On the Krupa-Zawisza ultraproduct of self-adjoint operators

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Fix $H \cong \ell^2$, $\mathrm{SA}(H)$=self-adjoint operators on H (possibly unbounded). Ongoing project aimed at understanding $\mathcal{U}(H) \curvearrowright \mathrm{SA}(H)$.

Why $\mathcal{U}(H)$ actions?

- Satisfactory non-$\leq_{\mathrm{B}}$ criterion to $S_\infty \curvearrowright X$ by Hjorth's turbulence theory.
- Very often classification problems (=equivalence relations) E are beyond S_∞-action.
- Many E's in operator algebras are related to $\mathcal{U}(H)$. (Törnquist suggests to find a "turbulence theory" of $\mathcal{U}(H)$)
- Ultimately we want to understand $\mathcal{U}(H) \curvearrowright \mathrm{vN}(H)$. (cf. Sasyk–Törnquist).
- One of the most important example is $\mathcal{U}(H) \curvearrowright \mathbb{B}(H)_{\mathrm{sa},1}$ by conjugation: it's known to be generically turbulent (Kechris–Sofronidis).

Why $\mathrm{SA}(H)$?

- While $\mathbb{B}(H)_{\mathrm{sa},1}$ is SOT-Polish, the restriction to the unit ball can be inconvenient.
- $\mathbb{B}(H)_{\mathrm{sa}}$ is standard Borel by SOT, but none of nice topologies (norm, (σ)SOT, (σ)WOT,$\cdots$) make it Polish.
- $\mathrm{SA}(H)$ is SRT-Polish in which $\mathbb{B}(H)_{\mathrm{sa}}$ is F_σ.

$$A_n \stackrel{\text{SRT}}{\to} A \stackrel{\text{def}}{\Leftrightarrow} (A_n - i)^{-1} \stackrel{\text{SOT}}{\to} (A - i)^{-1})$$

- It is sometimes crucial to use unbounded operators (e.g. modular theory of type III factors, affiliated operators in II_1 factors)
- Unbounded operators can produce substantially more complex E than bounded ones.

Theorem (A-Matsuzawa '14)

Let $G = \mathbb{K}(H)_{\mathrm{sa}} \rtimes \mathcal{U}(H)$ act on $\mathrm{SA}(H)$ by

$$(K, u) \cdot A := uAu^* + K, \quad u \in \mathcal{U}(H),\ K \in \mathbb{K}(H)_{\mathrm{sa}},\ A \in \mathrm{SA}(H).$$

Then $G \curvearrowright \mathrm{SA}(H)$ is unclassifiable by countable structures, while the restriction $G \curvearrowright \mathbb{B}(H)_{\mathrm{sa}}$ is smooth (Weyl–von Neumann).

We'll attempt the ultraproduct approach to $\mathcal{U}(H) \curvearrowright \mathrm{SA}(H)$.

Unbounded self-adjoint operator

- A densely defined operator on H is a linear map $A\colon D(A) \to H$, where $D(A) \subset H$ (domain of A) is a dense subspace.
- B is an extension of A, written $A \subset B$ if $D(A) \subset D(B)$ and $A\xi = B\xi,\ \xi \in D(A)$.
- $A = B$ if $A \subset B$ and $B \subset A$.
- The adjoint A^* of A is

$$D(A^*) = \{\xi \in H;\ \exists!\,\zeta \in H\,\forall \eta \in D(A)\ \ \langle \xi, A\eta\rangle = \langle \zeta, \eta\rangle\},$$
$$A^*\xi := \zeta.$$

- A is symmetric if $A \subset A^*$ i.e.,

$$\forall \xi, \eta \in D(A) \quad \langle \xi, A\eta\rangle = \langle A\xi, \eta\rangle.$$

- A is self-adjoint if $A = A^*$.

Example

Fix an orthonormal basis (ONB) $(\xi_n)_{n=1}^{\infty}$ for H and $(a_n)_{n=1}^{\infty}$ in $\mathbb{R}$.
Then $A=\sum_{n=1}^{\infty} a_n e_n \in \mathrm{SA}(H)$, where $e_n=\langle \xi_n,\ \cdot\ \rangle \xi_n$,

$$D(A)=\left\{\xi \in H; \sum_{n=1}^{\infty} a_n^2 |\langle \xi, \xi_n\rangle|^2<\infty\right\},\ \ A\xi := \sum_{n=1}^{\infty} a_n \langle \xi, \xi_n\rangle \xi_n,\ \ \xi \in D(A).$$

$B := A|_{\mathrm{span}\{\xi_n\}_{n\in\mathbb{N}}}$ is symmetric but not self-adjoint $B \subsetneq A \subsetneq B^*$.
We denote A by $\mathrm{diag}(a_1, a_2, \cdots)$.

Example

Fix $\mu \in \mathrm{Prob}(\mathbb{R})$ and $H=L^2(\mathbb{R},\mu)$. Then $A \in \mathrm{SA}(H)$, where

$$D(A)=\left\{f \in H; \int_{\mathbb{R}} x^2 |f(x)|^2 \mathrm{d}\mu(x)<\infty\right\}, [Af](x) := xf(x),\ \ \ f \in D(A).$$

Given $(a_n)_{n=1}^{\infty} \in \ell^{\infty}(\mathbb{B}(H))$, it's easy to define $(a_n)_{\mathcal{U}} \in \mathbb{B}(H_{\mathcal{U}})$:

$$(a_n)_{\mathcal{U}}\xi := (a_n\xi_n)_{\mathcal{U}}, \quad \xi = (\xi_n)_{\mathcal{U}} \in H_{\mathcal{U}}.$$

Here, we can take any representing seq $(\xi_n)_{n=1}^{\infty}$ for $\xi \in H_{\mathcal{U}}$.
How should we define $A_{\mathcal{U}}$ for $A \in \mathrm{SA}(H)$? What's $D(A_{\mathcal{U}}) \subset H_{\mathcal{U}}$?

Definition (doesn't work...)

$\tilde{D}(A_{\mathcal{U}}) = \{\xi \in H_{\mathcal{U}}; \xi = (\xi_n)_{\mathcal{U}},\ \forall n\ \xi_n \in D(A)\}$ and

$$A_{\mathcal{U}}\xi := (A\xi_n)_{\mathcal{U}}, \quad \xi \in D(A_{\mathcal{U}}). \tag{1}$$

where $\xi = (\xi_n)_{\mathcal{U}},\ \xi_n \in D(A)\ (n \in \mathbb{N})$.

- $\tilde{D}(A_{\mathcal{U}}) = H_{\mathcal{U}}$. ($D(A)$ is dense in H).
- This $A_{\mathcal{U}}\xi$ depends on the choice of representing $(\xi_n)_{n=1}^{\infty}$.

Example

Let $A = \mathrm{diag}(1, 2, 3, \cdots)$ relative to the ONB $\{\eta_n\}_{n=1}^{\infty}$, and consider

$$\xi_n := \eta_1, \quad \xi_n' := \eta_1 + \frac{1}{n}\eta_n \quad (n \geq 1).$$

Clearly $\xi_n, \xi_n' \in D(A)$, $\xi = (\xi_n)_{\mathcal{U}} = (\xi_n')_{\mathcal{U}} \in H_{\mathcal{U}}$, and
$(A\xi_n)_{n=1}^{\infty} = (\eta_1)_{n=1}^{\infty}$, $(A\xi_n')_{n=1}^{\infty} = (\eta_1 + \eta_n)_{n=1}^{\infty} \in \ell^{\infty}(H)$, but

$$\lim_{n \to \mathcal{U}} \|A\xi_n - A\xi_n'\| = \lim_{n \to \mathcal{U}} \|\eta_n\| = 1 \neq 0,$$

so $(A\xi_n)_{\mathcal{U}} \neq (A\xi_n')_{\mathcal{U}}$.
Should we define $A_{\mathcal{U}}\xi = (A\xi_n)_{\mathcal{U}}$ or $A_{\mathcal{U}}\xi = (A\xi_n')_{\mathcal{U}}$?

Krupa–Zawisza ('84) gave the right definition of $A_{\mathcal{U}}$ as a self-adjoint operator.

Definition (Krupa–Zawisza)

Let $\mathbf{A} = (A_n)_{n=1}^{\infty}$ be a sequence in $\mathrm{SA}(H)$.

(i) $(\xi_n)_{n=1}^{\infty} \in \ell^{\infty}(H)$ is an **A**-*sequence* if $\xi_n \in D(A_n)$ for each $n \in \mathbb{N}$.

(ii) $\mathscr{D}_{\mathbf{A}} = \{\xi \in H_{\mathcal{U}}; \exists (\xi_n)_{n=1}^{\infty}\ \mathbf{A}\text{-seq s.t. } \xi = (\xi_n)_{\mathcal{U}},\ \sup_n \|A_n \xi_n\| < \infty\}$. Such $(\xi_n)_{n=1}^{\infty}$ is called **A**-*bounded*.

(iii) The *Krupa-Zawisza subspace* is defined by $\mathscr{H}(\mathbf{A}) := \overline{\mathscr{D}_{\mathbf{A}}} \subset H_{\mathcal{U}}$.

Theorem (Krupa-Zawisza ('84))

(i) $\exists ! \mathbf{A}_{\mathcal{U}} \in \mathrm{SA}(\mathscr{H}(\mathbf{A}))$ *with* $D(\mathbf{A}_{\mathcal{U}}) = \mathscr{D}_{\mathbf{A}}$ *s.t. for* $\xi \in \mathscr{D}_{\mathbf{A}}, \eta \in \mathscr{H}(\mathbf{A})$, $\mathbf{A}_{\mathcal{U}}\xi = \eta$ *iff* $\xi = (\xi_n)_{\mathcal{U}}$ *(*$\mathbf{A}$*-bounded) and* $\eta = (A_n\xi_n)_{\mathcal{U}}$ *holds.*

(ii) $(\mathbf{A}_{\mathcal{U}} - i)^{-1} = ((A_n - i)^{-1})_{\mathcal{U}}|_{\mathscr{H}(\mathbf{A})}$ *and* $\mathrm{Ker}(((A_n - i)^{-1})_{\mathcal{U}})^{\perp} = \mathscr{H}(\mathbf{A})$.

Remark

- *The above example shows that different choice of* $\mathbf{A}$*-bounded rep seq* $(\xi_n)_{n=1}^{\infty}$ *of* $\xi \in \mathscr{D}_{\mathbf{A}}$ *may result in different* $(A_n\xi_n)_{\mathcal{U}}$*!*
- *The rep seq* $(\xi_n)_{n=1}^{\infty}$ *is determined to be the unique one (up to* $c_{\mathcal{U}}(H)$*) for which* $\eta = (A_n\xi_n)_{\mathcal{U}} \in \mathscr{H}(\mathbf{A})$ *holds.*
- *In the above example,* $(A\xi_n)_{\mathcal{U}} \in \mathscr{H}(\mathbf{A})$, *while* $(A\xi_n')_{\mathcal{U}} \notin \mathscr{H}(\mathbf{A})$.

We can't see if a chosen bounded one $(\xi_n)_{n=1}^{\infty}$ is the right one without computing $(A_n\xi_n)_{\mathcal{U}}$.

Intrinsic characterization of the right representative is desirable.

This motivates us to define the following notions:

Definition (A–Ojima–Saigo '13)

(i) $\mathbf{A}$-seq $(\xi_n)_{n=1}^{\infty}$ is *$\mathbf{A}$-regular* if $\mathbb{R}\ni t\mapsto e^{itA_n}\xi_n\in H$ $(n\in\mathbb{N})$ is *$\mathcal{U}$-equicontinuous*, i.e., $\forall\varepsilon>0\ \exists\delta>0\ \exists W\in\mathcal{U}$ s.t. $\forall t\in(-\delta,\delta)\ \forall n\in W$,

$$\|e^{itA_n}\xi_n-\xi_n\|<\varepsilon.$$

$$\widetilde{\mathscr{H}}(\mathbf{A}):=\{\xi\in\ell^{\infty}(H);\ \xi=(\xi_n)_{\mathcal{U}},\ (\xi_n)_{n=1}^{\infty}\ \mathbf{A}\text{-regular}\}$$

(ii) $\mathbf{A}$-seq $(\xi_n)_{n=1}^{\infty}$ is *$\mathbf{A}$-proper*, if $\forall\varepsilon>0\ \exists a>0\ \exists(\eta_n)_{n=1}^{\infty}\in\ell^{\infty}(H)$ s.t. $\forall n\in\mathbb{N}\ \xi_n\in\mathrm{ran}1_{[-a,a]}(A_n)$ and $\lim_{n\to\mathcal{U}}\|\xi_n-\eta_n\|_{A_n}<\varepsilon$, where

$$\|\xi\|_A:=(\|\xi\|^2+\|A\xi\|^2)^{\frac{1}{2}}$$

$$\widetilde{\mathscr{D}}_{\mathbf{A}}:=\{\xi\in\ell^{\infty}(H);\ \xi=(\xi_n)_{\mathcal{U}},\ (\xi_n)_{n=1}^{\infty}\ \mathbf{A}\text{-proper}\}$$
$$\mathscr{D}_0(\mathbf{A}):=\{(\xi_n)_{n=1}^{\infty}\in\ell^{\infty}(H);\ \exists a>0\ \forall n\in\mathbb{N}\ \xi_n\in\mathrm{ran}1_{[-a,a]}(A_n)\}.$$

Theorem (A–Ojima–Saigo '13)

(i) $\widetilde{\mathscr{D}}_{\mathbf{A}} = \mathscr{D}_{\mathbf{A}}$, $\widetilde{\mathscr{H}}(\mathbf{A}) = \mathscr{H}(\mathbf{A})$.

(ii) *If $\xi \in \mathscr{H}(\mathbf{A})$, then any rep $\mathbf{A}$-seq of ξ is $\mathbf{A}$-regular.*

(iii) *If $\xi \in \mathscr{D}_{\mathbf{A}}$, then an rep $\mathbf{A}$-seq $(\xi_n)_{n=1}^{\infty}$ of ξ is the "right one", i.e., $\mathbf{A}_{\mathcal{U}}\xi = (A_n\xi_n)_{\mathcal{U}}$ iff $(\xi_n)_{n=1}^{\infty}$ is $\mathbf{A}$-proper.*

(iv) *$\mathscr{D}_0(\mathbf{A})$ is a core for $\mathbf{A}_{\mathcal{U}}$.*

Now consider the unitary orbit $\mathcal{O}(A) := \{uAu^*; u \in \mathcal{U}(H)\} \subset \mathrm{SA}(H)$.

Problem

Let $A, B \in \mathrm{SA}(H)$.

(a) *When does $B \in \overline{\mathcal{O}(A)}$ hold?*

(b) *$\exists A$ with a dense orbit? $\overline{\mathcal{O}(A)} \stackrel{?}{=} \mathrm{SA}(H)$. Is $\mathcal{O}(A)$ meager?*

(c) *When is $\mathcal{O}(A)$ closed? (obvious if $A = \lambda 1_H$)*

(d) *Is it possible to have $A = A_{\mathcal{U}}$? (obviously no if A bounded)*

They can be answered by KZ ultraproduct (model theory?)

Definition

For H, H' Hilbert spaces and $A \in \mathrm{SA}(H), B \in \mathrm{SA}(H')$, we write $B \hookrightarrow A$ if $\exists K \subset H$ reducing A and $\exists u\colon H' \overset{\sim}{\to} K$ s.t. $uBu^* = A|_{K \cap D(A)}$.

Example ($2A \in \overline{\mathcal{O}(A)}$)

Let $A = \mathrm{diag}(1, 2, \cdots) \in \mathrm{SA}(H)$, w.r.t $\{\xi_n\}_{n=1}^{\infty}$. Then $\exists (u_n)_{n=1}^{\infty}$ in $\mathcal{U}(H)$ s.t. $u_n A u_n^* \overset{n \to \infty}{\to} 2A$.

OBS: $2A \in \overline{\mathcal{O}(A)}$ is impossible if A is bounded.

Outline.

$K :=$ proj onto $\overline{\mathrm{span}}\{\xi_{2n}; n \in \mathbb{N}\}$. Fix $v \in \mathbb{B}(H)$, $vv^* =$ proj onto K s.t.

- K reduces A, $v^*Av = 2A$ and $v^*Av \hookrightarrow A_{\mathcal{U}}$ (in fact $v^*Av \hookrightarrow A$).
- $\exists (u_n)_{n=1}^{\infty}$ in $\mathcal{U}(H)$ s.t. $u_n \overset{n \to \infty}{\to} v$ (SOT).
- $u_n^* A u_n \overset{n \to \infty}{\to} v^*Av = 2A$ (SRT). (Note: $u_n^* \not\to v^*$)

□

Theorem (Answer to (a) $B \in \overline{\mathcal{O}(A)}$)

Let $\mathbf{A} = (A_n)_{n=1}^{\infty},\ A_n,\ B \in \mathrm{SA}(H)\ (n \in \mathbb{N})$. *TFAE.*

(i) $B \hookrightarrow \mathbf{A}_{\mathcal{U}}$.

(ii) $\exists u_n \in \mathcal{U}(H)$ *s.t.* $\lim_{n \to \mathcal{U}} u_n A_n u_n^* = B$ *in* $\mathrm{SA}(H)$.

Proof of (i)⇒(ii).

(i)⇒(ii) $\exists K \subset \mathscr{H}(\mathbf{A})$, $\exists v \colon H \overset{\sim}{\to} K$ s.t. $vBv^* = (\mathbf{A}_{\mathcal{U}})|_K$. By Haagerup–Winsløw '00, $\exists w_n \colon K \overset{\sim}{\to} H$ s.t. $\forall \xi \in K\ (w_n \xi)_{\mathcal{U}} = \xi$. Then $u_n := v^* w_n^* \in \mathcal{U}(H)$ satisfies for $\xi, \eta \in H$ (use WRT=SRT),

$$\begin{aligned}
\lim_{n \to \mathcal{U}} \langle v^* w_n^* (A_n - i)^{-1} w_n v\xi, \eta \rangle &= \lim_{n \to \mathcal{U}} \langle (A_n - i)^{-1} w_n v\xi, w_n v\eta \rangle \\
&= \langle ((A_n - i)^{-1})_{\mathcal{U}} (w_n v\xi)_{\mathcal{U}}, (w_n v\eta)_{\mathcal{U}} \rangle \\
&= \langle ((A_n - i)^{-1})_{\mathcal{U}} v\xi, v\eta \rangle = \langle v(B - i)^{-1} v^* v\xi, v\eta \rangle \\
&= \langle (B - i)^{-1} \xi, \eta \rangle.
\end{aligned}$$

□

Let $(X,\mu),(X_n,\mu_n)$ be standard prob spaces, Γ a countable group.
$\mathrm{Act}(\Gamma,X,\mu)=\{a\colon \Gamma \overset{\text{pmp}}{\curvearrowright} (X,\mu)\}$ is Polish.
Given $(a_n)_{n=1}^{\infty}$ one can construct $(a_n)_{\mathcal{U}}\colon \Gamma \curvearrowright (X_{\mathcal{U}},\mu_{\mathcal{U}})$.

Theorem (Conley–Kechris–Tucker–Drob '13)

$b\in \mathrm{Act}(\Gamma,X,\mu)$, $a_n\in \mathrm{Act}(\Gamma,X_n,\mu_n)$ $(n\in\mathbb{N})$. TFAE.

(i) $b\prec (a_n)_{\mathcal{U}}\in \mathrm{Act}(\Gamma,X_{\mathcal{U}},\mu_{\mathcal{U}})$.

(ii) $\exists \tilde{a}_n\in \mathrm{Act}(\Gamma,X_n,\mu_n)$, $\tilde{a}_n\cong a_n$ *s.t.* $\displaystyle\lim_{n\to\mathcal{U}}\tilde{a}_n=b$ *in* $\mathrm{Act}(\Gamma,X,\mu)$.

$\mathrm{vN}(H)=\{M\overset{\text{vNalg}}{\subset}\mathbb{B}(H)\}$ is Polish w.r.t. the *Effros-Maréchal* topology.
Given $(M_n,\varphi_n)_{n=1}^{\infty}$ one can construct $(M_n,\varphi_n)^{\mathcal{U}}$.

Theorem (A–Haagerup–Winsløw '14)

Let $M_n,N\in\mathrm{vN}(H)$ $(n\in\mathbb{N})$. TFAE.

(i) $\exists N \overset{\leftarrow}{\hookrightarrow} (M_n,\varphi_n)^{\mathcal{U}}$.

(ii) $\exists \tilde{M}_n\in\mathrm{vN}(H)$, $\tilde{M}_n\cong M_n$ *s.t.* $\lim_{n\to\mathcal{U}}\tilde{M}_n=N$ *in* $\mathrm{vN}(H)$.

The above three theorems are proved by methods specific to each structures. It seems that there is some principle which leads to a unified proof for all of the above theorems and more.

Problem (???)

Given a category $\mathcal{C}$ of objects with a certain metric structure which admits an ultraproduct $(X_n)_{\mathcal{U}}$. Find a nice Polish space $\mathcal{X}$ of separable objects in $\mathcal{C}$ (which should contain all the isomorphism classes of separable objects) s.t. the following is true:

Let $X_n, Y \in \mathcal{X}$. Then TFAE.

(i) $Y \hookrightarrow (X_n)_{\mathcal{U}}$.

(ii) $\exists \tilde{X}_n \in \mathcal{X}$, $\tilde{X}_n \cong X_n$ s.t. $\lim_{n\to\mathcal{U}} \tilde{X}_n = Y$.

Let's go back to the remaining 3 questions about
$\mathcal{O}(A) = \{uAu^*;\ u \in \mathcal{U}(H)\}$.

(b) $\exists A$ with a dense orbit? $\overline{\mathcal{O}(A)} \overset{?}{=} \mathrm{SA}(H)$. Is $\mathcal{O}(A)$ meager?

Definition

For $A \in \mathrm{SA}(H)$, the essential spectrum $\sigma_{\mathrm{e}}(A)$ is the set of all $\lambda \in \sigma(A)$ which is either

(1) an accumulation point in $\sigma(A)$ or

(2) an isolated eigenvalues of ∞ multiplicity.

$\sigma_{\mathrm{e}}(A)$ is always closed, and nonempty if A bounded but can be empty in general.

Proposition

Let $A \in \mathrm{SA}(H)$ and let $\lambda \in \mathbb{R}$. TFAE.

(i) $\lambda \in \sigma_{\mathrm{e}}(A)$.

(ii) $\lambda 1_H \hookrightarrow A_{\mathcal{U}}$. *That is, λ is an eigenvalue of $A_{\mathcal{U}}$ of ∞ multiplicity.*

Theorem

Let $A \in \mathrm{SA}(H)$. TFAE.

(i) $\overline{\mathcal{O}(A)} = \mathrm{SA}(H)$.

(ii) $\sigma_{\mathrm{e}}(A) = \mathbb{R}$.

Proposition (Chokski–Nadkarni '98 (bounded case), A–Matsuzawa '16)

(1) The set of all $A \in \mathrm{SA}(H)$ which has purely singular continuous spectrum equal to $\mathbb{R}$, is a dense G_δ subset of $\mathrm{SA}(H)$.
In particular, $\sigma_{\mathrm{e}}(A) = \mathbb{R}$ is SRT-generic.

(2) $\forall A \in \mathrm{SA}(H)$ $\mathcal{O}(A)$ is SRT-meager.

The equivalence $\sigma_{\rm e}(A)=\mathbb{R}\Leftrightarrow \overline{\mathcal{O}(A)}=\mathrm{SA}(H)$ can be proved using Hadwin's result below, of which we can give a simple proof using KZ ultraproduct.

Theorem (Hadwin '75)

Let $A,B\in \mathbb{B}(H)$ be normal. TFAE.

(i) $B\in \overline{\mathcal{O}(A)}$.

(ii) $\mathrm{rank}(1_U(B))\le \mathrm{rank}(1_U(A))$ *for all* $U\subset \mathbb{C}$ *open.*

Hadwin's theorem is reformulated as such and is generalized to vNalg setting by Sherman '07.

Proposition

Hadwin's theorem holds for all $A,B\in \mathrm{SA}(H)$.

(c) When is $\mathcal{O}(A)$ closed? (obvious if $A = \lambda 1_H$)

Theorem

Let $A \in \mathrm{SA}(H)$. Then $\mathcal{O}(A)$ is SRT-closed iff $A = \lambda 1_H$ for some $\lambda \in \mathbb{R}$.

Outline.

When $\sigma_{\mathrm{e}}(A) = \emptyset$, then $A \cong \mathrm{diag}(a_1^{\oplus n_1}, a_2^{\oplus n_2}, \cdots)$ ($|a_1| < |a_2| \nearrow \infty$) and $B := \mathrm{diag}(a_2^{\oplus n_2}, a_3^{\oplus n_3}, \cdots) \hookrightarrow A_{\mathcal{U}}$ hence $B \in \overline{\mathcal{O}(A)}$ but $B \ncong A$.
So fix $\lambda \in \sigma_{\mathrm{e}}(A)$. If $\exists \mu \in \sigma(A) \setminus \{\lambda\}$, then $B := \lambda 1_H \hookrightarrow A_{\mathcal{U}}$ but $B \ncong A$.
Thus $\sigma(A) = \{\lambda\}$, which means $A = \lambda 1_H$. □

(d) Is it possible to have $A = A_{\mathcal{U}}$?

If $A \in \mathrm{SA}(H)$ is bounded, then $A \neq A_{\mathcal{U}}$, because $D(A) = H \subsetneq D(A_{\mathcal{U}}) = H_{\mathcal{U}}$.
For unbounded A, it can happen that $A = A_{\mathcal{U}}$.

Theorem

For $A \in \mathrm{SA}(H)$, TFAE.

(i) $A = A_{\mathcal{U}}$.

(ii) $\sigma_{\mathrm{e}}(A) = \emptyset$.

Note that $\sigma_{\mathrm{e}}(B) \neq \emptyset$ if B is bounded.

Example

$A = \mathrm{diag}(1, 2, 3, \cdots)$ satisfies $A = A_{\mathcal{U}}$ and $\sigma_{\mathrm{e}}(A) = \emptyset$.

Future direction

- Study dynamical properties of $\mathcal{U}(H) \curvearrowright \mathrm{SA}(H)$ and related equivalence relations on $\mathrm{SA}(H)$ (e.g. Weyl–von Neumann type relations)
- Adapt our method to von Neumann (or C*) framework. Cf. Sherman's work on unitary orbits for normal operators in a von Neumann algebra.
- Applications to $\mathbb{R}$-actions.
- Study of norm resolvent convergence in $\mathrm{SA}(H)$.
  Cf. Voiculescu's non-commutative Weyl-von Neumann Theorem.

Definition

Say that $A, B \in \mathrm{SA}(H)$ are *Weyl–von Neumann equivalent* $A \overset{\mathrm{WvN}}{\sim} B$ if $\exists u \in \mathcal{U}(H),\ K \in \mathbb{K}(H)_{\mathrm{sa}}\ [\ B = uAu^* + K\ ]$.

Namely $\overset{\mathrm{WvN}}{\sim} = E_G^{\mathrm{SA}(H)}$, where $G = \mathbb{K}(H)_{\mathrm{sa}} \rtimes \mathcal{U}(H)$. We know that $\overset{\mathrm{WvN}}{\sim}$ is unclassifiable by countable structures. What can we say more about it?

Although Weyl–von Neumann type theorem

$$A \overset{\mathrm{WvN}}{\sim} B \Leftrightarrow \sigma_{\mathrm{e}}(A) = \sigma_{\mathrm{e}}(B)$$

miserably fails for unbounded operators ($\Rightarrow$ is true), we do know that it holds on $\mathbb{R}$, i.e.,

$$\sigma_{\mathrm{e}}(A) = \sigma_{\mathrm{e}}(B) = \mathbb{R} \Rightarrow A \overset{\mathrm{WvN}}{\sim} B.$$

Similarly, W–vN holds e.g. on $\mathbb{R} \setminus (-2,3) \cup (6,9)$ but fails on $\mathbb{N}$ or $\emptyset$.

Question

On which closed subset $M \subset \mathbb{R}$ does the Weyl–von Neumann Theorem hold?

$$\forall A, B \in \mathrm{SA}(H)\ [\sigma_{\mathrm{e}}(A) = \sigma_{\mathrm{e}}(B) = M \overset{?}{\Rightarrow} A \overset{\mathrm{WvN}}{\sim} B].$$

Theorem (A–Matsuzawa '17)

Let M be a closed subset of $\mathbb{R}$. TFAE

(i) *The Weyl–von Neumann theorem holds on M. That is,*

$$\forall A, B \in \mathrm{SA}(H)\ [\sigma_{\mathrm{e}}(A) = \sigma_{\mathrm{e}}(B) = M \Rightarrow A \overset{\mathrm{WvN}}{\sim} B].$$

(ii) *M has no large gaps at infinity. That is, $M \neq \emptyset$ and*

$$(*)\quad d_M := \limsup_{|\lambda|\to\infty} \mathrm{dist}(\lambda, M) = 0.$$

Here we assume $\sup \emptyset = 0$.

Thank you for your attention!