Ultraproducts as a tool in the model theory of metric structures

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Preliminary comments

- Ultraproducts play a more significant role in model theory of metric structures than in classical model theory of discrete, algebraic structures.
- In the metric setting, ultraproducts do more than proving compactness.
- They provide an important experimental tool for studying metric structures model theoretically.
- Especially toward verifying axiomatizability of classes of structures and clarifying what is definable.

Early uses of ultraproducts

- hints in work of Skolem (1934) and von Neumann (1942).
- a metric ultraproduct occurs in work of Wright, a student of Kaplansky (1954; Ann. of Math.)
- for discrete structures, the work of Łoś was published in 1955; Robinson's nonstandard analysis in 1961.
- ultraproducts of some operator algebras in Sakkai (1962), McDuff (1969), Connes (1974), ...
- Banach space ultraproducts in Krivine (1967), nonstandard hulls in Luxemburg (1968); used extensively by Lindenstrauss, Pisier, Johnson, . . . and studied by Stern, Heinrich, ...
- metric geometry: (ultralimits) Gromov (1981), (asymptotic cones) van den Dries–Wilkie (1984).

A crash course in continuous logic

- Signatures $\mathcal{L}$ specify *moduli of uniform continuity* for their function symbols F and predicate symbols P.
- $\mathcal{L}$-structures: complete metric spaces equipped with constants, functions, and predicates that interpret the symbols of $\mathcal{L}$.
- predicates are $[0,1]$-valued; predicates and functions must satisfy the given moduli.
- $\mathcal{L}$-terms: built as usual from constants and function symbols.
- Atomic $\mathcal{L}$-formulas: of the form $P(t_1, \ldots, t_n)$ and $d(t_1, t_2)$.
- $\mathcal{L}$-formulas: obtained by closing under continuous connectives $u : [0,1]^n \to [0,1]$ and quantifiers sup and inf.
- A theory T in $\mathcal{L}$ is a set of axioms, which are expressions of the form "$\sigma = 0$" where σ is an $\mathcal{L}$-sentence (formula with no free variables). We identify T with the set of sentences σ.

Ultraproducts defined

- $(M_i \mid i \in I)$ are $\mathcal{L}$-structures, $\mathcal{U}$ an ultrafilter on I.
- We let $N_0 = \prod_{i \in I} M_i$ as a set; its members are $(a) = (a_i \mid i \in I)$, $a_i \in M_i$.
- We interpret the symbols on N_0:

$$\begin{aligned} F^{N_0}\big((a_i \mid i \in I), \dots\big) &= \big(F^{M_i}(a_i, \dots) \mid i \in I\big) && \in N_0 \\ P^{N_0}\big((a_i \mid i \in I), \dots\big) &= \lim_{i \to \mathcal{U}} P^{M_i}(a_i, \dots) && \in [0, 1] \end{aligned}$$

- d^{N_0} is only a pseudometric; $E((a), (b)) :\Leftrightarrow d^{N_0}((a), (b)) = 0$ is an equivalence relation and the quotient N_0/E has canonical metric and interpretations of the symbols of $\mathcal{L}$.
- We define the ultraproduct to be N_0/E (which is complete) and denote it by $N := \prod_{i \in I} M_i/\mathcal{U}$.
- The image of $(a) \in N_0$ in N is denoted $(a)_{\mathcal{U}}$; note that

$$(a)_{\mathcal{U}} = (b)_{\mathcal{U}} \Leftrightarrow 0 = \lim_{i \to \mathcal{U}} d(a_i, b_i) \quad \Big[= d^{N_0}\big((a), (b)\big)\Big].$$

Properties of ultraproducts

- Łoś's Theorem: for every $\mathcal{L}$-formula $\varphi(x, y, \ldots)$ and elements $(a)_\mathcal{U}, (b)_\mathcal{U}, \ldots \in N = \prod M_i/\mathcal{U}$:

$$\varphi^N\big((a)_\mathcal{U}, (b)_\mathcal{U}, \ldots\big) = \lim_{i \to \mathcal{U}} \varphi^{M_i}(a_i, b_i, \ldots).$$

- Corollary The Compactness Theorem for continuous logic of metric structures.

Elementary

Let M, N be $\mathcal{L}$-structures. Let $A \subseteq M$, $B \subseteq N$ and $f\colon A \to B$.

Definition

(a) M and N are elementarily equivalent if $\sigma^M = \sigma^N$ for all $\mathcal{L}$-sentences σ. We write $M \equiv N$.

(b) The map f is elementary if

$$(M, \langle a \mid a \in A \rangle) \equiv (N, \langle f(a) \mid a \in A \rangle).$$

Examples

Any isomorphism from M onto N is elementary.
The diagonal map $D_M\colon M \to M_{\mathcal{U}}$ is elementary.
Any restriction of an elementary map is elementary
The composition of elementary maps is elementary.
The inverse of an elementary map is elementary.

Let M, N be $\mathcal{L}$-structures. Let $A \subseteq M, B \subseteq N$ and $f\colon A \to B$.

- Keisler-Shelah Theorem $M \equiv N$ if and only if there is an ultrafilter $\mathcal{U}$ such that $M_{\mathcal{U}} \cong N_{\mathcal{U}}$.
- Corollary The map f is elementary iff there is an ultrafilter $\mathcal{U}$ and an isomorphism J from $M_{\mathcal{U}}$ onto $N_{\mathcal{U}}$ such that
$$J \circ D_M = D_N \circ f \quad \text{on } A.$$

Definition

A class $\mathcal{C}$ of $\mathcal{L}$-structures is axiomatizable if there is an $\mathcal{L}$-theory T such that $\mathcal{C}$ is the class of all models of T; that is,

$$\mathcal{C} = \{M \mid M \text{ is an } \mathcal{L}\text{-structure and } \sigma^M = 0 \text{ for all } \sigma \in T\}.$$

Theorem

For any class $\mathcal{C}$ of $\mathcal{L}$-structures, the following are equivalent:
(a) $\mathcal{C}$ is axiomatizable.
(b) $\mathcal{C}$ is closed under isomorphisms, ultraproducts, and ultraroots.

M is an ultraroot of N if there is an ultrafilter $\mathcal{U}$ such that N is isomorphic to $M_{\mathcal{U}}$.

T-formulas

Let T be an $\mathcal{L}$-theory and $x = x_1, \ldots, x_n$ a tuple of distinct variables.

- A T-formula $\varphi(x)$ is a sequence $(\varphi_n(x))$ of $\mathcal{L}$-formulas such that $\varphi_n^M(a)$ converges as $n \to \infty$, uniformly for all models M of T and all $a \in M^x$.
- The interpretation $\varphi^M$of a T-formula $\varphi(x) = (\varphi_n(x))$ in a model M of T is defined by
$$\varphi^M(a) := \lim_n \varphi_n^M(a) \text{ for all } a \in M^x.$$
- We refer to the interpretations of T-formulas in the models of T (as above) as T-predicates.
- Example: for any $\mathcal{L}$-formulas $(\theta_n(x))$, the weighted sum $\sum_n 2^{-n}\,\theta_n^M(x)$ is a T-predicate.

Uniform asssignments of sets to models of a theory

Let T be an $\mathcal{L}$-theory; fix a tuple of distinct variables $x = x_1, \ldots, x_n$.

We will denote by $\mathbb{X}$ any operation that assigns a subset $\mathbb{X}(M) \subseteq M^x$ to each model M of T. We refer to such an $\mathbb{X}$ as a uniform assignment of sets to models of T.

For example, if $\varphi(x)$ is a T-formula, we consider $\mathbb{X}_\varphi$ defined by

$$\mathbb{X}_\varphi(M) := \{a \in M^x \mid \varphi^M(a) = 0\}.$$

Such an operation will be referred to as a T-zeroset in the variables x.

We note the following properties of T-zerosets $\mathbb{X} = \mathbb{X}_\varphi$:

(iso) If M, N are models of T and J is an isomorphism of M onto N, then $J(\mathbb{X}(M)) = \mathbb{X}(N)$.

(diag) If M is a model of T and $\mathcal{U}$ is an ultrafilter, with $D\colon M \to M_\mathcal{U}$ the diagonal embedding, then

$$D(\mathbb{X}(M)) = D(M^x) \cap \mathbb{X}(M_\mathcal{U}).$$

These two conditions yield that if M, N are models of T and J is any elementary embedding from M into N, then

$$J(\mathbb{X}(M)) = J(M^x) \cap \mathbb{X}(N).$$

For model theorists: a uniform assignment $\mathbb{X}$ satisfies (iso) and (diag) if and only if there is a set X of x-types over the theory T such that for any model M of T we have

$$\mathbb{X}(M) = \{a \in M^x \mid \mathrm{tp}_M(a) \in X\}.$$

Properties of $\mathbb{X}$ that we discuss later correspond to properties of X in the topometric space $S_x(T)$.

Characterization of T-zerosets

Let T be a theory in a countable signature $\mathcal{L}$ and let $x = x_1, \ldots, x_n$ be a tuple of distinct variables. Let $\mathbb{X}$ be a uniform assignment of sets to models of T.

Theorem

The following are equivalent:
(1) There exists a T-formula $\varphi(x)$ such that $\mathbb{X} = \mathbb{X}_\varphi$.
(2) $\mathbb{X}$ satisfies (iso) and (diag), and the containment

$$\prod \mathbb{X}(M_i)/\mathcal{U} \subseteq \mathbb{X}(\prod M_i/\mathcal{U})$$

holds for every ultraproduct $\prod M_i/\mathcal{U}$ of models of T.

Note that the conditions in (2) can be verified by understanding how the assignment $\mathbb{X}$ behaves under isomorphisms and on ultraproducts of models of T; this will not generally require a full understanding of the behavior of $\mathbb{X}(M)$ for arbitrary models of T.

T-Definable sets

Let T be a theory in a countable signature $\mathcal{L}$ and let $x = x_1, \ldots, x_n$ be a tuple of distinct variables. Let $\mathbb{X}$ be a uniform assignment of sets to models of T.

Definition

We say $\mathbb{X}$ is T-definable if the class of T-predicates is closed under the quantifiers $\sup_{x \in \mathbb{X}(M)}$ and $\inf_{x \in \mathbb{X}(M)}$.

For a T-definable assignment $\mathbb{X}$, the central idea is that we have a natural concept of induced model theoretic structure on $\mathbb{X}(M)$ that is obtained uniformly from $\mathcal{L}$-formulas applied to M. Among other things, this can assist in providing an explicit axiomatization of the models of T.

Characterization of T-definable sets

Let T be a theory in a countable signature $\mathcal{L}$ and let $x = x_1, \ldots, x_n$ be a tuple of distinct variables. Let $\mathbb{X}$ be a uniform assignment of sets to models of T.

Theorem

The following are equivalent:
(1) $\mathbb{X}$ is T-definable.
(2) $\mathbb{X}$ satisfies (iso), and the equality

$$\prod \mathbb{X}(M_i)/\mathcal{U} = \mathbb{X}(\prod M_i/\mathcal{U})$$

holds for every ultraproduct $\prod M_i/\mathcal{U}$ of models of T.
(3) The predicate $P(x)$ defined by $P^M(a) := \mathrm{dist}(a, \mathbb{X}(M))$ for all models M of T and all $a \in M^x$ is a T-predicate.

Note that condition (diag) is omitted in (2); it follows from the ultraproduct equality in (2), applied to ultrapowers.

Let T be a theory in a countable signature $\mathcal{L}$ and let $x = x_1, \ldots, x_n$ be a tuple of distinct variables. Let $\mathbb{X}$ be a uniform assignment of sets to models of T.

Corollary

The following are equivalent:
(1) $\mathbb{X}$ *is* T*-definable.*
(2) $\mathbb{X}$ *is a* T*-zeroset and some* T*-predicate* $P(x)$ *with* $\mathbb{X}$ *as its zeroset has the almost-near property;*
namely, for all epsilon > 0 *there exists* $\delta > 0$ *such that for all models* M *of* T *and all* $a \in M^x$*, we have*

$$P^M(a) < \delta \textit{ implies } \operatorname{dist}(a, \mathbb{X}(M)) \leq \epsilon.$$

(3) $\mathbb{X}$ *is a* T*-zeroset and every* T*-predicate* $P(x)$ *with* $\mathbb{X}$ *as its zeroset has the almost-near property.*

Example 1

Let $\mathcal{C}$ be the class of pointed metric spaces (M, c) of diameter ≤ 1 in which the metric d is an ultrametric (i.e., d satisfies $d(x, z) \leq \max(d(x, y), d(y, z))$). Let T be the theory of $\mathcal{C}$.

Fix r in the interval $0 < r < 1$ and for every (M, c) in $\mathcal{C}$ let $\mathbb{X}(M) = \{a \in M \mid d(c, a) \leq r\}$.

Obviously $\mathbb{X}$ is a T-zeroset. However, $\mathbb{X}$ is not T-definable.

Proof: Let (M, c) be a member of $\mathcal{C}$ in which there exist elements (a_n) such that $\inf_n d(c, a_n) = r$ and $r < d(c, a_n)$ for all $n \in \mathbb{N}$. Let $\mathcal{U}$ be a nonprincipal ultrafilter on $\mathbb{N}$. The element $a = (a_n)_{\mathcal{U}}$ of the ultrapower $(M, c)_{\mathcal{U}}$ satisfies $d(c, a) = r$. However, the ultrametric inequality prevents the existence of any sequence (b_n) from M that satisfies $a = (b_n)_{\mathcal{U}}$ and $d(c, b_n) \leq r$ for all $n \in \mathbb{N}$.

Example 2

Let $\mathcal{L}$ be a countable signature and T an $\mathcal{L}$-theory. Recall that we say T is ω-categorical if T has a unique separable model, which is not compact.

Proposition

The following are equivalent:
(1) T is ω-categorical.
(2) Every T-zeroset is T-definable.

Proof.

For each model M_0 of T and $a_0 \in M_0^x$, let $\mathbb{X}$ be defined on models M of T by

$$\mathbb{X}(M) := \{a \in M^x \mid (M, a) \equiv (M_0, a_0)\}.$$

Obviously $\mathbb{X}$ is a T-zeroset. The omitting types theorem for continuous logic implies that $\mathbb{X}$ is T-definable iff $\mathbb{X}(M) \neq \varnothing$ for all models M of T. The characterization of separable categoricity then implies the equivalence of (1) and (2). □

Example 3

This example comes from work in progress with Yves Raynaud, in which we are constructing new families of uncountably categorical Banach spaces. These spaces are asymptotically hilbertian in a strong sense.

Definition

A Banach space E is asymptotically 2-hilbertian if every ultrapower of E has the form $E_{\mathcal{U}} = D(E) \oplus_2 H$ for some Hilbert space $H \subseteq E_{\mathcal{U}}$, where D is the diagonal embedding of E into $E_{\mathcal{U}}$.

Here we only have time to discuss how the Hilbert part of $E_{\mathcal{U}}$ can be identified.

Let T_b denote a theory that axiomatizes the class of all unit balls of Banach spaces over $\mathbb{R}$. For each such Banach space E, let $B(E)$ denote the unit ball of E, considered as a model of T_b.

For each $a \in B(E)$ we let $\langle a \rangle$ denote the linear subspace of E generated by a. We call $\langle a \rangle$ a 2-summand of E if there is a subspace F of E such that $E = F \oplus_2 \langle a \rangle$; note that if this equation holds, then F is uniquely determined by a. Note further that if $E = F \oplus_2 H$ and H is a Hilbert space, then $\langle a \rangle$ is a 2-summand of E for every $a \in H$.

For each model $B(E)$ of T_b, we define

$$\mathbb{X}(B(E)) := \{a \in B(E) \mid \langle a \rangle \text{ is a 2-summand of E}\}.$$

Proposition

The set assignment $\mathbb{X}$ is a T_b-zeroset but it is not T_b-definable.

To prove that $\mathbb{X}$ is a T_b-zeroset, we verify the conditions in the ultraproduct characterization of zerosets. Condition (iso) is obvious. Condition (diag) follows from the easy stronger fact that if $a \in B(E_1)$ and $E_1 \subseteq E_2$, and if $E_2 = F \oplus_2 \langle a \rangle$, then $E_1 = (F \cap E_1) \oplus_2 \langle a \rangle$. The ultraproduct containment condition holds because the $\oplus_2$ direct sum operation commutes with the ultraproduct construction.

To prove that $\mathbb{X}$ is not T_b-definable, consider $E = \oplus_2(E_n \mid n \in \mathbb{N})$, where each E_n is polyhedral and 2-dimensional, with the unit sphere of E_n approximating the unit circle more and more closely as $n \to \infty$. Then $\mathbb{X}(B(E)) = \{0\}$, whereas E is asymptotically 2-hilbertian, so $\mathbb{X}(B(E_{\mathcal{U}}))$ will be very large. Thus the ultraproduct equality condition fails to hold.