DIVIDING AND WEAK QUASI-DIMENSIONS IN ARBITRARY THEORIES

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Abstract. We show that any countable model of a model complete theory has an elementary extension with a "pseudofinite-like" quasi-dimension that detects dividing.

1. Introduction

In a pseudofinite structure, every set S has a size $|S|$, a nonstandard cardinality. It is reasonable to say that S and T are "similar in size" if $|\log|S| - \log|T||$ is bounded (by a natural number). This gives the notion of *fine pseudofinite dimension* [3, 2], the quotient of $\log|S|$ by a suitable convex set. García shows [1] that the fine pseudofinite dimension detects dividing: roughly speaking, if $\phi(x,b)$ divides over $\psi(x,a)$ then there is a b' with $tp(b'/a) = tp(b/a)$ so that the dimension of $\phi(x,b')$ is strictly stronger than the dimension of $\psi(x,a)$.

We give a limited extension of this to model complete theories in relational languages (and, via Morleyization, to any theory): any countable model whose theory is model complete embeds elementarily in a "large" fragment of a pseudofinite structure in such a way that the notion of dimension pulls back to the original model; moreover, if $\phi(x,b)$ divides over $\psi(x,a)$ then there is a b' in an elementary extension with $tp(b'/a) = tp(b/a)$ so that the dimension of $\phi(x,b')$ is strictly stronger than the dimension of $\psi(x,a)$.

There is a straightforward way to embed a countable structure in a pseudofinite structure, namely embed M in an ultraproduct of its finite restrictions. That being said, this embedding need not be elementary. It is also easy to obtain a dimension-like function that detects dividing by linearizing the partial order on definable sets given by dividing. The dimension here, however, is an abelian group, and even a quotient of $\mathbb{R}^*$.

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2. Construction

Let $\mathcal{L}$ be a countable first-order relational signature and let T be a complete, model complete theory in $\mathcal{L}$. Set $\mathcal{L}' := \mathcal{L} \cup \{V_\alpha \ : \ \alpha < \omega + \omega\}$, where the V_α are fresh unary relation symbols. For the sake of readability, if M is

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an $\mathcal{L}'$-structure and $\alpha < \omega+\omega$, we let $V_\alpha(M)$ denote the interpretation of the symbol V_α in M. Occasionally we might abuse notation and write a formula in the form $\forall \vec{x} \in V_\alpha(\cdots)$ to mean $\forall \vec{x}(\bigwedge_i V_\alpha(x_i) \to \cdots)$. All $\mathcal{L}'$-structures considered will have the property that the interpretations of the V_α's will form a chain: if $\alpha < \beta < \omega + \omega$, then $V_\alpha(M) \subseteq V_\beta(M)$.

By a *partitioned $\mathcal{L}$-formula* we mean a triple $(\varphi, \vec{x}, \vec{y})$, where φ is a $\mathcal{L}$-formula, $\vec{x}$ and $\vec{y}$ are disjoint finite tuples of variables (taken from some fixed countably infinite list of variables), and the free variables of φ are amongs those appearing in $\vec{x}$ and $\vec{y}$. We follow traditional model-theoretic notation by writing $\varphi(\vec{x}; \vec{y})$ for the partitioned formula $(\varphi, \vec{x}, \vec{y})$. We let $\mathcal{F}$ denote the set of partitioned *quantifier-free* $\mathcal{L}$-formulae.

Let $(\sigma_i \ : \ i < \omega)$ denote an enumeration of $\mathcal{F} \times (\omega + \omega)$. For $i < \omega$, we write $\sigma_i = (\varphi_i, \alpha_i)$ and sometimes refer to α_i by $\alpha(\sigma_i)$.

We say that an $\mathcal{L}'$-structure M *strongly satisfies* σ_i if, whenever $\vec{a} \in V_{\alpha_i}(M)$ is such that there is $N \supseteq M$ with $N \models T$ and $\vec{b} \in N$ such that $N \models \varphi_i(\vec{a}; \vec{b})$, then there is $\vec{c} \in V_{\alpha_i+1}(M)$ such that $M \models \varphi_i(\vec{a}; \vec{c})$.

For each $n \in \omega$, we define an $\mathcal{L}'$-structure $M_n \models T_\forall$ with the property that if $i < n$, M_n strongly satisfies σ_i. Let M_0 denote a one-element substructure of a model of T whose unique element satisfies each V_α.

Suppose we have constructed M_{n-1}. Consider the first n pairs $\sigma_0, \ldots, \sigma_{n-1}$ and fix a permutation $\sigma_{r_0}, \ldots, \sigma_{r_{n-1}}$ so that $i \leq j$ implies that $\alpha(\sigma_{r_i}) \leq \alpha(\sigma_{r_j})$. We construct $\mathcal{L}'$-structures $M_n^i \models T_\forall$, for $i = 0, \ldots, n$, by recursion on i in such a way that M_n^i strongly satisfies $\sigma_{r_0}, \ldots, \sigma_{r_{i-1}}$. We will then set $M_n := M_n^n$.

Let $M_n^0 = M_{n-1}$. Suppose that M_n^i has been constructed and set $\alpha := \alpha(\sigma_{r_i})$ and let $\varphi(\vec{x}; \vec{y})$ be the formula in σ_{r_i}. Enumerate the tuples of length $|\vec{x}|$ in $V_\alpha(M_n^i)$ as $\vec{a}_1, \ldots, \vec{a}_k$. We now recursively construct a sequence of models $M_n^{i,j}$; we begin with $M_n^{i,0} = M_n^i$. Given $M_n^{i,j}$, we proceed as follows:

- If there is a $\vec{b} \in V_{\alpha+1}(M_n^{i,j})$ such that $M_n^{i,j} \models \phi(\vec{a}_j; \vec{b})$, then set $M_n^{i,j+1} := M_n^{i,j}$,
- Otherwise, if there is an extension M of $M_n^{i,j}$ and a tuple $\vec{b}$ from M such that $M \models \phi(\vec{a}_j; \vec{b})$, then set $M_n^{i,j+1} := M_n^{i,j} \cup \{\vec{b}\}$ and declare that any element of $\vec{b}$ which is not in $V_{\alpha+1}(M_n^{i,j})$ belongs to $V_{\alpha+1}(M_n^{i,j+1}) \setminus V_\alpha(M_n^{i,j+1})$.
- If neither of the first two cases apply, set $M_n^{i,j+1} = M_n^{i,j}$.

Set $M_n^{i+1} := M_n^{i,k+1}$. Since $V_\alpha(M_n^{i+1}) = V_\alpha(M_n^i)$, we see that M_n^{i+1} still strongly satisfies $\sigma_{r_0}, \ldots, \sigma_{r_i-1}$; moreover, by design, M_n^{i+1} also strongly satisfies σ_{r_i}, thus finishing the recursive construction.

Fix a nonprincipal ultrafilter $\mathcal{U}$ on ω and set $M := \prod_{\mathcal{U}} M_n$. By definition, $V_\alpha(M) = \{x \in M \ : \ M \vDash V_\alpha(x)\}$. We also set $V_{<\omega}(M) := \bigcup_{n<\omega} V_n(M)$ and $V(M) := \bigcup_{\alpha<\omega+\omega} V_\alpha(M)$, both considered as $\mathcal{L}'$-structures in the obvious way.

Since T is model-complete, it has a set of $\forall\exists$-axioms. Suppose that $\forall\vec{x}\exists\vec{y}\varphi(\vec{x};\vec{y})$ is such an axiom. Fix $\alpha < \omega + \omega$ and take i such that $\sigma_i = (\varphi(\vec{x};\vec{y}), \alpha)$. Fix $n > i$ and consider $N \models T$ such that $M_n \subseteq N$. Since $N \models T$ and M_n strongly satisfies σ_i, we have $M_n \models \forall\vec{x} \in V_\alpha \exists\vec{y} \in V_{\alpha+1}\varphi(\vec{x};\vec{y})$. It follows that $M \models \forall\vec{x} \in V_\alpha \exists y \in V_{\alpha+1}\varphi(\vec{x};\vec{y})$. This proves:

Proposition 2.1. *$V_{<\omega}(M)$ and $V(M)$ are both models of T.*

There is no guarantee that $V_\omega(M)$ nor M are models of T. Note that $V(M)$ is existentially closed in M. Indeed, $M \models T_\forall$ and every model of T is an existentially closed model of $T_\forall$ (by model-completeness of T).

Lemma 2.2. *Suppose that $p(\vec{x})$ is a countable set of existential $\mathcal{L}$-formulae with parameters from $V_\alpha(M)$ that is finitely satisfiable in $V(M)$ (equivalently, in M). Then there is $\vec{c} \in V_{\alpha+1}(M)$ such that $V(M) \models p(\vec{c})$.*

Proof. Let $p'(\vec{x}) := p(\vec{x}) \cup \{V_{\alpha+1}(\vec{x})\}$. Since M is countably saturated, it suffices to show that $p'(\vec{x})$ is finitely satisfiable in M. Indeed, we then get $\vec{c} \in V_{\alpha+1}(M)$ such that $M \models p(\vec{c})$; since $V(M)$ is existentially closed in M (as remarked above), we have that $V(M) \models p(\vec{c})$. In addition, since the formulae in p are existential and, to show that $M \models \exists\vec{x} \in V_{\alpha+1}\exists\vec{z}\phi(\vec{x},\vec{z},\vec{a})$ it clearly suffices to show $M \models \exists\vec{x} \in V_{\alpha+1}\exists\vec{z} \in V_{\alpha+1}\phi(\vec{x},\vec{z},\vec{a})$, it suffices to assume that the formulae in p are actually quantifier-free.

We are left with showing the following: if $\varphi(\vec{x},\vec{a})$ is a quantifier-free formula with parameters from $V_\alpha(M)$ such that $V(M) \models \exists\vec{x}\varphi(\vec{x},\vec{a})$, then $V(M) \models \exists\vec{x} \in V_{\alpha+1}\varphi(\vec{x},\vec{a})$. Since $M \models \exists\vec{x}\varphi(\vec{x},\vec{a})$, we get $M_n \models \exists\vec{x}\varphi(\vec{x},\vec{a}_n)$ for $\mathcal{U}$-almost all n, where $(\vec{a}_n)$ is a representative sequence for $\vec{a}$. Fix i such that $\sigma_i = (\varphi(\vec{x},\vec{y}), \alpha)$. Since M_n strongly satisfies σ_i for $n > i$, it follows that, for $\mathcal{U}$-almost all n, we can find $\vec{c}_n \in V_{\alpha+1}(M_n)$ such that $M_n \models \varphi(\vec{c}_n, \vec{a}_n)$. □

Lemma 2.3. *Suppose N is a model of T and $A \subseteq N$ is countable. Then there is $A' \subseteq V_{<\omega}(M)$ such that $\mathrm{tp}^N_{\mathcal{L}}(A) = \mathrm{tp}^{V(M)}_{\mathcal{L}}(A')$.*

Proof. Enumerate $A = \{a_0, a_1, \ldots\}$ and construct $A' \subseteq M'$ inductively: given $a'_0, \ldots, a'_n \in V_{n+1}(M)$ with $tp^N_{\mathcal{L}}(a_0, \ldots, a_n) = \mathrm{tp}^{V(M)}_{\mathcal{L}}(a'_0, \ldots, a'_n)$, by the previous lemma and the model-completeness of T (so every type is determined by its existential formulae), there is an $a'_{n+1} \in V_{n+2}(M)$ with $\mathrm{tp}^N_{\mathcal{L}}(a_{n+1}/a_0, \ldots, a_n) = \mathrm{tp}^{V(M)}_{\mathcal{L}}(a'_{n+1}/a'_0, \ldots, a'_n)$. □

We now fix a countable model N of T and take $A = N$ in the above lemma, yielding an elementary embedding $a \mapsto a' : N \to V_{<\omega}(M)$ with image N'.

Definition 2.4. For an $\mathcal{L}$-formula $\varphi(\vec{x})$, we let $\varphi_\omega(\vec{x}) := \varphi(\vec{x}) \wedge V_\omega(\vec{x})$.

Lemma 2.5. *Suppose that $\varphi(\vec{x},\vec{y})$ and $\psi(\vec{x},\vec{z})$ are existential $\mathcal{L}$-formulae. If $N \models \forall\vec{x}(\varphi(\vec{x},\vec{a}) \leftrightarrow \psi(\vec{x},\vec{b}))$, then $M \models \forall\vec{x}(\varphi_\omega(\vec{x},\vec{a}') \leftrightarrow \psi_\omega(\vec{x},\vec{b}'))$.*

Proof. Since $a \mapsto a'$ is elementary, we get $V_{<\omega}(M) \models \forall\vec{x}(\varphi(\vec{x},\vec{a}') \leftrightarrow \psi(\vec{x},\vec{b}'))$. Since $V_{<\omega}(M) \preceq V(M)$ (by model-completeness of T), the same equivalence holds in $V(M)$. Finally, if $\vec{c} \in V_\omega(M)$, then $M \models \varphi_\omega(\vec{c}, \vec{a}')$ if and only if $V(M) \models \varphi_\omega(\vec{c}, \vec{a}')$ and likewise with ψ_ω. □

We recall the notion of pseudofinite dimension, especially as considered in [3, 2]. Since M is an ultraproduct of finite sets, any definable set D has a nonstandard cardinality $|D|$ in $\mathbb{R}^*$ (the ultrapower of the reals). We let $\mathcal{C}$ be the convex hull of $\mathbb{Z}$ in $\mathbb{R}^*$. Then for any definable set X, we can define

$$\delta_M(X) = \log |X| / \mathcal{C},$$

the image of $\log |X|$ in $\mathbb{R}^*/\mathcal{C} \cup \{-\infty\}$ (where $\log |X| = -\infty$ if $|X| = 0$). This is the *fine pseudofinite dimension.*

The fine pseudofinite dimension satisfies the quasi-dimension axioms:

- $\delta_M(\emptyset) = -\infty$ and $\delta_M(X) > -\infty$ implies $\delta_M(X) \geq 0$,
- $\delta_M(X \cup Y) = \max\{\delta_M(X), \delta_M(Y)\}$,
- For any definable function $f : X \to Z$ and every $\alpha \in \mathbb{R}^*/\mathcal{C} \cup \{-\infty\}$, if $\delta_M(f^{-1}(z)) \leq \alpha$ for all $z \in Z$ then $\delta_M(X) \leq \alpha + \delta_M(Z)$.

One of the features of fine pseudofinite dimension is that if we fix any definable set X, we may define a measure $\mu_X(Y)$ on definable Y by $\mu_X(Y) = st(\frac{|Y|}{|X|})$ so that $\delta_M(Y) = \delta_M(X)$ if and only if $\mu_X(Y) \in (0, \infty)$.

In light of the lemma above, the following definition makes sense.

Definition 2.6. Suppose $X \subseteq N^k$ is definable. Without loss of generality, we may suppose that X is defined by $\varphi(\vec{x}, \vec{a})$, where $\varphi(\vec{x}, \vec{y})$ is quantifier-free. We then define $\delta_N(X) = \delta_M(\varphi_\omega(\vec{x}, \vec{a}'))$, where the latter dimension is computed in the pseudofinite structure M.

Lemma 2.7. $\delta_N(X \times Y) = \delta_N(X) + \delta_N(Y)$

Proof. Suppose X and Y are defined by $\varphi(\vec{x}, \vec{a})$ and $\psi(\vec{y}, \vec{b})$ respectively. Then $X \times Y$ is defined by $\rho(\vec{x}, \vec{y}, \vec{a}, \vec{b}) = \varphi(\vec{x}, \vec{a}) \wedge \psi(\vec{y}, \vec{b})$. Then

$$\delta_N(X \times Y) = \delta_M(\rho_\omega(\vec{x}, \vec{y}, \vec{a}, \vec{b})) = \delta_M(\varphi_\omega(\vec{x}, \vec{a})) + \delta_M(\psi_\omega(\vec{y}, \vec{b})) = \delta_N(X) + \delta_N(Y).$$

using the pseudofinite axioms for δ_M. □

δ_N need not satisfy the final quasi-dimension axiom, however—it is possible that there are many values $z \in Z_\omega$ so that $\delta(f^{-1}(z))$ is large and so $\delta_M(X_\omega)$ is large as well, but that none of these are in the image of M, so $\delta_N(X)$ is large even though $\delta_N(f^{-1}(z))$ is small for all $z \in Z$.

Nonetheless, there is a connection between δ_N and dividing, essentially the one shown by García in [1] for pseudofinite dimension.

Proposition 2.8. *Suppose that $\psi(x, a)$ and $\varphi(x, b)$ are existential $\mathcal{L}$-formulae with parameters from $V_\omega(M)$ such that $\varphi(x, b)$ implies $\psi(x, a)$ and $\varphi(x, b)$ divides over a. Then there is $b^\# \in V_\omega(M)$ with $b^\# \equiv_{\mathcal{L},a} b$ and $\delta_M(\varphi(x, b^\#)) < \delta_M(\psi(x, a))$.*

Proof. Assume that no $b^\#$ exists as in the conclusion. We then use that to get $K \in \mathbb{N}$ such that $K|\varphi_\omega(x, b^\#)| \geq |\psi_\omega(x, a)|$ for all $b^\# \in V_\omega(M)$ with $b^\# \equiv_{\mathcal{L},a} b$. In fact, by saturation again, there is $\chi(x, a) \in \mathrm{tp}^M_{\mathcal{L}}(b/a)$ such that $K|\varphi_\omega(x, b^\#)| \geq |\psi_\omega(x, a)|$ for all $b^\# \models \chi_\omega(x, a)$.

Fix L sufficiently large (depending only on k and K) and take $(b_i)_{i<L}$ from $V_{<\omega}(M)$ satisfying $\chi_\omega(x, a)$ and such that $\{\varphi(x, b_i) : i < L\}$ is k-inconsistent. In particular, we have $K|\varphi_\omega(x, b_i)| \geq |\psi_\omega(x, a)|$ for all $i < L$. As in [1], if L is sufficiently large, we get $i_1 < \ldots < i_k < L$ such that $\bigcap_{j=1}^k \mu_{\psi_\omega}(\varphi_\omega(x, b_{i_j})) > 0$. In particular, there is $c \in V_\omega(M)$ such that $M \models \varphi_\omega(c, b_{i_j})$ for all $j = 1, \ldots, k$. It follows that $V(M) \models \varphi(c, b_{i_j})$ for $j = 1, \ldots, k$, a contradiction. □

In the previous result, if we have $\psi(x, a)$ and $\varphi(x, b)$ formulae with parameters from N such that $\varphi(x, b)$ implies $\psi(x, a)$ and $\varphi(x, b)$ forks over a, then we can apply the previous result with $\psi(x, a')$ and $\varphi(x, b')$. It should not be too surprising that, even in this situation, we need to look in $V(M)$ for the desired witness to dimension drop as N is usually not saturated enough to see this dimension drop.

Combining Proposition 2.8 with the remarks made in the previous paragraph yields the main result of this note:

Theorem 2.9. *Suppose that $\psi(x, a)$ and $\phi(x, b)$ are existential $\mathcal{L}$-formulae with parameters from N such that $\phi(x, b)$ implies $\psi(x, a)$ and $\phi(x, b)$ divides over a. Then there is an elementary extension $N^\#$ of N, an extension of δ_N to a quasidimension $\delta_{N^\#}$ on $N^\#$, and $b^\# \in N^\#$ with $b^\# \equiv_{\mathcal{L},a} b$ and $\delta_{N^\#}(\phi(x, b^\#)) < \delta_{N^\#}(\psi(x, a)) = \delta_N(\psi(x, a))$.*

Remark 2.10. Note that a similar argument applies to an arbitrary relational language by taking $\mathcal{L}_0$ and an $\mathcal{L}_0$-structure N and letting T be the theory of the Morleyization of N.

Remark 2.11. Note that the same construction applies, with only the obvious changes, to theories in continuous logic.

References

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