

ROBINSON FORCING AND THE QUASIDIAGONALITY PROBLEM

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ABSTRACT. We introduce weakenings of two of the more prominent open problems in the classification of C^* -algebras, namely the quasidiagonality problem and the UCT problem. We show that a positive solution of the conjunction of the two weaker problems implies a positive solution of the original quasidiagonality problem as well as allows us to give a local, finitary criteria for the MF problem, which asks whether every stably finite C^* -algebra is MF.

1. INTRODUCTION

A prominent open problem in the theory of C^* -algebras, raised in the work of Blackadar and Kirchberg [5, section 7], is whether or not every stably finite nuclear C^* -algebra is quasidiagonal, what we will refer to as the *quasidiagonality problem*. Recent spectacular progress on this problem was made in [22], where it was shown that every separable, simple, unital, stably finite, nuclear C^* -algebra in the so-called UCT class is quasidiagonal. (UCT stands for “Universal Coefficient Theorem” and one can consult [3] for more information on this class.) Thus, a positive solution to the *UCT problem*, which asks whether every separable nuclear C^* -algebra is in the UCT class, yields a positive solution to the quasidiagonality problem for simple algebras. In Section 3, we introduce weaker versions of the UCT problem and the quasidiagonality problem, and we prove that a positive solution to the conjunction of the two weaker problems implies a positive solution to the quasidiagonality problem in full. The key model-theoretic notions involved in this proof are *existentially closed algebras*, *locally universal algebras*, and *Robinson forcing*. A brief review of these notions is the content of Section 2.

Related to the quasidiagonality problem is the *MF problem*, which asks whether every stably finite C^* -algebra is MF, that is, embeds into an ultrapower of the universal, separable UHF algebra \mathcal{Q} . By the Choi-Effros lifting theorem, a nuclear C^* -algebra is quasidiagonal if and only if it is MF, whence it is readily seen that the MF problem is in fact a generalization of the quasidiagonality problem.

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In [15], the current set of authors considered a problem of a similar nature to the MF problem, namely the *Kirchberg embedding problem*, which asks whether every C^* -algebra embeds into an ultrapower of the Cuntz algebra \mathcal{O}_2 . Using Robinson forcing, the authors were able to give a local, finitary reformulation of the Kirchberg embedding problem, namely that every C^* -algebra has *good nuclear witnesses* (see Definition 10 below or [15, Definition 3.6]). It is reasonable to wonder whether or not a stably finite version of the above local criteria would similarly provide a reformulation of the MF problem. In Section 3, we prove, once again assuming that both weak problems have positive solutions, that this is indeed the case, namely that the MF problem is equivalent to the problem of whether every stably finite C^* -algebra has good stably finite nuclear witnesses.

In [15], the authors proved that the only possible nuclear C^* -algebra that is existentially closed amongst all C^* -algebras is \mathcal{O}_2 (and this is the case if and only if the Kirchberg embedding problem has a positive answer). It is natural to wonder whether or not the corresponding statement is true in the stably finite situation, namely whether or not \mathcal{Q} is the only possible nuclear stably finite C^* -algebra that is existentially closed amongst all stably finite C^* -algebras. In Section 4, we make some progress on this question by proving that a simple, nuclear stably finite C^* -algebra in the UCT class that is existentially closed amongst all stably finite C^* -algebras is AF. Central to this result is the study of existentially closed subalgebras of II_1 factors, which is an interesting topic in its own right. We also point out how these techniques also give some insight into the *AF-embedding problem*, which asks whether every stably finite nuclear C^* -algebra embeds into an AF algebra [5, Question 7.3.3].

The *trace problem* asks whether every stably finite C^* -algebra has a trace. (We were unable to find a precise name for this problem in the literature so we have chosen to give it this ad hoc name here.) Since MF algebras have a trace, a positive solution to the MF problem implies a positive solution to the trace problem. Thus, assuming the aforementioned weak problems have a positive solution, the existence of good stably finite nuclear witnesses implies that every stably finite C^* -algebra has a trace. In Section 5, by adapting work of Haagerup from [17], we show directly, without any further assumptions, that the existence of good stably finite nuclear witnesses implies that every stably finite C^* -algebra has a trace. We also show that a quasitracial version of the notion of good stably finite nuclear witnesses yields a sufficient local condition for a positive solution of a question of Kaplansky, namely whether or not every quasitrace on a C^* -algebra is actually a trace.

In order to keep this note fairly short, we make no attempt to keep the note self-contained. A nice recent survey giving details on the context of this note is [24]. We also assume that the reader has seen model-theoretic methods as applied to C^* -algebras. The monograph [13] has become the canonical reference; one can also consult our earlier paper [15].

Let us conclude this section by introducing some notation. We let \mathcal{SF} denote the class of separable, stably finite, unital C^* -algebras. We let \mathcal{SF}_n (resp. \mathcal{SF}_{ns}) denote the subclass of \mathcal{SF} consisting of the nuclear (resp. simple, nuclear) algebras. To apply model-theoretic methods, it is important to note that \mathcal{SF} consists of the separable models of a universal theory of C^* -algebras, whilst the classes \mathcal{SF}_n and \mathcal{SF}_{ns} are classes definable by uniform families of existential formulae (in the lingo from [13]).

We use ω to denote an arbitrary nonprincipal ultrafilter on \mathbb{N} ; given a family (A_n) of C^* -algebras indexed by \mathbb{N} , we write $\prod_{\omega} A_n$ for the corresponding ultraproduct and A^{ω} in case $A_n = A$ for each n .

Finally, as usual \otimes with no further decoration denotes the minimal tensor product in the category of C^* -algebras, operator systems, or operator spaces.

2. EXISTENTIALLY CLOSED AND LOCALLY UNIVERSAL ALGEBRAS

Recall that if $\theta : A \rightarrow B$ is a (unital) embedding between (unital) C^* -algebras, then θ is said to be *existential* if, for any quantifier-free formula $\phi(x, y)$ in the language of (unital) C^* -algebras and any tuple a from A , we have

$$\inf_{b \in A_1} \phi(a, b) = \inf_{b \in B_1} \phi(\theta(a), b).$$

If A is a subalgebra of B and the inclusion map is existential, we say that A is *existentially closed in B* . If A is existentially closed in all extensions belonging to a particular class of C^* -algebras (e.g., the class of all stably finite C^* -algebras), then we say that A is existentially closed for that class.

The key tool used throughout this paper is the construction of algebras using *Robinson forcing*. The details behind Robinson forcing (especially in the continuous case) can be quite cumbersome and we refer the interested reader to [2], [11], and [15, Appendix A] for complete treatments. The main idea behind Robinson forcing is that one constructs separable algebras by specifying the operator norms of $*$ -polynomial combinations of a distinguished countable dense set in a Baire category style fashion that allows one to construct these algebras “slowly.” For particularly nice classes, we can ensure that the “generic” algebra thus constructed has various desirable properties (e.g. nuclearity, simplicity, etc...) and is also existentially closed.

Before continuing further, we offer a perspective on existential embeddings which may seem more familiar to readers with a background in classification theory. Let $\theta : A \rightarrow B$ be an embedding of C^* -algebras. We say that θ is *approximately split injective* if there exists a directed set I , an ultrafilter \mathcal{U} on I , and an embedding $\sigma : B \rightarrow A^{\mathcal{U}}$ such that $\sigma \circ \theta : A \rightarrow A^{\mathcal{U}}$ equals the diagonal embedding of A into $A^{\mathcal{U}}$. Note that if B is separable, then one may take \mathcal{U} to be a nonprincipal ultrafilter on the natural numbers. This notion was isolated and systematically studied in the context of C^* -algebras by Barlak and Szabo [1] (not quite in the formulation given and with a different name), though it implicitly appears in the work of Gardella [14] and the authors [15]. Similar notions appear even earlier, e.g., [5, Definition 7.1.5].

It turns out that a $*$ -embedding $\theta : A \rightarrow B$ is approximately split injective *if and only if* θ is positively existential [15, p. 170]. (See [1, Theorem 4.19] for one direction: the full equivalence in the category of operator systems is implicit in [15, section 2.4] and [16, section 5].) Being positively existential is strictly weaker than being existential. However, one can easily check that all formulae referred to in this note are positive, whence the reader is free to replace “existential” with “approximately split injective” with little loss of essential meaning. Note also that when we say that a map between C^* -algebras is positively existential, we always mean this in the model-theoretic sense (as opposed to the map being existential and also a positive map between C^* -algebras).

Suppose that \mathcal{K} is a class of separable, unital C^* -algebras. We say that $A \in \mathcal{K}$ is *locally universal for \mathcal{K}* if every element of \mathcal{K} unitaly embeds into an ultrapower of A . Thus, the QD problem (resp. MF problem) asks whether or not \mathcal{Q} is locally universal for \mathcal{SF}_n (resp. \mathcal{SF}).

The following lemma is well-known and straightforward. It is also implicitly used in [13, Section 6.5].

Lemma 1. *Suppose that $A \in \mathcal{K}$ is existentially closed for \mathcal{K} . Then the following are equivalent;*

- (1) A is locally universal for \mathcal{K} ;
- (2) \mathcal{K} has the joint embedding property: for every $B, C \in \mathcal{K}$, there is $D \in \mathcal{K}$ and unital embeddings of B and C into D .
- (3) for every $B \in \mathcal{K}$, there is $D \in \mathcal{K}$ and unital embeddings of A and B into D .

One can apply the previous lemma to the case $\mathcal{K} = \mathcal{SF}_{\text{ns}}$. Indeed, \mathcal{SF}_{ns} has an existentially closed element (see [13]). Moreover, the tensor product of two elements of \mathcal{SF}_{ns} belongs to \mathcal{SF}_{ns} again by a result of Haagerup [17]. Consequently, \mathcal{SF}_{ns} has a locally universal element.

However, the situation with the class \mathcal{SF} (or \mathcal{SF}_n) seems to be somewhat more nebulous. Indeed, whilst \mathcal{SF} also admits an existentially closed element, the following question seems to be open:

Question 2. *Does the class \mathcal{SF} (or \mathcal{SF}_n) have the joint embedding property?*

Thus, at the moment, it is unclear whether or not there is any locally universal stably finite (nuclear) C^* -algebra!

One approach to Question 2 might be to show that the minimal tensor product of two elements of \mathcal{SF} is once again an element of \mathcal{SF} . However, in the case both are simple this statement is equivalent to a positive solution to the trace problem again by a result of Haagerup (Fact 25 below). In fact, the following weaker question still seems to be open:

Question 3. *Suppose that $A \in \mathcal{SF}_n$ and $B \in \mathcal{SF}$. Is $A \otimes B \in \mathcal{SF}$?*

For our purposes below, the following related lemma will prove useful.

Lemma 4. *Suppose that $A, B \in \mathcal{SF}$ and A is QD. Then $A \otimes B$ is stably finite.*

Proof. Suppose $A \otimes B$ is not stably finite. It follows that there exists $v \in M_k(A \otimes B)$ such that $v^*v = 1$ and $\|vv^* - 1\| = 1$. There is thus a u.c.p. map $\phi : A \rightarrow M_n$ such that for $\theta := \phi_k \otimes \text{id}_B$ we have that $\theta(v) \in M_{kn}(B)$ satisfies $\|\theta(v)^*\theta(v) - 1\| < 1/4$ and $\|\theta(v)\theta(v)^* - 1\| > 3/4$. By functional calculus, there is $w \in M_{kn}(B)$ such that $w^*w = 1$ and $ww^* \neq 1$, contradicting that B is stably finite. \square

Note also that, by Lemma 4, we see that if the quasidiagonality problem has a positive solution, then Question 3 has a positive answer.

3. THE QUASIDIAGONALITY PROBLEM

We start this section by defining the aforementioned weakening of the UCT problem.

Definition 5. The *weak UCT problem* is the statement that every element of \mathcal{SF}_n is contained in an element of \mathcal{SF}_n that belongs to the UCT class.

Definition 6. The *weak quasidiagonality problem* is the statement that there is a simple locally universal element of \mathcal{SF}_n .

Note that in the statement of the weak quasidiagonality problem, we really do need to use an ultrapower of B as there are stably finite, (nuclear) unital C^* -algebras that are not contained in a simple, stably finite, (nuclear) unital C^* -algebra.

Here is the main result of this section.

Theorem 7. *Suppose that both the weak UCT problem and weak quasidiagonality problem have positive solutions. Then the quasidiagonality problem has a positive solution.*

Proof. By Lemmas 1 and 4, it is enough to prove that there exists $A \in \mathcal{SF}_n$ that is existentially closed for \mathcal{SF}_n and quasidiagonal. The version of the Omitting Types theorem via Robinson forcing (see [2, Corollary 4.7]), together with the assumption that the weak quasidiagonality problem has a positive answer and the fact that simplicity is definable by a uniform family of existential formulae (see [13, Theorem 5.7.3(6)]), allows us to construct a separable algebra A that is simple and “finitely generic” for \mathcal{SF}_n . (Finitely generic simply means that the structure is obtained from finite forcing as opposed to infinite forcing.) The discussion in [15, Appendix A] shows that if \mathcal{SF}_n were the class of models of an $\forall\exists$ -theory, then structures that are finitely generic for \mathcal{SF}_n belong to \mathcal{SF}_n and are existentially closed for \mathcal{SF}_n . In our case, \mathcal{SF}_n is not the class of models of an $\forall\exists$ -theory, but rather is a class of structures definable by a uniform family of existential formulae. It is readily verified that the proof given in [15, Appendix A] goes through in this setting as well (see also [11, Corollary 4.6]), whence the separable,

simple C^* -algebra A constructed belongs to \mathcal{SF}_n and is existentially closed for \mathcal{SF}_n . Since we are assuming that the weak UCT problem has a positive solution, A is a subalgebra of a separable $B \in \mathcal{SF}_n$ that belongs to the UCT class. Since A is existentially closed for \mathcal{SF}_n , by [1, Theorem 2.10] (see also [14, Theorem 3.13]), A also belongs to the UCT class. By the main result of [22] referred to in the introduction, A is quasidiagonal. \square

Remark 8. To run the preceding proof, one does not really need the full strength of the weak quasidiagonality problem for \mathcal{SF}_n . Indeed, for each integer n , let $\varphi_n(a, b)$ denote the formula

$$\varphi_n(a, b) := \inf_{\{x \in A^n : \|\sum x_j^* x_j\| \leq 2\}} \left\| \sum x_j^* a x_j - b \right\|.$$

It is shown in [13, Proposition 5.10.3] that A is simple if and only if, for each $a, b \in A$, we have $\inf_n \varphi_n(a, b) = 0$. Consequently, to be able to apply the Omitting Types Theorem as in the proof of the previous theorem, one needs to be able to verify the following: for each $A \in \mathcal{SF}_n$ and each $a_1, \dots, a_m \in A$, one can find $B \in \mathcal{SF}_n$ satisfying the UCT and $b_1, \dots, b_m \in B$ whose operator norm microstates are close to those of a_1, \dots, a_m and for which $\inf_n \varphi_n(b_i, b_j)$ is small for each i, j . (Of course the B and b_1, \dots, b_m depend on the level of precision of the microstates and the tolerance for $\inf_n \varphi_n$.)

Remark 9. The situation is simpler if one works in \mathcal{SF}_{ns} . Indeed, let A be any element of \mathcal{SF}_{ns} that is existentially closed for \mathcal{SF}_{ns} . Then a positive solution to the weak UCT problem implies that A is in the UCT class, whence quasidiagonal. (Note here we only need the weak UCT problem to have a positive solution for simple algebras.) It follows that all elements of \mathcal{SF}_{ns} are quasidiagonal. In other words, a positive solution to the weak UCT problem implies what Winter calls $\text{QDQ}_{\text{simple},1}$ in [24]. As Winter points out, it is plausible that $\text{QDQ}_{\text{simple},1}$ could imply a positive solution to the quasidiagonality problem.

We now turn to the local, finitary equivalent of the MF problem. First, we recall some terminology from [15]. Recall that for an operator system E , $\text{ex}(E) := \inf_X d_{\text{cb}}(E, X)$, where X ranges over all matricial operator systems and d_{cb} is the completely bounded version of the Banach-Mazur distance. (This is not literally the definition given in [15] but is readily seen to be equivalent.)

Definition 10. We say that a unital C^* -algebra A has *good nuclear witnesses* if, for every finite-dimensional operator system $F \subset A$, there is a $*$ -embedding $\rho := (\rho_n)^\bullet : A \rightarrow \mathcal{B}(H)^\omega$ such that $\text{ex}(\rho_n(F)) \rightarrow 1$ as $n \rightarrow \omega$.

In the context of this paper, it is natural to extend the above definition to the stably finite context. Indeed, if in the notation above each F_n is isomorphic (as an operator system) to a subsystem of a C^* -algebra Q_n that belongs to \mathcal{SF} (respectively \mathcal{SF}_s), then we say that A has *good (resp. simple) stably finite nuclear witnesses*. Note that if A has good stably finite nuclear witnesses, then A is itself stably finite.

Corollary 11. *Suppose that the weak UCT problem and weak quasidiagonality problem both have positive solutions. Then the MF problem is equivalent to the statement that every element of \mathcal{SF} has good stably finite nuclear witnesses.*

Proof. As in [15], the forward direction is immediate (and does not use either of the weak problems). Now assume that every element of \mathcal{SF} has good stably finite nuclear witnesses. Then by Robinson forcing, we can find an existentially closed element A of \mathcal{SF} that is also nuclear. (See [15, Section 3.2] for more details.) By Theorem 7, we know that A is quasidiagonal. By Lemmas 1 and 4, A is locally universal for \mathcal{SF} , whence the MF problem has a positive solution. \square

4. NUCLEAR EXISTENTIALLY CLOSED ELEMENTS OF \mathcal{SF}

In [15], it is shown that the only possible C*-algebra that is nuclear and existentially closed amongst all C*-algebras is \mathcal{O}_2 . It is natural to wonder whether or not there is a stably finite version of this result. Here is the natural guess:

Conjecture 12. *If $A \in \mathcal{SF}$ is existentially closed for \mathcal{SF} and nuclear, then $A \cong \mathcal{Q}$.*

In this section, we make some progress towards settling the previous conjecture. We begin by enumerating a list of general properties that hold of a unital C*-algebra that admits an existential (unital) embedding into a II_1 factor. Before doing so, we pause to recall a few standard definitions and results.

Definition 13. A *quasitrace* (really, a *2-quasitrace*) on a C*-algebra A is a function $\tau : A \rightarrow \mathbb{C}$ which satisfies:

- (1) $\tau(xx^*) = \tau(x^*x) \geq 0$ for all $x \in A$;
- (2) $\tau(a + ib) = \tau(a) + \tau(b)i$ for all self-adjoint $a, b \in A$;
- (3) $\tau(ab) = \tau(ba)$ for all self-adjoint and commuting $a, b \in A$;
- (4) there is a function $\tau_2 : M_2(A) \rightarrow \mathbb{C}$ satisfying (1)-(3) above such that $\tau(x) = \tau_2(x \otimes e_{11})$ for all $x \in A$.

By [4, Proposition II.4.1] any quasitrace τ on A extends to a quasitrace τ_n on $M_n(A)$ for all $n \in \mathbb{N}$. For a positive element $a \in M_k(A)$ and a quasitrace τ define the dimension function by $d_\tau(a) := \lim_{n \rightarrow \infty} \tau_k(a^{1/n})$. The C*-algebra A is then said to have *strict comparison* if for all positive $a, b \in M_k(A)$ such that $d_\tau(a) < d_\tau(b)$ for all quasitraces τ on A there is a sequence $r_n \in M_k(A)$ such that $\|r_n^* b r_n - a\| \rightarrow 0$. The reader may consult [19] for instance for the history and context of comparison properties in the classification of stably finite nuclear C*-algebras.

We are now ready to state and prove the main proposition of this section. As we will remark, some statements are known to hold by alternate means.

However, we wish to illustrate how through the use of an existential embedding into a von Neumann algebra, one can simply and directly transfer von Neumann algebraic comparison theory to the embedded C*-algebra, thereby avoiding many of the difficulties one confronts in working with comparison theory at the level of C*-algebras. In the future, one could hope that model theoretic techniques could be used to “regularize” a stably finite C*-algebra by embedding it into a suitably “nice” C*-algebra with much better comparison properties.

Proposition 14. *Suppose that A is a separable, unital C*-algebra that admits an (positive) existential unital embedding into a II_1 factor M . Then:*

- (1) A is simple, monotracial, and the unique trace is definable;
- (2) there is a simple, monotracial AF-algebra B such that $K_*(A) \cong K_*(B)$.
- (3) for any two projections $p, q \in A$ either $p \preceq q$ or $q \preceq p$ (“ \preceq ” denotes Murray-von Neumann subequivalence);
- (4) A has unique quasitrace;
- (5) A has strict comparison.

Proof. As in [15, Proposition 2.5], we have that A has the uniform Dixmier property in the parlance of [13, section 7.2], whence it is simple and definably monotracial by [13, Lemma 7.2.2], proving (1). In fact, the analysis given in [13] shows that the trace is existentially definable and the embedding of A into M is existential in the language of *tracial* C*-algebras, that is, the language of C*-algebras enlarged with a predicate symbol naming the trace. Now since the embedding of A in M is existential, the induced map on K-theory is injective. (See, for instance, [15, section 2.3] or [1, Theorems 2.8 and 4.19].) In particular, $K_1(A) = 0$. Since the embedding of A into M is also existential in the language of *tracial* C*-algebras, the induced injection $K_0(A) \rightarrow K_0(M) = \mathbb{R}$ is an embedding of scaled ordered groups. Moreover, once again using existentiality in the language of *tracial* C*-algebras, we see that the image of the embedding is dense in \mathbb{R} . It follows from the Effros-Handelman-Shen Theorem [9] (more precisely, the version stated in [3, Theorem 7.4.3]) that $K_0(A) \cong K_0(B)$ as scaled ordered groups for some simple, unital monotracial AF algebra B , proving (2).

For (3), fix projections $p, q \in A$. Without loss of generality, we may assume that $p \preceq q$ in M ; we show that $p \preceq q$ in A . Let $\phi(x, y)$ be the formula $\inf_z \max(\|x - z^*z\|, \|y - zz^*\|)$. Since $p \preceq q$ in M and A is existentially closed in M , we have

$$\left(\inf_{p'} \max(\|p'q - p'\|, \phi(p', p))\right)^A = 0.$$

In the above formula, the infimum ranges over projections. Fix $\epsilon > 0$ sufficiently small and let $\delta = \Delta_\phi(\epsilon)$, where Δ_ϕ is the modulus of uniform continuity for ϕ . Let $\eta > 0$ be sufficiently small and take a projection $p' \in A$ such that $\max(\|p'q - p'\|, \phi(p', p)) < \eta$. Then if η is sufficiently small, there is a projection $p'' \in A$ with $p'' \leq q$ and $\|p'' - p'\| < \delta$, whence $\phi(p'', p) < \epsilon$;

if $\epsilon > 0$ was chosen sufficiently small, then we have that p'' is Murray-von Neumann equivalent to p , witnessing that $p \preceq q$.

By part (3) and the density of the trace values on projections in A , the standard argument for showing that the trace is the unique dimension function on the projection lattice of M applies, and we know that $\phi(p) = \tau(p)$ for any projection $p \in A$ and any quasitrace ϕ on A . Since quasitraces are norm continuous and are almost linear on almost commuting elements of A by [4] and any self-adjoint contraction in M may be $1/n$ -norm approximated by a linear combination of $2n$ orthogonal projections in M , the conclusion of (4) follows.

For (5), let $a, b \in M_n(A)_+$ be contractions; by part (4), we only need to show that if the support projection of a is subequivalent to the support projection of b , then there exists a sequence $r_n \in A$ such that $\|r_n^* b r_n - a\| \leq 1/n$ for all n . We can easily find such a sequence in M (even with $\|r_n\| \leq n$), so we are done by existentiality. \square

Remarks 15.

- (1) The proof of statement (3) boils down to the quite useful fact that Murray-von Neumann equivalence is existential and weakly stable in the language of C^* -algebras.
- (2) By the main result of [17], statement (4) is immediately implied by statement (1) in the case that A is exact.
- (3) In [13, Theorem 8.2.1], it is shown that strict comparison is $\forall\exists$ -axiomatizable, yielding a different proof of item (5) in the previous proposition.

Corollary 16. *Suppose that $A \in \mathcal{SF}_{\text{ns}}$ is in the UCT class and existentially closed for \mathcal{SF} . Then A is AF.*

Proof. We have that A is separable, simple, and nuclear, whence A admits a faithful trace by [17]. It follows that the von Neumann algebraic closure of A with respect to the trace is semidiscrete, whence is hyperfinite by Connes' classification of injective factors [8]. Thus we have a unital embedding of A into \mathcal{R} (the hyperfinite II_1 factor), which is moreover existential since A is existentially closed for \mathcal{SF} . Let B be the simple, monotracial AF algebra given by the previous proposition. By [22, Corollary D], we can conclude that $A \cong B$ since we know that A is also of finite nuclear dimension, which follows from item (5) in Proposition 14 and the resolution of the Toms-Winter conjecture in the monotracial case [21]. \square

Remark 17. By [22, Corollary 6.5], if $A \in \mathcal{SF}_{\text{ns}}$ is monotracial and UCT, then A embeds into a simple, monotracial AF algebra. Since being AF is definable by a uniform family of existential formulae [7, Theorem 2], an existential substructure of an AF algebra is AF again, whence we obtain a different proof of Corollary 16. However, we prefer the above proof as it follows from the main result of [22] in a more elementary fashion.

Question 18. *Is there a proof of Corollary 16 that avoids the use of the ideas in [22]?*

Remark 19. A positive solution to the previous question could lead to an alternate approach to (the nontrivial direction of) Corollary 11 that is more elementary in nature. Indeed, from good stably finite nuclear witnesses, we know that there is $A \in \mathcal{SF}$ that is existentially closed for \mathcal{SF} and nuclear. If Question 3 has a positive answer, then by Lemma 1, A is locally universal for \mathcal{SF} . By a positive solution to the weak QD problem, it follows that there is $C \in \mathcal{SF}_{\text{ns}}$ such that is locally universal for \mathcal{SF} . One can now apply Robinson forcing again to conclude that there is $D \in \mathcal{SF}$ that is existentially closed for \mathcal{SF} that is also *simple* and nuclear. By a positive solution to the weak UCT problem, D is also UCT, whence D is AF by Corollary 16. By Lemma 2, D is locally universal for \mathcal{SF} , whence the MF problem has a positive solution.

In [15], it is shown that \mathcal{O}_2 is the only possible existentially closed C^* -algebra that is a tensor square. We can (assuming weak UCT) establish the stably finite version of this statement:

Corollary 20. *Assume that the weak UCT problem has a positive solution. Suppose that A is existentially closed for \mathcal{SF} and that there is a C^* -algebra B such that $A \cong B \otimes B$. Then $A \cong \mathcal{Q}$ (whence the MF problem has a positive solution). If, in addition, B itself is also existentially closed for \mathcal{SF} , then $B \cong \mathcal{Q}$.*

Proof. Let $\alpha : A \rightarrow A$ denote the flip automorphism, namely $\alpha(a \otimes b) = b \otimes a$ for all $a, b \in B$. Note that the associated crossed product C^* -algebra, $A \rtimes_{\alpha} \mathbb{Z}_2$, is again stably finite as it can be realized as a subalgebra of $M_2(A)$. Since A is existentially closed for \mathcal{SF} , it follows that α is approximately inner (see [12, Section 3]), whence B has *approximately inner flip*.

At this point, there are a couple of (more or less equivalent) ways to conclude that $A \cong \mathcal{Q}$. One way is to note that $A \cong B \otimes B$ also has approximately inner flip (see [10]), so is simple and nuclear (again, see [10]). By the weak UCT, Corollary 16 implies that A is AF. In [10], it is shown that an AF algebra with approximately inner flip is UHF. Since A is existentially closed amongst the UHF algebras, we must have $A \cong \mathcal{Q}$.

Alternatively, by [23, Proposition 1.9(ii)], we have that $B^{\otimes \infty}$ (the tensor product of B with itself countably many times) is *strongly self-absorbing* (see [23] for the definition). However, in [1, Theorem 2.9(6)], it is shown that if D is strongly self-absorbing, then any algebra existentially closed in D must be isomorphic to D . Since $A \cong B \otimes B$ is existentially closed in $B^{\otimes \infty}$, we can conclude that $A \cong B^{\otimes \infty}$. By the positive solution to the weak UCT problem, A satisfies UCT, whence $A \subseteq \mathcal{Q}$ as [22] implies that all strongly self-absorbing C^* -algebras satisfying the UCT embed in \mathcal{Q} . Finally, since A is existentially closed in \mathcal{Q} , we can conclude that $A \cong \mathcal{Q}$.

If B is existentially closed for \mathcal{SF} , then since $B \subseteq A \cong \mathcal{Q}$, we can conclude that $B \cong \mathcal{Q}$. \square

Remark 21. The use of [1, Theorem 2.9(6)] in the proof above allows us the opportunity to present a model-theoretic proof of that result. Indeed, suppose that D is strongly self-absorbing and E is existentially closed in D . Fix an embedding $D \rightarrow E^\omega$ that restricts to the diagonal embedding on E . By considering the chain

$$E \subseteq D \rightarrow E^\omega \subseteq D^\omega \rightarrow (E^\omega)^\omega \dots$$

and using the fact that the maps between the successive ultrapowers of E are elementary (being ultrapowers of the diagonal map) and the maps between the successive ultrapowers of D are elementary (as the initial map $D \rightarrow D^\omega$ is elementary as D is strongly self-absorbing), we see that D and E are elementarily equivalent. It follows that the map $D \rightarrow E^\omega$ is elementary (again, since D is strongly self-absorbing), whence E is actually elementary in D . Since D is strongly self-absorbing, it is the prime model of its theory, whence so is E . By uniqueness of prime models, we see that $D \cong E$.

We point out that this result gives a new proof of the aforementioned result of the authors that any separable, nuclear C^* -algebra, existentially closed amongst all C^* -algebras, is $*$ -isomorphic to \mathcal{O}_2 . This proof has the advantage of only relying on the Kirchberg-Phillips embedding theorem, which states separable nuclear C^* -algebra embeds in \mathcal{O}_2 , rather than the more difficult “ $A \otimes \mathcal{O}_2$ ” theorem of Kirchberg (as well as the axiomatizability of simple, purely infinite algebras).

We conclude this section by making a remark on the *AF-embeddability problem*, which asks whether every element of \mathcal{SF}_n embeds into an AF algebra. (See, for example, [6, Section 8.5].) By Corollary 16, we immediately have:

Corollary 22. *Suppose that B is an element of \mathcal{SF}_n that embeds into an element of \mathcal{SF}_{ns} that is in the UCT class and that is existentially closed for \mathcal{SF} . Then B is AF-embeddable.*

By relativizing the above concepts to a fixed element B of \mathcal{SF}_n and using the preceding corollary, we can derive a sufficient local, finitary criteria for the AF-embeddability problem (for simple algebras). If A is an element of \mathcal{SF} that contains B , we say that A has *good simple stably finite nuclear witnesses over B* if the algebras Q_n are simple, contain B , and the embedding $\rho : B \rightarrow \prod_\omega Q_n$ restricts to the diagonal embedding on A .

Corollary 23. *Assume that the weak UCT problem has a positive solution. Suppose that $B \in \mathcal{SF}$ is such that every element of \mathcal{SF} containing B has good simple stably finite nuclear witnesses over B . Then B is AF-embeddable.*

Proof. The hypotheses allow us to run the model-theoretic forcing machinery in the language expanded by constants to name elements of B . The result is a unital C^* -algebra C that is existentially closed for the elements of \mathcal{SF} that contain B that is also simple and nuclear. In particular, C is existentially

closed for \mathcal{SF} . By the positive solution of the weak UCT problem for \mathcal{SF} , C is also UCT. It follows that C is AF. \square

5. TRACES ON STABLY FINITE C^* -ALGEBRAS

In this section, we show that a positive solution to the trace problem follows from every stably finite C^* -algebra admitting good stably finite nuclear witnesses without needing any weak UCT or weak QD assumptions. The following result is the key technical modification of Haagerup's work that we will need to accomplish this.

Proposition 24. *Suppose that A and B are separable, unital, simple C^* -algebras, A has good stably finite nuclear witnesses, B is exact, and $\rho = (\rho_n)^\bullet : B \rightarrow \prod^\omega M_n$ is a $*$ -embedding of B into a tracial ultraproduct of matrix algebras. Then $A \otimes B$ is stably finite.*

Proof. Fix an operator subsystem F of A spanned by finitely many unitaries. Let $F_k \subset Q_k$ be a sequence of pairs of operator systems and stably finite C^* -algebras as guaranteed by good stably finite nuclear witnesses for $F \subset A$ such that $\text{ex}(F_k) \leq 1 + 1/k$. For each k , we have a finite AW*-algebra $N_k := \prod^\omega (Q_k \otimes M_n)$, where the ultraproduct is taken with respect to the quasitrace ideal; see [17, Proposition 4.2 and Lemma 5.6]. It follows from [20] that $\tilde{\rho}_k : F_k \otimes B \rightarrow N_k$ satisfies $\|\tilde{\rho}_k\|_{\text{cb}} \leq 1 + 1/k$. By the exactness of B , $\theta : F \otimes B \rightarrow \prod_\omega (F_k \otimes B)$ is a cb-isometry [20, Theorem 17.7]. We now easily see that $\rho := (\tilde{\rho}_k)^\bullet \circ \theta : F \otimes B \rightarrow \prod_\omega N_k$ is a u.c.p. map which maps unitaries to unitaries. By considering larger finite sets of unitaries there is a u.c.p. map $\rho' : F' \otimes B \rightarrow \prod_{\omega'} N_k$ which maps unitaries to unitaries where $F' \otimes B$ is an operator system which generates $A \otimes B$ as C^* -algebra. Pisier's linearization trick [20, Proposition 13.6] states that we can extend any such u.c.p. map ρ' to a $*$ -homomorphism $A \otimes B \rightarrow \prod_{\omega'} N_k$. Since $A \otimes B$ is simple, the aforementioned $*$ -homomorphism is injective, whence we have the desired conclusion since $\prod_{\omega'} N_k$ is stably finite. \square

We need the following fact, which is [17, Theorem 2.4]:

Fact 25. *For an arbitrary unital C^* -algebra A , A has a trace if and only if $A \otimes C_r^*(\mathbb{F}_\infty)$ is stably finite.*

The following follows immediately from the Proposition 24 and Fact 25.

Corollary 26. *If A is unital, simple, and has good stably finite nuclear witnesses, then A has a trace.*

Corollary 27. *If every stably finite C^* -algebra has good stably finite nuclear witnesses, then every stably finite C^* -algebra has a trace.*

Remark 28. If Question 2 has a positive answer, then there is a shorter model-theoretic proof of the previous corollary. Indeed, by the assumption that every stably finite C^* -algebra has good stably finite nuclear witnesses together with the fact that \mathcal{SF} has the joint embedding property, we have

that there is a nuclear, stably finite C^* -algebra A that is locally universal for \mathcal{SF} . By the main result of [17], A has a trace. It follows that A^ω has a trace, whence so does every stably finite C^* -algebra, being a subalgebra of A^ω .

First, recall the definition of a quasitrace from Definition 13 above. It follows from [18, 4] that all stably finite C^* -algebras admit a quasitrace. A famous question of Kaplansky asks whether or not every quasitrace on a stably finite C^* -algebra is linear, that is, is actually a trace. By [4, section II], this question reduces to the question of whether every AW^* II_1 -factor is a II_1 -factor. It follows that a positive answer to the Kaplansky question implies that all stably finite C^* -algebras admit a trace. The main result of [17] is that every quasitrace on an exact C^* -algebra is a trace.

Let \mathcal{SF}_τ denote the universally axiomatizable class of structures (A, τ) , where $A \in \mathcal{SF}$ and τ is a quasitrace on A . Given $(A, \tau) \in \mathcal{SF}$, we say that (A, τ) has *good quasitracial stably finite nuclear witnesses* if, in the definition of good stably finite nuclear witnesses, the stably finite algebras Q_n can be equipped with quasitraces τ_n such that $\rho|_{C^*(F)} : C^*(F) \rightarrow \prod_\omega Q_n$ is quasitrace-preserving. By further applying the reasoning of [17, Lemma 5.10 and Theorem 5.11] we obtain:

Corollary 29. *If A is unital and simple, τ is an extremal quasitrace on A , and (A, τ) admits good quasitracial stably finite nuclear witnesses, then τ is a trace.*

Corollary 30. *If every element of \mathcal{SF}_τ has good quasitracial stably finite nuclear witnesses, then every quasitrace on a stably finite C^* -algebra is a trace.*

We now show that quasitraces can be put into a model-theoretic framework. In order to do this, we first recall that every quasitrace on a unital C^* -algebra is Lipschitz continuous with Lipschitz constant $\sqrt{2}$ by [4, Corollary II.2.5(iii)]. Thus, we may consider the language of unital C^* -algebras augmented by a new predicate symbol τ whose modulus of uniform continuity is $\Delta_\tau(\epsilon) = \epsilon/\sqrt{2}$.

Proposition 31. *The class of structures (A, τ) , where A is a unital C^* -algebra and τ is a quasitrace on A , is universally axiomatizable.*

Proof. Items (1) and (2) in the definition of quasitrace are easily seen to be universally axiomatizable. In order to axiomatize item (3), we use an “approximate-near” version of (3). Indeed, it is shown in [4, Corollary II.2.6] that for any quasitrace τ on a C^* -algebra A and for any $\epsilon > 0$, there is a $\delta > 0$ such that for all self-adjoint $a, b \in A$ with $\|ab - ba\| < \delta$, one has $|\tau(a + b) - \tau(a) - \tau(b)| < \epsilon$. This result is proven by using an ultrapower construction for dimension functions. However, the same arguments apply to yield an *ultraproduct* construction for dimension functions, whence the proof of [4, Corollary II.2.6] can be adapted to show that the choice of δ

depends only on ϵ and not on the structure (A, τ) . Consequently, axiom (3) in the definition of quasitrace can be expressed by

$$\sup_{x, y \in A_{\text{sa}}} \min(\delta \div \|xy - yx\|, |\tau(x + y) - \tau(x) - \tau(y)| \div \epsilon) = 0,$$

where $r \div s := \max(r - s, 0)$.

Finally, item (4) of the definition of quasitrace can be axiomatized using the quantifier-free definability of $M_2(A)$ in A^{eq} ; see [13, Lemma 4.2.3]. \square

Question 32. *Does the class \mathcal{SF}_τ have the joint embedding property?*

Once again, if the previous question has a positive answer, then there is a shorter, model-theoretic proof of Corollary 30. Indeed, the assumption of good quasitracial stably finite nuclear witnesses allows us to find $(A, \tau) \in \mathcal{SF}_\tau$ that is existentially closed for \mathcal{SF}_τ and with A nuclear. By the main result of [17], τ is a trace. Since every element of \mathcal{SF}_τ embeds into the ultrapower of A in a quasitrace-preserving way, the desired result follows.

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