BOOK REVIEW: "HILBERT'S FIFTH PROBLEM AND RELATED TOPICS" BY TERENCE TAO

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1. Book overview

The book under review (based primarily on the author's lecture notes and blog posts) centers around three major theorems: the positive resolution of **Hilbert's fifth problem**, due to the combined efforts of Gleason, Montgomery, and Zippin [7] and [13]; the **structure theorem for finite approximate groups**, in its full generality due to Breuillard, Green, and Tao [1], building upon a major breakthrough by Hrushovski [12]; and **Gromov's theorem for polynomial growth** [10]. Each of these theorems are widely considered to be jewels in their respective areas of mathematics: the theory of locally compact groups, additive combinatorics, and geometric group theory respectively. The book proves these theorems in full details, draws analogies and connections between them, and, most importantly to the logic audience, explains the use of ultraproduct/nonstandard techniques in deriving these results.

The cornerstone around which the rest of the book revolves is the positive solution to Hilbert's fifth problem. Recall that a **Lie group** is a smooth manifold G endowed with the structure of a group for which the group operations are smooth functions. (In other words, a Lie group is a group object in the category of smooth manifolds.) In particular, a Lie group is a **locally euclidean topological group**, that is, a topological space equipped with a group structure for which the group operations are continuous and which further possesses an open neighborhood of the identity homeomorphic to some open subset of some (finite-dimensional) euclidean space. Perhaps the most common interpretation of Hilbert's fifth problem asks whether or not this a priori weaker structure, namely that of being a locally euclidean topological group, already implies the stronger structure of being a Lie group, that is, whether or not the locally euclidean group can be equipped with the structure of a smooth manifold (compatible with the given topology on the group) for which the group operations are now upgraded from being mere continuous functions to being smooth functions. The positive resolution to Hilbert's fifth problem can be stated as follows:

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Theorem 1.1 (Gleason, Montgomery, and Zippin)**.** *If* G *is a locally compact group, the following are equivalent:*

(1) G *is locally euclidean.*
(2) G *is a Lie group.*
(3) G *has the* ***no small subgroups*** (***NSS***) *property: there is an open neighborhood of the identity in* G *containing no nontrivial subgroup.*

The proof of Theorem 1.1 uses a combination of analytic and representation theoretic techniques (amongst other things). A beautiful simplification of part of the story is due to Hirschfeld [11], who used techniques from nonstandard analysis to give an elegant reinterpretation of the technical Gleason-Yamabe lemmas used to prove the hardest direction in Theorem 1.1, namely the implication that the NSS property implies being a Lie group. In particular, given some infinite hypernatural number $N \in {}^*\mathbb{N}$, Hirschfeld considered the following two subsets of the nonstandard extension ${}^*G$ of G:

$$G[N] := \left\{a \in {}^*G \ : \ a^i \approx e \text{ for all } i \in {}^*\mathbb{Z} \text{ with } \frac{i}{N} \approx 0\right\},$$

and

$$G^0[N] := \left\{a \in {}^*G \ : \ a^i \approx e \text{ for all } i \in {}^*\mathbb{Z} \text{ with } \frac{i}{N} \text{ finite}\right\}.$$

Here, we use the $\approx$ symbol to mean "infinitely close." Hirschfeld's nonstandard interpretation of the Gleason-Yamabe lemmas is that $G[N]$ is a subgroup of G^*, $G^0[N]$ is a normal subgroup of $G[N]$, and $G[N]/G^0[N]$ is an abelian group which one can further enrich with the structure of a Lie algebra which can in turn be used to equip G with the structure of a Lie group. (Hirschfeld carried this line of reasoning out under the assumption that G has the NSS property; van den Dries and Goldbring show how to prove these facts solely under the local compactness assumption in [4].) Hirschfeld mentions that the idea of giving a nonstandard analysis account of Hilbert's 5th problem was was passed on to him by Moshe Machover as a possible topic for an M.Sc. thesis.

An interesting sidenote for logicians is to recall one use of Theorem 1.1 in model theory. In [14], Anand Pillay showed how to construct, for any group G definable in an o-minimal structure $(M, <, \ldots)$, a topology on G (called the t-topology) which makes G a topological group and which has an open neighborhood of the identity homeomorphic to some open neighborhood of M^n (with the product topology). In particular, when $(M, <, \ldots) = (\mathbb{R}, <, \ldots)$ is an o-minimal expansion of the reals, this topological group is locally euclidean and thus a Lie group.

Perhaps even more important to the rest of mathematics than the positive resolution of Hilbert's fifth problem is the more general structure theorem for an arbitrary locally compact group, which roughly says that an arbitrary locally compact group can be "approximated" by Lie groups:

Theorem 1.2 (Gleason and Yamabe). *Suppose that* G *is a locally compact group. Then for every open neighborhood* U *of the identity in* G*, there is an open subgroup* G' *of* G *and a compact subgroup* N *of* G' *contained in* U *such that* G'/N *is a Lie group.*

Before moving on, we relay an attractive heuristic that Tao uses to tie together all of the major theorems discussed in this book. Namely, each of the theorems presented involve some "group-like" object satisfying some "weak regularity" assumption. In the case of the two theorems above, the group-like assumption is that G is a topological group; in Theorem 1.1, the weak regularity assumption is that the group is locally euclidean, while in the case of Theorem 1.2, the weak regularity assumption is that the group is locally compact. The conclusion in each of the major theorems in the book is that the input object (or some related object) is then "close" to (or perhaps even equal to) some object possessing either "Lie-type" structure in the continuous case or "nilpotent-type" structure in the discrete cases to follow. It is clear how these conclusions manifest themselves in the two theorems presented above.

The next major topic discussed is the overall structure theorem for approximate groups. Given a group G and a natural number $K \in \mathbb{N}$, a subset A of G is called a K**-approximate group** if A is symmetric (that is, $1 \in A = A^{-1}$) and $A \cdot A \subseteq X \cdot A$ for some $X \subseteq G$ with $|X| \leqslant K$. It is clear that a 1-approximate subgroup of G is the same as an actual subgroup of G, whence, for K small compared to the size of A, being a K-approximate group is one vague way of asking that A "almost" be a subgroup of G. The study of approximate groups is of central interest in additive combinatorics. Approximate groups have applications to a wide variety of areas of mathematics, including number theory and the construction of expander graphs. The definitive word on approximate subgroups of *abelian groups* is known as **Freiman's theorem**:

Theorem 1.3 (Green and Rusza [9]). *Suppose that* A *is a finite* K*-approximate subgroup of the abelian group* G*. Then there is a finite subgroup* H *of* G *and a* ***generalized arithmetic progression*** $P \subseteq G/H$ *such that* $\pi^{-1}(P) \subseteq 4A := A + A + A + A$*, where* $\pi : G \to G/H$ *is the usual quotient map, and with* $|P| \gg_K \frac{|A|}{|H|}$.

Here, given any abelian group G (which is G/H in the statement of the theorem above), any $v_1, \ldots, v_r \in G$, and any $N_1, \ldots, N_r \in \mathbb{N}$, one can consider the generalized arithmetic progression

$$\{a_1 v_1 + \cdots + a_r v_r \ : \ a_i \in \mathbb{Z}, \ |a_i| \leqslant N_i \text{ for all } i = 1, \ldots, r\},$$

which is easily seen to be a 2^r-approximate subgroup of G. Thus Theorem 1.3 shows that any finite K-approximate group is "close to" (in the precise sense described above) to a particular kind of approximate group. Thus, the previous theorem once again fits the scheme suggested by Tao appearing in Theorems 1.1 and 1.2, this time the Lie-theoretic structure being replaced by the "nilpotent-type" structure of a generalized arithmetic progression.

Progress on the structure of approximate subgroups of nonabelian groups had stalled somewhat until Hrushovski's breakthrough work in [12]. Of particular importance was Hrushovski's idea that an *ultraproduct of finite approximate groups*, naturally termed an **ultra-approximate group**, can naturally be "modelled" by a locally compact group. (This idea is reminiscent of the **asymptotic cone** construction appearing in the proof of Gromov's theorem, that we discuss next.) Slightly more precisely, if A^* is such an ultra-approximate subgroup of the (nonstandard) group G^* and we set H^* to be the subgroup of G^* generated by A^* (a so-called $\bigvee$-definable group, that is, a subgroup of G^* that is a countable union of definable sets), then **Hrushovski's stabilizer theorem** (adapting methods from stability theory) produces a normal subgroup N^* of H^* which is $\bigwedge$-definable (that is, a normal subgroup of H^* that is a countable intersection of definable sets) and of "small" index in H^*. General model theory then provides the quotient group H^*/N^* with a locally compact topology compatible with the group structure, and it is this locally compact group that models the original ultra-approximate group A^* in an appropriate manner. By using the Gleason-Yamabe theorem (Theorem 1.2 above), one can then approximate this locally compact group by a Lie-group (whilst simultaneously replacing the original ultra-approximate group by a closely related one), allowing one to bring in tools from Lie theory and, in particular, allowing one to prove facts about ultra-approximate groups by induction on the dimension of the corresponding Lie model.

Breuillard, Green, and Tao [1] reworked this construction using ideas coming from additive combinatorics (what they call Sanders-Croot-Sisask theory) and at the same time completed the project of the structure theory of finite approximate groups, obtaining the desired noncommutative analog of Freiman's theorem:

Theorem 1.4 (Breuillard, Green, and Tao, informally stated)**.** *If* A *is a finite* K*-approximate subgroup of* G*, then there is a finite subgroup* H *of* G *and a* ***noncommutative progression*** $P \subseteq N(H)/H$ (*where* $N(H)$ *denotes the normalizer of* H *in* G) *of rank* $O_K(1)$ *and which generates a nilpotent subgroup of* G *of nilpotency class* $O_K(1)$ *such that* $\pi^{-1}(P)$ *is "close to"* A^4 (*in essentially the same sense as in Theorem 1.3*).

A nice treatment of the above theorem can also be found in van den Dries' Bourbaki seminar notes [3].

It is interesting to note that the proof of Theorem 1.4 uses the "local" version of Hilbert's fifth problem and the Gleason-Yamabe theorem, that is, the versions of these results for the class of **local groups**, which are topological spaces equipped with continuous, partially defined group operations satisfying natural axioms. The local version of these results was first established by Goldbring [8] using nonstandard analysis.

The final major theorem discussed in this book is Gromov's theorem on polynomial growth. Suppose now that G is a finitely generated group with finite generating set S, which we assume is symmetric for simplicity. Associated to the pair (G, S) is a metric $d = d_{G,S}$ on G given by defining $d(g, h)$ to be the minimal $m \in \mathbb{N}$ for which there are $s_1, \ldots, s_m \in S$ such that $g = hs_1 \cdots s_m$. For $r \in \mathbb{N}$, we then let $B(r) := \{g \in G : d(g, e) \leqslant r\}$ denote the closed ball in G centered at the identity of radius r with respect to this metric. Of course, $|B(r)| \leqslant |S|^r$; however, for certain groups, this exponential upper bound is far from being sharp. In fact, we say that G has **polynomial growth** if there is $d \in \mathbb{N}$ and $C > 0$ such that $|B(r)| \leqslant Cr^d$ for all $r \in \mathbb{N}$. (A priori this notion looks like it depends on the choice of generating set, but one can show that in fact it does not.) A theorem of Wolf shows that any finitely generated nilpotent group has polynomial growth as does any finitely generated group that contains a finite-index nilpotent subgroup (that is, any **virtually nilpotent group**) as a group that is virtually of polynomial growth is in fact of polynomial growth. A remarkable theorem of Gromov is that the converse holds:

Theorem 1.5 (Gromov)**.** *If* G *is a finitely generated group of polynomial growth, then* G *is virtually nilpotent.*

Once again, this theorem fits the general mold suggested by Tao: the group-like object this time is a finitely generated group, the weak regularity assumption is the polynomial growth assumption, and the conclusion shows that a nilpotent-like object exists, namely a nilpotent subgroup, that is close to the original object in that it is of finite index un the original group.

Historically, Gromov's theorem predates the theorem on approximate subgroups and has nontrivial connections with logic as well as the Gleason-Yamabe theorem. Indeed, after Gromov's initial proof, van den Dries and Wilkie [6] gave a precise account of the proof while developing the machinery behind the **asymptotic cone** construction that is now ubiquitous throughout geometric group theory. Given an infinite hypernatural number $N \in {}^*\mathbb{N}$, they consider those elements $g \in {}^*G$ such that $\frac{d(g,e)}{N}$ is finite. Denoting this set by G_N, they considered the standard metric space C_N whose underlying set is the quotient of G_N by the pseudometric given by $(g, h) \mapsto \mathrm{st}(d(g, h))$, where st denotes the standard part operation. While the cone C_N is always a homogeneous, connected, locally connected metric space, the assumption of polynomial growth allows one to find an infinite $N \in {}^*\mathbb{N}$ such that C_N is furthermore proper (closed balls are compact) and "finite-dimensional" in an appropriate sense. An extension of the work on Hilbert's fifth problem described above allows one to conclude that the isometry group of C_N is in fact a Lie group. Since elements of G are naturally isometries of C_N, this allows one to bring in the structure theory of Lie groups in order to conclude that G is virtually nilpotent.

In the book under review, Tao shows how to instead derive Gromov's theorem from the structure theory of approximate groups (as was first established by Hrushovski in [12]). Indeed, a simple pigeonhole argument allows one to show that, given a finitely generated group G, there is a $K > 1$ such that, for some sequence (r_n) from $\mathbb{N}$ tending to infinity, the the sets $B(r_n)$ are K-approximate subgroups of G. In fact, using the techniques established in the proof of the approximate group theorem, one can prove the following stronger version of Gromov's theorem:

Theorem 1.6. *For every $C > 0$ and $d \in \mathbb{N}$, there exists $M \in \mathbb{N}$ such that: for any group G with finite generating set S, if there is $m \geqslant M$ such that $|S^m| \leqslant Cm^d|S|$, then G is virtually nilpotent.*

Note that this theorem does not require full polynomial growth but only the a priori weaker condition that the growth rate is bounded for a sufficiently large scale which, furthmerore, only depends on the constants C and d (but importantly not on $|S|$).

2. A brief tour of the book

Readers familiar with the author's blog posts will recognize the usual traits which make reading his blogs so enjoyable: the exposition is casual yet simultaneously precise, guiding examples are plentiful and extremely useful in forming one's intuitions, and examples (varying in difficulty) appear to complement the main text, fleshing out details of earlier arguments as well as providing further elaborations on topics appearing throughout the text.

The book attempts to be fairly self-contained and for that reason, after a lengthy introductory section (akin to the summary occurring in the previous section of this review, but much longer and with many more elucidating examples), the book begins with a section on preliminaries needed from Lie theory. Sections 1.3-1.6 form the technical core of the solution of Hilbert's fifth problem and the Gleason-Yamabe theorem. This portion of the book is written purely in standard language (much to this author's chagrin); the reader interested in seeing the nonstandard point of view can consult the survey article [4] by van den Dries and Goldbring.

Before moving on to the theory of approximate groups and its consequences (including the aforementioned approach to Gromov's theorem as well as other consequences in the theory of Riemannian manifolds), the author makes a lengthy segue into the use of ultraproduct arguments as a "bridge" between "soft" and "hard" analysis. The main use of this technique is the aformentioned use of ultraproducts of finite approximate groups (a subject fitting in the "hard" category) to obtain Lie models (a subject fitting in the "soft" category). This section should be extremely useful to someone who has never before seen these techniques so ubiquitous in applications of nonstandard analysis. While the reader already familiar with nonstandard techniques might be a bit put off by Tao's choice of notation, this section is a lovely introduction to basic ultraproduct techniques, including discussions of saturation, Łos' theorem, and nonstandard hulls of metric spaces.

The second part of the book contains a handful of "miscellaneous" topics related to the main thread of the text appearing in the first part. Of particular interest to logicians will be Tao's discussion of a nonstandard proof of the triangle-removal lemma (a key result used in the graph-theoretic proof of Roth's theorem on arithmetic progressions in positive density sets), which uses the "ultraproduct of counterexamples" technique discussed above together with the **Loeb measure** construction, which is developed from scratch.

Another topic of interest to logicians is the section on polynomial bounds via nonstandard analysis. Many logicians are aware of the approach of using nonstandard analysis to derive (albeit nonexplicit) bounds in polynomial rings as discussed at length in the paper of van den Dries and Schmidt [5]. In this book, Tao provides another example of this technique by giving such a soft proof of a recent (at the time of writing) result in this area:

Theorem 2.1 (Chang [2]). *Given polynomials* $P_1, \ldots, P_r : \mathbb{C}^n \to \mathbb{C}$ *of degree at most* d *and with rational coefficients of height at most* h*, if there is a solution to the system*

$$P_1 = \cdots = P_r = 0$$

in $\mathbb{C}^n$*, then there is a solution in* $\overline{\mathbb{Q}}^n$ (*the algebraic numbers*) *of degree at most* C *and height at most* Ch^C*, where* C *is a universal constant that depends only on* r*,* n*, and* d*.*

The proof of this theorem without the bounds is a simple application of the Nullstellensatz and Tao proceeds to show how the previous theorem is proven, once again using a "compactness and contradiction" argument, showing that if no such C existed, then one could take an ultraproduct of counterexamples and contradict the corresponding qualitative version of the theorem (applied to the algebraic closure of a particular external subfield of the hypercomplex field $^*\mathbb{C}$).

3. Concluding remarks

The material presented in this book covers a wide landscape of extraordinarily breathtaking mathematics as presented in a manner that only Tao could. The reader taking the time to make their way through this book, attempting the exercises as they proceed, will feel infinitely rewarded. Logicians looking for interesting applications of their techniques will appreciate Tao's fresh take and advocacy for the nonstandard method. This book is a must-read!

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