise syndetic} = it has bounded gaps on arbitrarily large intervals.
Basically, nonstandard analysis consists of two properties:

1. **STAR MAP:**
   Every mathematical object of interest $X$ is extended to an object $^*X$, called hyper-extension or nonstandard extension. (When $X$ is infinite, $^*X$ is a proper extension.)

2. **TRANSFER:**
   $^*X$ is a sort of weakly isomorphic copy of $X$, in the sense that it satisfies exactly the same elementary properties as $X$.

The notion of “elementary property” is made precise by using the formal language of 1st order logic.
Transfer principle

If $P(A_1, \ldots, A_n)$ is any elementary property of $A_1, \ldots, A_n$ then

$$P(A_1, \ldots, A_n) \iff P(*A_1, \ldots, *A_n)$$

Examples:

- The hyperintegers $\mathbb{Z}$ are a discretely ordered ring.
- The hyperreal numbers $\mathbb{R}$ are an ordered field that properly extends the real line $\mathbb{R}$.

$\mathbb{Z}$ and $\mathbb{Z}$, and similarly $\mathbb{R}$ and $\mathbb{R}$, cannot be distinguished by any elementary property.
• A property of $X$ is **elementary** if it talks about elements of $X$ ("first-order" property).
  *E.g.*, the properties of ordered field are elementary properties of $\mathbb{R}$.

• A property of $X$ is **not elementary** if it talks about subsets or functions of $X$ ("second-order" property).
  *E.g.*, the well-ordering property of $\mathbb{N}$ and the completeness property of $\mathbb{R}$ are not elementary.
The hypernatural numbers

Usually, nonstandard analysis focuses on the hyperreal numbers $\mathbb{R}^\ast$. For applications in combinatorial number theory, one focuses on the discrete setting given by the hyperintegers $\mathbb{Z}^\ast$.

$\mathbb{Z}^\ast$ is a discretely ordered ring whose positive part are the hypernatural numbers $\mathbb{N}^\ast$.

$$\mathbb{N}^\ast = \{ 1, 2, \ldots, n, \ldots \} \quad \text{finite numbers} \quad \ldots, N - 2, N - 1, N, N + 1, N + 2, \ldots$$

infinite numbers
Why NSA in combinatorics?

- Arguments of elementary finite combinatorics can be used in a hyperfinite setting to prove results about infinite sets of integers, also in the case of null asymptotic density.

- Nonstandard proofs for density-depending results usually work also in more general settings, such as amenable groups or semigroups.

- The nonstandard integers (or hyperintegers) \(*\mathbb{Z}\) may serve as a “bridge” between the discrete and the continuum.
• Tools from analysis and measure theory, such as Birkhoff Ergodic Theorem and Lebesgue Density Theorem, can be used in $\mathbb{N}$. 

• Hypernatural numbers can play the role of ultrafilters on $\mathbb{N}$ and be used in Ramsey Theory problems (e.g., partition regularity of Diophantine equations).

• Hypernatural numbers $^*\mathbb{N}$ have a natural compact topology and one can consider the (discrete) topological dynamics in the system ($^*\mathbb{N}, S$) where $S : \xi \mapsto \xi + 1$ is the shift operator.

• Model-theoretic tools are available, most notably saturation. E.g., saturation is needed for the Loeb measure construction.
The \( u \)-equivalence relation

**Definition**

For \( \xi, \zeta \in *\mathbb{N} \), we say that \( \xi \sim \zeta \) are \textit{\( u \)-equivalent} if they are indistinguishable by any “standard property”:

- For every \( A \subseteq \mathbb{N} \) one has either \( \xi, \zeta \in *A \) or \( \xi, \zeta \notin *A \).

- Any \textit{hypernational number} \( \xi \in *\mathbb{N} \) generates an ultrafilter on \( \mathbb{N} \):
  \[
  \mathcal{U}_\xi = \{ A \subseteq \mathbb{N} \mid \xi \in *A \}.
  \]

- \( \xi \sim \eta \iff \mathcal{U}_\xi = \mathcal{U}_\eta \).

- By \textit{saturation}, every ultrafilter on \( \mathbb{N} \) is generated by some \( \xi \in *\mathbb{N} \) (actually, by at least \( \mathfrak{c}^+ \)-many \( \xi \)).

- In some sense, in our nonstandard setting every ultrafilter \( \mathcal{U} = \mathcal{U}_\xi \) is a \textit{principal} ultrafilter.
Independent and indiscernible pairs

**Definition**

\((\alpha, \beta) \in {}^*\mathbb{N} \times {}^*\mathbb{N}\) is an **independent pair** if for every \(A \subseteq \mathbb{N} \times \mathbb{N}\):

- \((\alpha, \beta) \in {}^*A \implies (n, \beta) \in {}^*A\) for some \(n \in \mathbb{N}\).

An **indiscernible pair** is an independent pair \((\alpha, \beta)\) where \(\alpha \sim \beta\).

**Fact**

\((\alpha, \beta)\) is an **independent pair** if and only if

\[
\mathcal{U}_{(\alpha, \beta)} = \{ A \subseteq \mathbb{N} \times \mathbb{N} \mid (\alpha, \beta) \in {}^* A \} = \mathcal{U}_\alpha \otimes \mathcal{U}_\beta.
\]

By considering **iterated hyper-extensions**, one easily obtains indiscernible pairs \((\alpha, {}^*\alpha) \in {}^{**}\mathbb{N} \times {}^{**}\mathbb{N}\).
Ramsey’s Theorem: a nonstandard proof

**Theorem (Ramsey 1928 – Infinite version)**

Let $[X]^k = C_1 \cup \ldots \cup C_r$ be a finite coloring of the $k$-tuples of an infinite set $X$. Then there exists an infinite homogeneous set $H$, i.e. all $k$-tuples from $H$ are monochromatic: $[H]^k \subseteq C_i$. 
Nonstandard proof \((k = 2)\)

- Pick an indiscernible pair \((\alpha, \beta)\) and let \(C = C_i\) be the color such that \(\{\alpha, \beta\} \in \ast C\).

- Then there exists \(h_1 \in \mathbb{N}\) such that \(\{h_1, \beta\} \in \ast C\), and hence \(\{h_1, \alpha\} \in \ast C\).

- Let \(\Gamma_1 := \{(n, m) \in C \mid n > h_1 \text{ and } \{h_1, m\} \in C\}\). Since \(\{\alpha, \beta\} \in \ast \Gamma_1\), we have \(\{h_2, \beta\} \in \ast \Gamma_1\) for some \(h_2 \in \mathbb{N}\). So, \(\{h_2, \beta\} \in \ast C\), and hence \(\{h_2, \alpha\} \in \ast C; h_2 > h_1; \{h_1, h_2\} \in C\).

- Let \(\Gamma_2 := \{(n, m) \in C \mid n > h_2 \text{ and } \{h_1, m\}, \{h_2, m\} \in C\}\). \(\{\alpha, \beta\} \in \ast \Gamma_1 \Rightarrow \{h_3, \beta\} \in \ast \Gamma_1\) for some \(h_3 \in \mathbb{N}\). So, \(\{h_3, \beta\} \in \ast C\), and hence \(\{h_3, \alpha\} \in \ast C; h_3 > h_2; \{h_1, h_3\}, \{h_2, h_3\} \in C\).

- Iterate the construction to define the infinite homogeneous set \(H = \{h_i\}_{i \in \mathbb{N}}\) such that \([H]^2 \subseteq C\).
Part 4: Ramsey Theory of Diophantine equations

Ramsey Theory of Diophantine equations

**Definition**
An equation $F(X_1, \ldots, X_n) = 0$ is partition regular (PR) on $\mathbb{N}$ if for every finite coloring of $\mathbb{N}$ there exist a monochromatic solution, i.e. monochromatic elements $x_1, \ldots, x_n$ such that $F(x_1, \ldots, x_n) = 0$.

- **Schur’s Theorem**: In every finite coloring of $\mathbb{N}$ one finds monochromatic triples $a, b, a + b$.
  So, the equation $X + Y = Z$ is PR.

- **van der Waerden’s Theorem**: In every finite coloring of $\mathbb{N}$ one finds arbitrarily long arithmetic progressions.
  So, the equation $X + Y = 2Z$ is PR.
  (Solutions are the 3-term arithmetic progressions.)

- Not all linear equations are PR! *E.g.*, $X + Y = 3Z$ is not PR.
Part 4: Ramsey Theory of Diophantine equations

The problem of partition regularity of linear Diophantine equations was completely solved by Richard Rado.

**Theorem (Rado 1933)**

The Diophantine equation $c_1X_1 + \ldots + c_nX_n = 0$ is PR if and only if $\sum_{i \in I} c_i = 0$ for some (nonempty) $I \subseteq \{1, \ldots, n\}$.

Numerous PR results have been proved for linear equations (especially about infinite systems), but the study on the nonlinear case has been sporadic, until very recently.
An equation $F(X_1, \ldots, X_n) = 0$ is partition regular on $\mathbb{N}$ if there exist $\xi_1 \sim \ldots \sim \xi_n$ in $\mathbb{N}$ such that $F(\xi_1, \ldots, \xi_n) = 0$.

So, Schur’s Theorem states the existence of $\xi \neq \zeta$ in $\mathbb{N}$ such that:

$$\xi \sim \zeta \sim \xi + \zeta$$
Idempotent elements

**Definition**

An element $\nu \in ^*\mathbb{N}$ is idempotent if the generated filter $\mathcal{U}_\nu = \mathcal{U}_\nu \oplus \mathcal{U}_\nu$ is idempotent, i.e.,

$$A \in \mathcal{U}_\nu \iff \{n \mid A - n \in \mathcal{U}_\nu\} \in \mathcal{U}_\nu.$$  

By using iterated hyper-extension, one obtains the following

**Characterization of idempotents**

An element $\nu \in ^*\mathbb{N}$ is idempotent $\iff \nu \sim \nu + ^*\nu$. 
Theorem (Bergelson-Hindman 1990)

Let $\mathcal{U}$ be an idempotent ultrafilter. Then every $A \in 2\mathcal{U} \oplus \mathcal{U}$ contains an arithmetic progression of length 3. In consequence, $X - 2Y + Z = 0$ is PR.

Nonstandard proof.

Let $\nu$ be the idempotent element such that $\mathcal{U} = \mathcal{U}_\nu$. Then

- $\xi = 2\nu + ++**\nu$
- $\zeta = 2\nu + *\nu + **\nu$
- $\vartheta = 2\nu + 2*\nu + **\nu$

are $u$-equivalent numbers of $***\mathbb{N}$ that generate $2\mathcal{U} \oplus \mathcal{U}$. For every $A \in 2\mathcal{U} \oplus \mathcal{U}$, the elements $\xi, \zeta, \vartheta \in ***A$ form a 3-term arithmetic progression and so, by transfer, there exists a 3-term arithmetic progression in $A$. 
The previous nonstandard argument admits a strong generalization.

**Theorem ("Idempotent Ultrafilter Rado" - DN 2015)**

Let \( c_1 X_1 + \ldots + c_n X_n = 0 \) be a Diophantine equation with \( n \geq 3 \).

If \( c_1 + \ldots + c_n = 0 \) then there exist integers \( a_1, \ldots, a_{n-1} \) such that for every idempotent ultrafilter \( \mathcal{U} \), the ultrafilter

\[
\mathcal{V} = a_1 \mathcal{U} \oplus \ldots \oplus a_{n-1} \mathcal{U}
\]

is an injective PR-witness, i.e. for every \( A \in \mathcal{V} \) there exist distinct \( x_1, \ldots, x_n \in A \) with \( c_1 x_1 + \ldots + c_k x_k = 0 \).
### Theorem

1. If $\xi \sim \zeta$ and $\xi \neq \zeta$ then $|\xi - \zeta|$ is infinite.

   Let $f : \mathbb{N} \to \mathbb{N}$ be any function. Then

2. If $\xi \sim \zeta$ then $^*f(\xi) \sim ^*f(\zeta)$.

3. If $^*f(\xi) \sim \xi$ then $^*f(\xi) = \xi$.

### Remark

The last property corresponds to the following basic (non-trivial) fact about ultrafilters:

$$f(\mathcal{U}) = \mathcal{U} \implies \{ n \mid f(n) = n \} \in \mathcal{U}$$
In the last few years, research on the partition regularity of nonlinear equations has made significant progress. In particular, the nonstandard framework of hypernatural numbers and \( u \)-equivalence relation, combined with the use of idempotent elements, revealed quite effective.

To introduce the method, let us see in details two simple examples.

→ Luperi Baglini’s talk.
Theorem (Csikvári - Gyarmati - Sárközy 2012)

The equation $X + Y = Z^2$ is not partition regular (except for $X = Y = Z = 2$).

Proof. By contradiction, let $\alpha \sim u \beta \sim u \gamma$ be such that $\alpha + \beta = \gamma^2$. Let $f : \mathbb{N} \rightarrow \mathbb{N}_0$ be the function defined by $2^{f(n)} \leq n < 2^{f(n)+1}$, and set $a := f(\alpha)$. Notice that $a$ is infinite, as otherwise $\alpha$ would be finite and $\alpha \sim \beta \sim \gamma$ would imply $\alpha = \beta = \gamma = 2$. Now assume WLOG $\alpha \geq \beta$. Then

$$2^a \leq \alpha < \alpha + \beta = \gamma^2 \leq 2\alpha < 2 \cdot 2^{a+1} \Rightarrow 2^{\frac{a}{2}} < \gamma < 2^{\frac{a}{2}+1}.$$ 

So, either $f(\gamma) = \lfloor \frac{a}{2} \rfloor$ or $f(\gamma) = \lfloor \frac{a}{2} \rfloor + 1$. Since $a = f(\alpha) \sim f(\gamma)$, we have either $a \sim \lfloor \frac{a}{2} \rfloor$ or $a \sim \lfloor \frac{a}{2} \rfloor + 1$. But then either $a = \lfloor \frac{a}{2} \rfloor$ (which cannot occur since $a$ is positive), or $a = \lfloor \frac{a}{2} \rfloor + 1$ (and hence $a = 1$ or $a = 2$, contradicting $a$ infinite).
Theorem (DN)

The equation $X^2 + Y^2 = Z$ is not partition regular.

Proof. By contradiction, let $\alpha \sim \beta \sim \gamma$ be such that $\alpha^2 + \beta^2 = \gamma$. $\alpha, \beta, \gamma$ are even numbers, since they cannot all be odd; then

$$\alpha = 2^a \alpha_1, \quad \beta = 2^b \beta_1, \quad \gamma = 2^c \gamma_1$$

where $a \sim b \sim c$ are positive and $\alpha_1 \sim \beta_1 \sim \gamma_1$ are odd.

Case 1: If $a < b$ then $2^{2a}(\alpha_1^2 + 2^{2b-2a}\beta_1^2) = 2^c \gamma_1$. Since $\alpha_1^2 + 2^{2b-2a}\beta_1^2$ and $\gamma_1$ are odd, it follows that $2a = c \sim a$. But then $2a = a$ and so $a = 0$, a contradiction. (Same proof if $b > a$.)

Case 2: If $a = b$ then $2^{2a}(\alpha_1^2 + \beta_1^2) = 2^c \gamma_1$. Since $\alpha_1, \beta_1$ are odd, $\alpha_1^2 + \beta_1^2 \equiv 2 \mod 4$, and so $2^c \gamma_1 = 2^{2a+1} \alpha_2$ where $\alpha_2$ is odd. But then $2a + 1 = c \sim a$ and so $2a + 1 = a$, a contradiction.
OPEN QUESTION
Is the Pythagorean equation partition regular?

\[ X^2 + Y^2 = Z^2 \]

- \( X^2 + Y^2 = Z^2 \) is PR for 2-colorings (computer-assisted proof by Heule - Kullmann - Marek 2016).
- \( X + Y = Z \) is PR (Schur’s Theorem).
- \( X^2 + Y = Z \) is PR (corollary of Sarkozy - Fürstenberg 1978).
- \( X + Y = Z^2 \) is not PR (Csikvári - Gyarmati - Sárközy 2012), but it is PR for 2-colorings (Green-Lindqvist 2016).
- \( X^2 + Y^2 = Z \) is not PR (DN 2016).
- \( X^2 + Y = Z^2 \) is PR (Moreira 2016)
- \( X_1X_2 + Y^2 = Z_1Z_2 \) is PR (DN - Luperi Baglini 2017).
- \( X^n + Y^n = Z^k \) where \( k \neq n \) are not PR (DN - Riggio 2016).
- \( X^n + Y^n = Z^n \) where \( n > 2 \) has no solutions (Fermat’s Theorem!).
In the following we will assume the property of $c^+$-enlargement.

There is a natural topology $\tau_S$ on $\mathbb{N}^*$, named the $S$-topology (standard topology) whose basic (cl)open sets are the hyper-extensions: $\{*A \mid A \subseteq \mathbb{N}\}$.

While the Stone-Čech compactification $\beta \mathbb{N}$ is – in a precise sense – the largest possible Hausdorff compactification of the discrete space $\mathbb{N}$, the hypernatural numbers $\mathbb{N}^*$ are an even larger compactification that retains the nice algebraic properties of the natural numbers.
1. *\( \mathbb{N} \) is **compact** and **completely regular** (but not Hausdorff).
   
   \([X \text{ is completely regular if for every closed } C \text{ and } x \notin C \text{ there is a continuous } f : X \to \mathbb{R} \text{ with } f(x) = 0 \text{ and } f \equiv 1 \text{ on } C.]\)

2. *\( \mathbb{N} \) is **dense** in *\( \mathbb{N} \).

3. Every \( f : \mathbb{N} \to K \) where \( K \) is compact Hausdorff is naturally extended to a continuous \( \bar{f} : *\mathbb{N} \to K \).

   \([\bar{f}(\xi) \text{ be the unique } x \in K \text{ that is “near” to } *f(\xi), \text{ in the sense that } *f(\xi) \in *U \text{ for all neighborhoods } U \text{ of } x.]\)

4. By means of hyper-extensions, every function \( f : \mathbb{N}^k \to \mathbb{N} \) is extended to a continuous \( *f : *\mathbb{N}^k \to *\mathbb{N} \) that satisfies the same “elementary properties” as \( f \). In particular, sum and product on *\( \mathbb{N} \) are extended to commutative operations on *\( \mathbb{N} \).
Let us remark that \( \ast \mathbb{N} \) is not Hausdorff, and in fact, not even \( T_0 \).

Notice that two elements \( \xi, \zeta \in \ast \mathbb{N} \) are not separated by the S-topology precisely when they generate the same ultrafilter \( U_\xi = U_\zeta \).

In case of topological spaces \( X \) that are not \( T_0 \), one considers the Kolmogorov equivalence \( \equiv_K \) between points that are “topologically indistinguishable”, that is:

\[
x \equiv_K y \text{ if and only if } x \text{ and } y \text{ have exactly the same neighborhoods.}
\]
The quotient space $X/\equiv_K$, called the Kolmogorov quotient of $X$, is of course always a $T_0$-space.

In the case of $^*\mathbb{N}$, the Kolmogorov equivalence coincides with the $u$-equivalence: $\xi \sim u \eta \iff \mathcal{U}_\xi = \mathcal{U}_\eta$, and the following holds:

- The Kolmogorov quotient $^*\mathbb{N}/\sim_u$ is (isomorphic to) the Stone-Čech compactification $\beta\mathbb{N}$. 
Part 5: Topological dynamics in $^\ast \mathbb{N}$: Hausdorff S-topology?

**OPEN QUESTION**

Are there hypernatural numbers $^\ast \mathbb{N}$ where the S-topology is Hausdorff? (That is, where $\xi \sim \zeta \Leftrightarrow \xi = \zeta$.)

The question above has an equivalent reformulation in terms of ultrafilters.

- Are there ultrafilters $\mathcal{U}$ on $\mathbb{N}$ with the following property?

  $$f(\mathcal{U}) = g(\mathcal{U}) \quad \implies \quad \{ n \mid f(n) = g(n) \} \in \mathcal{U}.$$  

The property is consistent with ZFC; indeed, it is satisfied when $\mathcal{U}$ is selective (and selective ultrafilters are consistent). However, it is still unknown whether Hausdorff ultrafilters exist in ZFC.
Let us consider the shift operator $S$ on the compact space $\mathbb{N}$:

$$S : \nu \mapsto \nu + 1.$$ 

- $\mathbb{N}$, $S$) is a discrete topological dynamical system.
- The (positive) orbit of a point $\nu \in \mathbb{N}$ is the set

$$\text{orb}(\nu) := \{(S \circ \ldots \circ S)(\nu) \mid k \in \mathbb{N}\} = \{\nu + k \mid k \in \mathbb{N}\}$$

- $\nu \in \mathbb{N}$ is recurrent if $\text{orb}(\nu) \cap \mathbb{N} \neq \emptyset$ for every neighbourhood $\mathbb{N}$ of $\nu$, that is:

$$\nu \in \mathbb{N} \implies \nu + k \in \mathbb{N} \quad \text{for some positive integer } k.$$
Part 5: Topological dynamics in \(*\mathbb{N}\)  

Self-recurrent points

**Definition**

A point \(\nu \in *\mathbb{N}\) is **self-recurrent** if:

\[ \nu \in *A \implies \nu + a \in *A \text{ for some } a \in A. \]

- A point \(\nu\) is self-recurrent if and only if it generates an *idempotent ultrafilter* \(\mathcal{U}_\nu = \mathcal{U}_\nu \oplus \mathcal{U}_\nu\).

By using a self-recurrent point, one can give a short proof of Hindman’s Theorem.

**Theorem (Hindman 1974)**

*For every finite coloring of \(\mathbb{N}\) there exists an infinite sequence \((x_i)\) such that all finite sums \(FS(X) = \{x_F := \sum_{i \in F} x_i \mid F \subset \mathbb{N} \text{ finite}\}\) are monochromatic.*
Hindman’s Theorem via self-recurrence

- Pick a self-recurrent point $\nu \in *\mathbb{N}$ and let $C$ be the color such that $\nu \in *C$.
- Pick $x_1 \in C$ with $\nu + x_1 \in *C$.
- Inductively, assume that we found $x_1 < \ldots < x_n$ such that $x_F = \sum_{i \in F} x_i \in C$ and $\nu + x_F \in *C$ for every $F \subseteq \{1, \ldots, n\}$.
- Denote $C - x_F = \{n \mid n + x_F \in C\}$.
- Since $\nu \in \bigcap_F * (C - x_F) = * \bigcap_F (C - x_F)$, we can pick $x_{n+1} \in \bigcap_F (C - x_F)$ such that $\nu + x_{n+1} \in \bigcap_F * (C - x_F)$. Then, $x_G$ and $\nu + x_G \in *C$ for every $G \subseteq \{1, \ldots, n + 1\}$.
A point \( \nu \in ^*\mathbb{N} \) is uniformly recurrent (almost periodic) if \( \text{orb}(\nu) \cap ^*A \) is syndetic for every neighborhood \(^*A\).

A set \( B \subseteq \mathbb{N} \) is syndetic if one has a “bounded return time”, that is, there exists \( N \) such that for every \( n \in \mathbb{N} \) there exists \( k \leq N \) with \( n + k \in B \).

If the space \( X \) is compact Hausdorff, then by Zorn’s Lemma it is proved that in every dynamical system \((X, T)\) one finds uniformly recurrent points. The same proof also works in the non-Hausdorff case, provided the space is regular, and so:

- In the dynamical system \((^*\mathbb{N}, S)\) there exist uniformly recurrent points.
van der Waerden’s Theorem

Uniformly recurrent points in the dynamical system \((*\mathbb{N}, S)\) can be used to give a proof of the classic van der Waerden’s Theorem.

**Theorem (van der Waerden 1926)**

*In every finite coloring \(\mathbb{N} = C_1 \cup \ldots \cup C_r\) there are arbitrarily long monochromatic arithmetic progressions.*
We will prove the following:

**Theorem**

*Let $\nu$ be a uniformly recurrent point of $(^{\ast}\mathbb{N}, S)$. If $\nu \in ^{\ast}A$ then $\text{orb}(\nu) \cap ^{\ast}A$ contains arbitrarily long arithmetic progressions. So, by transfer, $A$ contains arbitrarily long arithmetic progressions.*

Given a finite coloring $\mathbb{N} = C_1 \cup \ldots \cup C_r$, pick the color $C = C_i$ such that $\nu \in ^{\ast}C$. Then $C$ contains arbitrarily long arithmetic progressions.
By induction on $k$, we prove that if $\nu + m \in *A$ for some $m \geq 0$, then there exists an arithmetic progression of length $k$ in $\text{orb}(\nu + m) \cap *A$. The basis $k = 1$ is trivial ($A - m$ is syndetic).

**Successor step $k + 1$:** By the hypothesis, $\{n \in \mathbb{N} \mid \nu + n \in *A\}$ is syndetic, and so there exists $N \in \mathbb{N}$ such that for every $x \in \text{orb}(\nu)$ one has $x + t \in *A$ for some $t \leq N$.

Pick $t_0 \leq N$ such that $\nu + t_0 \in *A$. By the inductive hypothesis there exist $\ell_1$ and $y_1$ such that $\nu + \ell_1 + t_0 + iy_1 \in *A$ for $i = 1, \ldots, k$.

Pick $t_1 \leq N$ with $\nu + \ell_1 + t_1 \in *A$. If $t_1 = t_0$, we have found the desired arithmetic progression of length $k + 1$. 
Otherwise, consider the set

\[ A_1 = (A - t_1) \cap \bigcap_{i=1}^{k} (A - t_0 - iy_1). \]

Since \( \nu + \ell_1 \in *A_1 \), by the inductive hypothesis there exist \( \ell_2 \) and \( y_2 \) such that \( \nu + \ell_1 + \ell_2 + iy_2 \in *A_1 \) for \( i = 1, \ldots, k \). So,

\[ \nu + \ell_1 + \ell_2 + t_1 + iy_2, \nu + \ell_1 + \ell_2 + t_0 + i(y_1 + y_2) \in *A. \]

Pick \( t_2 \leq N \) such that \( \alpha + \ell_1 + \ell_2 + t_2 \in *A \). Notice that if \( t_2 = t_1 \) or \( t_2 = t_0 \) then we have obtained the desired arithmetic progression of length \( k + 1 \).
Otherwise, consider the set

\[ A_2 = (A - t_2) \cap \bigcap_{i=1}^{k} (A - t_1 - iy_2) \cap \bigcap_{i=1}^{k} (A - t_0 - i(y_1 + y_2)). \]

Since \( \nu + \ell_1 + \ell_2 \in *A_2 \), we can apply the inductive hypothesis and proceed as above.

Now \( \{1, \ldots, N\} \) is finite, and so after finitely many steps we find \( t_n = t_m \) where \( n > m \), and we obtain the following arithmetic progression of length \( k + 1 \) in \(*A*\):

\[ \nu + \ell_1 + \ldots + \ell_n + t_n + i(y_{m+1} + \ldots + y_n) \quad i = 0, 1, \ldots, k. \]
Part 5: Topological dynamics in \( *N \) van der Waerden’s Theorem