A Nonstandard Proof of Roth's Theorem

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Let $X \subseteq \mathbb{N}$. The upper Banach density of X is defined by

$$BD(X) = \lim_{n\to\infty} \sup_{k\in\mathbb{N}} \frac{|(X \cap [k, k+n-1])|}{n}.$$

A k-term AP (arithmetic progression), denoted by P for, is a set of the form

$$P := \{a + id \mid i = 0, 1, \ldots, k-1\}$$

for some integer a and positive integer d.

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Nonstandard Interpretations

The letter A, B, C, V represent internal subsets of integers, H, K, L, M, N represent hyperfinite integers, and $[n] := [0, n-1]$ for any nonnegative integer n. All sets considered from now on are either standard sets in $\mathbb{N}$ or hyperfinite sets in ${}^*\mathbb{N}$.

For any hyperfinite set A and hyperfinite integer N we will use the notation $\mu_N(A) := st\left(\frac{|A|}{N}\right)$.

Thus if $A \subseteq \Omega$ and $|\Omega| = H$, then $\mu_H(A)$ is the Loeb measure of A in Ω.

Proposition Let $X \subseteq \mathbb{N}$. Then $BD(X) \geq \alpha$ if and only if there exists a hyperfinite interval $a + [H] := [a, a+H-1]$ in ${}^*\mathbb{N}$ such that $\mu_H({}^*X \cap (a + [H])) \geq \alpha$.

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Proposition Let $X \subseteq \mathbb{N}$. Then X contains a 3-term AP if and only if ${}^*X$ contains a 3-term AP.

Blank Assumption We assume now that Roth's Theorem is not true and will derive a contradiction.

Let α be the standard real number defined by

$$\alpha := \sup \{\mu_H(A) \mid A \subseteq [H], A \text{ contains no } 3\text{-term AP}\}.$$

Proposition There exists $A \subseteq [H]$ such that $\mu_H(A) = \alpha$, A contains no 3-term AP, and for any K-term AP, say $P_K \subseteq [H]$ for some hyperfinite integer K, it must be true that $\mu_K(A \cap P_K) \leq \alpha$.

We now fix this A in $[H]$ and derive a contradiction by locating a 3-term AP in A.

Without loss of generality we can assume that $H = NL$ with $L/N \approx 0$ and consider $[H]$ as $[L] \times [N]$ with a lexicographical order.

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Proof

Lemma There is an internal set $S \subseteq [N]$ such that $\mu_N(S) = 1$ and $\mu_L(A \cap ([L] \times \{s\})) = \alpha$ for every $s \in S$.

Lemma There exist $x \in S$, $N' < N/4$, and $M < N'$ such that $x + N' + [M] \subseteq S$ and $x + 2N' + 2[M] \subseteq S$ where $2[M] := \{2m \mid m \in [M]\}$.

Proof: Fix an $N' < N/4$. For any standard $m \in \mathbb{N}$

$$S \cap \bigcap_{i=0}^{m} (S - N' - i) \cap \bigcap_{i=0}^{m} (S - 2N' - 2i) \neq \emptyset$$

by the fact that $\mu_N(S) = 1$. Now the lemma follows from countable saturation and the fact that S is internal.

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Proof continued

For each $h \in [2L/3, L-1] \subseteq [L]$ define

$$E_h := \{x \in [L] \mid a, x, h \text{ is a 3-term AP and } (a, 0) \in A\}.$$

Then $\mu_L(E_h) > 0$. (Note that $\mu_L(E_h)$ is at least $\alpha/6$.)

Mixing Lemma

(i) (double counting) If $V \subseteq [L]$, then there is an $m \in [M]$ with

$$\mu_L(A \cap (V \times \{N' + m\})) \geq \alpha\mu_L(V);$$

(ii) (Van der Waerden) If $\{V_l \subseteq [L] \mid l \in [n]\}$ is a finite collection of internal sets with $\mu_L(V_l) > 0$ for each $l < n$, then there exists a K-term AP P in $N' + [M]$ such that for all $p \in P$ and all $l \in [n]$,

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(iii) (Szemerédi) There is a K-term AP $P \subseteq N' + [M]$ and a $T_p \subseteq [2L/3, L-1]$ with $\mu_L(T_p) = 1/3$ for each $p \in P$ such that

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Let $P \subseteq [M]$ and $T_p \subseteq [2L/3, L-1]$ for each $p \in P$ be obtained in (iii) of the mixing lemma. Fix an $p \in P$. For each $h \in T_p$ recall the following.

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Since $2p \in S$, we have $\mu_L(A \cap ([2L/3, L-1] \times \{2p\}) = \alpha/3 > 0$ by the maximality of α.

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2. $x \in E_h$ implies that there is an $a \in [L]$ and d such that $(a, 0) \in A$, $a + d = x$, and $a + 2d = h$.

Since $2p \in S$, we have $\mu_L(A \cap ([2L/3, L-1] \times \{2p\}) = \alpha/3 > 0$ by the maximality of α.

Since $\mu_L(T_p) = 1/3$, we have that $A \cap (T_p \times \{2p\}) \neq \emptyset$. Fix $(h, 2p) \in A \cap (T_p \times \{2p\})$. Since $A \cap (E_h \times \{p\}) \neq \emptyset$, we can find $(x, p) \in A \cap (E_h \times \{p\})$. Since $x \in E_h$, we can find $a \in [L]$ and d such that $(a, 0) \in A$, $a + d = x$, and $a + 2d = h$. Finally, we conclude that $(a, 0)$, $(a + d, p)$, and $(a + 2d, 2p)$ form a 3-term AP in A.

Last Step

Let $P \subseteq [M]$ and $T_p \subseteq [2L/3, L-1]$ for each $p \in P$ be obtained in (iii) of the mixing lemma. Fix an $p \in P$. For each $h \in T_p$ recall the following.

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Mixing Lemma 1

Recall that L, N are hyperfinite integers, $H = LN$ and $[H] = [L] \times [N]$ with the lexicographical order, $L/N \approx 0$, $A \subseteq [H]$ contains no 3-term AP, $\mu_H(A) = \alpha$ is maximal, $S \subseteq [N]$ with $\mu_N(S) = 1$, $\mu_L(A \cap ([L] \times \{s\}) = \alpha$ for all $s \in S$, $0 \in S$, $N' + [M] \subseteq S$, and $2N' + 2[M] \subseteq S$.

Mixing Lemma 1 (double counting)

If $V \subseteq [L]$, then there is an $p \in N' + [M]$ with

$$\mu_L(A \cap (V \times \{p\})) \geq \alpha\mu_L(V).$$

Proof For μ_L-almost all $v \in V$, $\mu_M(A \cap (\{v\} \times (N' + [M]))) = \alpha$. Let $B = A \cap (V \times (N' + [M]))$. Now applying Fubini's Theorem,

$$\int_{N'+[M]} \int_V \chi_B(v, p) d\mu_L d\mu_M = \alpha \int_V d\mu_L = \alpha\mu_L(V).$$

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Mixing Lemma 2 (Van der Waerden)

Let n be a finite positive integer and $\{V_l \subseteq [L] \mid l \in [n]\}$ is a collection of internal sets with $\mu_L(V_l) > 0$, then there exists a K-term AP P in $N' + [M]$ such that for all $p \in P$ and all $l \in [n]$,

$$\mu_L(A \cap (V_l \times \{p\})) = \alpha\mu_L(V_l).$$

Proof For a finite positive integer k we color each $p \in N' + [M]$ by one of 3^n colors depending on whether $\mu_L(A \cap (V_l \times \{p\}))$ is greater than (i) $\alpha\mu_L(V_l) + 1/k$, (ii) less than $\alpha\mu_L(V_l) - 1/k$, or (iii) between $\alpha\mu_L(V_l) \pm 1/k$. By Mixing Lemma 1 we can find a homogeneous K_k-term AP in $N' + [M]$ with the color of the type (iii) for all $l \in [n]$. Now repeat the process with $N' + [M]$ being replaced by K_k for $k = 1, 2, \ldots$ and use countable saturation to obtain the final K-term AP P for some hyperfinite integer K.

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Regularity Lemma Given standard real $\epsilon > 0$, one can find a partition $[L] = \bigsqcup_{l=0}^{n-1} V_l$ for some finite $n = O_\epsilon(1)$ and constants $0 \leq c_{l,h} \leq 1$ such that for any $F \subseteq [L]$, there exists $T_{F,\epsilon} \subseteq [2L/3, L-1]$ with $\mu_L(T_{F,\epsilon}) > \frac{1-\epsilon}{3} L$ such that for any $h \in T_{F,\epsilon}$

$$\left| |F \cap E_h| - \sum_{l<n} c_{l,h} |F \cap V_l| \right| \leq \epsilon L.$$

Mixing Lemma 3

There is a K-term AP $P \subseteq N' + [M]$ and a $T_p \subseteq [2L/3, L-1]$ with $\mu_L(T_p) = 1/3$ for each $p \in P$ such that for all $h \in T_p$

$$\mu_L(A \cap (E_h \times \{p\})) = \alpha \mu_L(E_h).$$

Proof Given a standard real $\epsilon > 0$, let $[L] = \bigsqcup_{l=0}^{n-1} V_l$ and constants $0 \leq c_{l,h} \leq 1$ obtained in the regularity lemma. Without loss of generality we assume that $\mu_L(V_l) > 0$ for all $l < n$. By Mixing Lemma 2 there is a K_ϵ-term AP $P_\epsilon \subseteq N' + [M]$ such that $\mu_L(A \cap (V_l \times \{p\})) = \alpha \mu_L(V_l)$ for each $p \in P_\epsilon$. For a fixed $p \in P_\epsilon$ there is a $T_{p,\epsilon} \subseteq [2L/3, L-1]$ with $\mu_L(T_{p,\epsilon}) > \frac{1-\epsilon}{3} L$ such that for every $h \in T_{p,\epsilon}$

$$\left| |A \cap (E_h \times \{p\})| - \sum_{l<n} c_{l,h} |A \cap (V_l \times \{p\})| \right| \leq \epsilon L$$

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$$\left| |A \cap (E_h \times \{p\})| - \sum_{l<n} c_{l,h} |A \cap (V_l \times \{p\})| \right| \leq \epsilon L$$

Now for every $h \in T_{p,2\epsilon} = T'_{p,\epsilon} \cap T''_{p,\epsilon}$ ($T'_{p,\epsilon} = T_{F,\epsilon}$ for $F = A \cap ([L] \times \{p\})$ and $T''_{p,\epsilon} = T_{[L],\epsilon}$ in the regularity lemma)

$$
\begin{aligned}
&||A \cap (E_h \times \{p\})| - \alpha|E_h|| \\
&\quad \le \left| |A \cap (E_h \times \{p\})| - \sum_{l<n} c_{l,h}|A \cap (V_l \times \{p\})| \right| \\
&\quad + \left| \sum_{l<n} c_{l,h}(|A \cap (V_l \times \{p\})| - \alpha|V_l|) \right| + \alpha \left| \sum_{l<n} c_{l,h}|V_l| - |E_h| \right| \\
&\quad \le \epsilon L + \sum_{l<n} c_{l,h}\iota_l|V_l| + \epsilon L
\end{aligned}
$$

where $\iota_l \approx 0$. Hence $|\mu_L(A \cap (E_h \times \{p\})) - \alpha\mu_L(E_h)| \le 2\epsilon$ for every $h \in T_{p,2\epsilon}$. Now let $\epsilon = 1/k$ and $k \to \infty$. Note that P_ϵ can be made nested. By countable saturation one has a K-term AP P inside these P_ϵ's and T_p as nonstandard extension of $T_{p,\epsilon}$'s with μ_L measure $1/3$ for each $p \in P$ such that $\mu_L(A \cap (E_h \times \{p\})) = \alpha\mu_L(E_h)$ for every $h \in T_p$.

The End

Thank you for your attention.