The Erdős Sumset Conjecture

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Definition

An *infinite sumset* in $\mathbb{N}$ is a set of the form $B + C := \{b + c : b \in B, c \in C\}$ where B and C are infinite subsets of $\mathbb{N}$.

Proposition

Any finite coloring of $\mathbb{N}$ *has a monochromatic infinite sumset.*

Proof.

Exercise. [Hint: use (the infinite) Ramsey's theorem] □

Conjecture (Erdős)

Any set $A \subset \mathbb{N}$ *with positive density contains an infinite sumset.*

Definition

A set $A \subset \mathbb{N}$ has positive upper Banach density (and we write $d(A) > 0$) if there exists a sequence of intervals $(I_N)_{N \in \mathbb{N}}$ with lengths $|I_N| \to \infty$ as $N \to \infty$ and such that

$$\lim_{N \to \infty} \frac{|A \cap I_N|}{|I_N|} > 0$$

Theorem (M-F. Richter-D. Robertson)

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J. M., F. Richter, D. Robertson, 2019, Ann. of Math.

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Notation

Let $f : \mathbb{N} \to \mathbb{C}$. We denote by

$$\mathop{\mathbb{E}}_{n\in\mathbb{N}} f(n) := \lim_{N\to\infty} \frac{1}{|I_N|} \sum_{n\in I_N} f(n)$$

for some sequence of intervals $(I_N)_{N\in\mathbb{N}}$ along which the limit exists.

J. M., F. Richter, D. Robertson, 2019, Ann. of Math.

Fact

If $A \subset \mathbb{N}$ *has* $d(A) > 0$ *then* $d(A \cap (A - n)) > 0$ *for some* $n \in \mathbb{N}$.

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- This implies that A contains $B + C$ where $B = \{0, n\}$ and $C = A \cap (A - n)$ has positive density.
- (Nathanson) Iterating we get for every $k \in \mathbb{N}$ sets B and C with $|B| = k$, $d(C) > 0$ and $B + C \subset A$.

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- (Nathanson) Iterating we get for every $k \in \mathbb{N}$ sets B and C with $|B| = k$, $d(C) > 0$ and $B + C \subset A$.

Lemma (Bergelson)

Let $A \subset \mathbb{N}$ have $d(A) > 0$. Then $\exists L \subset \mathbb{N}$ with $d(L) \geq d(A)$ such that for every finite $B \subset L$, there exists $C = C(B) \subset \mathbb{N}$ with $d(C) > 0$ and $B + C \subset A$.

V. Bergelson, 1985, J. London Math. Soc.

Theorem

If $A \subset \mathbb{N}$ has $d(A) > 1/2$, then A contains an infinite sumset.

M. Di Nasso, I. Goldbring, R. Jin, S. Leth, M. Lupini, K. Mahlburg, 2015, Canad. J. Math.

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Proposition

Let $A \subset \mathbb{N}$. If there exists $L \subset \mathbb{N}$ and $\varepsilon > 0$ such that for every finite $F \subset L$

$$\bigcap_{\ell \in F} (A - \ell) \cap \left\{ n \in \mathbb{N} : d\big((A - n) \cap L\big) > \varepsilon \right\} \qquad \text{is infinite}$$

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Lemma (Bergelson, again)

Let $A \subset \mathbb{N}$ have $d(A) > 0$. Then $\exists L \subset \mathbb{N}$ with $d(L) \geq d(A)$ such that for every finite $F \subset L$,

$$d\left(\bigcap_{\ell \in F} (A - \ell)\right) > 0$$

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Reduced main theorem to

Theorem

For every $A \subset \mathbb{N}$ *with* $d(A) > 0$ *there exist* $L \subset \mathbb{N}$ *and* $\varepsilon > 0$ *such that for every finite* $F \subset L$

$$\bigcap_{\ell \in F} (A - \ell) \cap \Big\{ n \in \mathbb{N} : d\big((A - n) \cap L\big) > \varepsilon \Big\} \qquad \textit{is infinite}$$

Example

- If A is random, then take $L = \mathbb{N}$ and any $\varepsilon < d(A)$.
- If A is a Bohr set, say $A = \{x \in \mathbb{N} : \|x\alpha\|_{\mathbb{T}} < \rho\}$, take $L = \{\ell \in \mathbb{N} : \|\ell\alpha\|_{\mathbb{T}} < \rho/2\}$.

Definition

An *ultrafilter* is a collection p of subsets of $\mathbb{N}$ such that

- $\emptyset \notin p$, $\mathbb{N} \in p$.
- If $A \in p$ and $A \subset B$ then $B \in p$.
- If $A, B \in p$ then $A \cap B \in p$.
- $A \notin p$ if and only if $\mathbb{N} \setminus A \in p$.

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For each $n \in \mathbb{N}$, the collection $p_n := \{A \subset \mathbb{N} : n \in A\}$ is a *principal* ultrafilter.

$A - p := \{x \in \mathbb{N} : A - x \in p\}$. Note that $A - p_n = A - n$.

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Let $f : \mathbb{N} \to \mathbb{C}$ be bounded and p an ultrafilter.

$$\lim_n f(n) = y \iff \forall \varepsilon > 0,\ \{n : |f(n) - y| < \varepsilon\} \text{ is cofinite.}$$

$$p - \lim_n f(n) = y \iff \forall \varepsilon > 0,\ \{n : |f(n) - y| < \varepsilon\} \in p.$$

Theorem

For every $A \subset \mathbb{N}$ *with* $d(A) > 0$ *there exist* $L \subset \mathbb{N}$ *and* $\varepsilon > 0$ *such that for every finite* $F \subset L$

$$\bigcap_{\ell \in F} (A - \ell) \cap \Big\{ n \in \mathbb{N} : d\big((A - n) \cap L\big) > \varepsilon \Big\} \qquad \textit{is infinite}$$

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Can be written as

Theorem

For every $A \subset \mathbb{N}$ *with* $d(A) > 0$ *there exist a non-principal ultrafilter* p *such that*

$$p - \lim_{n} d\big((A - n) \cap (A - p)\big) > 0$$

For $f : \mathbb{N} \to [0,1]$ define $T^n f(x) = f(x+n)$ and $T^p f : \mathbb{N} \to [0,1]$ by

$$T^p f(x) = p - \lim_n f(x+n).$$

Note that $1_{A-n} = T^n 1_A$ and $1_{A-p} = T^p 1_A$.

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Note that $1_{A-n} = T^n 1_A$ and $1_{A-p} = T^p 1_A$.

The main theorem follows from:

Theorem (Final reduction)

For every bounded $f : \mathbb{N} \to [0,1]$ and every $\varepsilon > 0$ there exists a non principal ultrafilter p such that

$$p - \lim_n \mathop{\mathbb{E}}_{x \in \mathbb{N}} T^n f(x) T^p f(x) \geq \left(\mathop{\mathbb{E}}_{x \in \mathbb{N}} f(x) \right)^2 - \varepsilon.$$

Definition

$f : \mathbb{N} \to \mathbb{C}$ is *weak mixing* if $\forall\ g : \mathbb{N} \to \mathbb{C}$,

$$\mathop{\mathbb{E}}_{n\in\mathbb{N}} \left| \mathop{\mathbb{E}}_{x\in\mathbb{N}} T^n f(x) g(x) \right| = 0$$

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Definition

$f : \mathbb{N} \to \mathbb{C}$ is *(Besicovitch) almost periodic* if $\forall \varepsilon > 0$ there exists a trig. polynomial $p(x) = \sum_{j\in J} c_j e(\theta_j x)$ such that

$$\mathop{\mathbb{E}}_{x\in\mathbb{N}} \left| f(x) - p(x) \right| < \varepsilon.$$

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Remark

Unfortunately, it is not true that every bounded $f : \mathbb{N} \to \mathbb{R}$ can be decomposed as $f = f_{wm} + f_{bes}$!

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Definition

$f : \mathbb{N} \to \mathbb{C}$ is *compact* if $\forall\, \varepsilon > 0$ there exists a Bohr_0 set $B \subset \mathbb{N}$ such that

$$\forall n \in B \qquad \mathop{\mathbb{E}}_{x\in\mathbb{N}} \left| T^n f(x) - f(x) \right|^2 < \varepsilon.$$

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$B = \{n \in \mathbb{N} : \|n\alpha_1\|_{\mathbb{T}} < \rho, \ldots, \|n\alpha_d\|_{\mathbb{T}} < \rho\}$ for some $\alpha_i \in \mathbb{R}$ and $\rho > 0$.

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Theorem

Every bounded $f : \mathbb{N} \to \mathbb{C}$ can be decomposed as $f = f_{wm} + f_c$ where f_{wm} is weak mixing and f_c is compact.

Definition

$f : N \to \mathbb{C}$ is *(Besicovitch) almost periodic* if $\forall \varepsilon > 0$ there exists a trig. polynomial $p(x) = \sum_{j \in J} c_j e(\theta_j x)$ such that

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Definition

$f : \mathbb{N} \to \mathbb{C}$ is in $Bes^{\perp}$ if for every $\theta \in \mathbb{R}$

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Theorem

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Theorem

For every bounded $f : \mathbb{N} \to [0,1]$ *and every* $\varepsilon > 0$ *there exists a non principal ultrafilter p such that*

$$p - \lim_{n} \mathop{\mathbb{E}}_{x \in \mathbb{N}} T^n f(x) T^p f(x) \geq \left(\mathop{\mathbb{E}}_{x \in \mathbb{N}} f(x) \right)^2 - \varepsilon.$$

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Proof.

- Split $f = f_c + f_{wm} = f_{bes} + f_{\perp}$.
- $T^n f T^p f = T^n f_{wm} T^p f + T^n f_c T^p f_{bes} + T^n f_c T^p f_{\perp}$.

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- $\mathbb{E}_{n\in\mathbb{N}} \left| \mathbb{E}_{x\in\mathbb{N}} T^n f_{wm}(x) T^p f(x) \right| = 0$, so

$$p - \lim_n \mathop{\mathbb{E}}_{x\in\mathbb{N}} T^n f_{wm}(x) T^p f(x) = 0$$

  as long as p contains no set with 0 density.

Theorem

For every bounded $f:\mathbb{N}\to[0,1]$ *and every* $\varepsilon>0$ *there exists a non principal ultrafilter* p *such that*

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Such ultrafilters are called *essential*.

□

Theorem

For every bounded $f : \mathbb{N} \to [0,1]$ *and every* $\varepsilon > 0$ *there exists an essential ultrafilter* p *such that*

$$p\text{-}\lim_{n} \mathop{\mathbb{E}}_{x\in\mathbb{N}} T^n f_c(x) T^p f_{bes}(x) + T^n f_c(x) T^p f_{\perp}(x) \geq \left(\mathop{\mathbb{E}}_{x\in\mathbb{N}} f(x) \right)^2 - \varepsilon.$$

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Fact

- $T^p e(\theta x) = (p - \lim_n e(n\theta))\, e(\theta x) = \lambda e(\theta x).$

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Fact

- $T^p e(\theta x) = (p-\lim_n e(n\theta))\, e(\theta x) = \lambda e(\theta x)$.
- $\forall\delta>0$, *there is a* $Bohr_0$ *set* B *such that if* $B\in p$ *then*

$$|T^p e(\theta x) - e(\theta x)| < \delta \text{ for all } x\in\mathbb{N}.$$

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For every bounded $f : \mathbb{N} \to [0,1]$ and every $\varepsilon > 0$ there exists an essential ultrafilter p such that

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- *There exists a* $Bohr_0$ *set* B *such that for almost every ultrafilter* p *with* $B \in p$,

$$\mathop{\mathbb{E}}_{x\in\mathbb{N}} \left|T^p f_{bes}(x) - f_{bes}(x)\right|^2 < \varepsilon/2$$

Theorem

For every bounded $f : \mathbb{N} \to [0,1]$, every Bohr$_0$ set B and every $\varepsilon > 0$ there exists a (positive measure set of) essential ultrafilter p such that $B \in p$ and

$$p - \lim_{n} \mathop{\mathbb{E}}_{x \in \mathbb{N}} T^n f_c(x) f_{bes}(x) + T^n f_c(x) T^p f_{\perp}(x) \geq \left(\mathop{\mathbb{E}}_{x \in \mathbb{N}} f(x) \right)^2 - \varepsilon.$$

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- There is a Bohr_0 set $\tilde{B}$ such that $\forall n \in \tilde{B}$,

$$\mathop{\mathbb{E}}_{x \in \mathbb{N}} \left| T^n f_c(x) - f_c(x) \right|^2 < \varepsilon^2/4.$$

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For every bounded $f : \mathbb{N} \to [0,1]$, every Bohr$_0$ set B and every $\varepsilon > 0$ there exists a (positive measure set of) essential ultrafilter p such that $B \in p$ and

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$$\mathop{\mathbb{E}}_{x\in\mathbb{N}} \left| T^n f_c(x) - f_c(x) \right|^2 < \varepsilon^2/4.$$
- If $\tilde{B} \in p$ then
$$p - \lim_n \mathop{\mathbb{E}}_{x\in\mathbb{N}} T^n f_c(x) f_{bes}(x) \geq \mathop{\mathbb{E}}_{x\in\mathbb{N}} f_c(x) f_{bes}(x) - \varepsilon/2$$

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- There is a Bohr$_0$ set $\tilde{B}$ such that $\forall n \in \tilde{B}$,

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$$\mathop{\mathbb{E}}_{x\in\mathbb{N}} f_c(x) f_{bes}(x) = \mathop{\mathbb{E}}_{x\in\mathbb{N}} f_{bes}(x)^2 \geq \left(\mathop{\mathbb{E}}_{x\in\mathbb{N}} f_{bes}(x)\right)^2 = \left(\mathop{\mathbb{E}}_{x\in\mathbb{N}} f(x)\right)^2$$

Theorem

Let $f : \mathbb{N} \to \mathbb{R}$ *be bounded and in* $Bes^{\perp}$. *For every Bohr set* $B \subset \mathbb{N}$ *and bounded* $h : \mathbb{N} \to \mathbb{R}$ *there exists a (positive measure set of) essential ultrafilter* p *such that* $B \in p$ *and*

$$\mathop{\mathbb{E}}_{x \in \mathbb{N}} h(x) T^p f(x) \geq 0.$$

M. Beiglböck, 2011, Israel J. Math.

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- Bergelson's intersectivity: For all $A \subset \mathbb{N}$, there is an non-principal ultrafilter p such that

$$d(A - p) \geq d(A).$$

- Beiglbock: For all $A, B \subset \mathbb{N}$, there is an non-principal ultrafilter p such that

$$d\big(B \cap (A - p)\big) \geq d(A)d(B).$$

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Question

Is it true that every set $A \subset \mathbb{N}$ with positive density contains a set of the form

$$t + B \oplus B := \{t + b_1, b_2 : b_1, b_2 \in B, b_1 \neq b_2\}?$$

Question

Is it true that every set $A \subset \mathbb{N}$ with positive density contains a set of the form $B + C + D$?

Question

Do the primes contain an infinite sumset?

- Granville showed that yes conditionally on the Hardy-Littlewood tuples conjecture!
- That the primes contain $B + C$ where $|C| = \infty$ and $|B| = 2$ is equivalent to Zhang's theorem that the primes have bounded gaps infinitely often.

The Erdős Sumset Conjecture

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