Most C*-algebras do not admit quantifier elimination

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Throughout, all C*-algebras are unital, and by "embedding" we mean an injective unital $*$-homomorphism.

If $F \subseteq K$ are fields and $F = F^{\text{alg}}$, then any algebraic equation with coefficients in F that is satisfied by an element of K is satisfied by an element of F.

Question

Is there a similar notion of "algebraically closed" C-algebras?*

Existentially closed models

$M \models T$ is **existentially closed for** T if whenever $M \subseteq N$ and $N \models T$, then M and N agree on the truth values of all existential sentences with parameters from M.

Theorem (Steinitz, 1910 (for fields))

If the class of models of T is closed under direct limits, then every model of T can be extended to an existentially closed model of T.

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Question

Can we describe the class of existentially closed C-algebras by a theory?*

Model companions

Suppose T is a theory whose class of models is closed under direct limits.

Definition

The **model companion** of T is the theory T^* whose models are precisely the existentially closed models of T.

- The model companion of T might not exist. If it does, it is unique.

Classical examples

Theory	Model companion
Fields	Algebraically closed fields
Linear orders	Dense linear orders without endpoints
Boolean algebras	Atomless Boolean algebras
Abelian groups	Divisible abelian groups
	with infinitely many elements of each finite order
Groups	None

Theorem (E.-Farah-Goldbring-Kirchberg-Vignati)

The theory of C-algebras does not have a model companion.*

Theorem (Robinson, 1956)

Suppose that the class of models of T is closed under substructure and direct limit, and suppose that T^* is the model companion of T. Then the following are equivalent:

- *T has the amalgamation property*
- *T^* has* **quantifier elimination**

Corollary

If the theory of C-algebras has a model companion, then every C*-algebra embeds into a C*-algebra whose theory has quantifier elimination.*

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If the theory of C-algebras has a model companion, then every C*-algebra embeds into a C*-algebra whose theory has quantifier elimination.*

Definition

A theory T has **quantifier elimination** if for every formula $\varphi(\overline{x})$ and every $\epsilon > 0$ there is a formula $\psi_\epsilon(\overline{x})$ that does not use sup or inf and such that for all $M \models T$ and all $\overline{b} \in M^n$,

$$\left|\varphi^M(\overline{b}) - \psi_\epsilon^M(\overline{b})\right| < \epsilon.$$

Theorem (E.-Farah-Kirchberg-Vignati, E.-Amador-Farah-Hart-Kawach-Kim-Kirchberg-Vignati, E.-Goldbring-Vignati)

The complete theories of C-algebras with quantifier elimination are exactly the theories of the following algebras:*

- $\mathbb{C}$
- $\mathbb{C}^2$
- $M_2(\mathbb{C})$
- $C(\textit{Cantor set})$

Corollary

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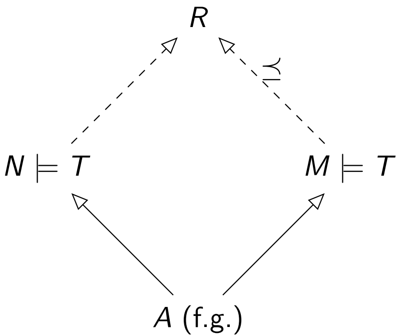
The non-commutative case

Quantifier elimination reformulated



Positive results

Theorem

$M_2(\mathbb{C})$ has quantifier elimination.

Proof.

Up to isomorphism, $M_2(\mathbb{C})$ is the only model of its theory.

$$A = \mathbb{C}, \mathbb{C}^2, M_2(\mathbb{C})$$

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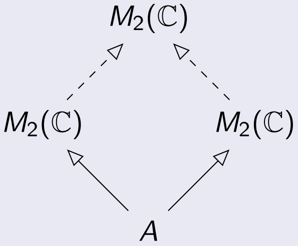

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Definition

Let A be a C*-algebra, and $\overline{a} = (a_1, \ldots, a_n) \in A^n$.

- The **type** of $\overline{a}$ is

$$tp(\overline{a}) = \{\varphi(\overline{x}) : A \models \varphi(\overline{a})\}.$$

- The **quantifier-free type** of $\overline{a}$ is

$$qftp(\overline{a}) = \{\varphi \in tp(\overline{a}) : \varphi \text{ does not use sup or inf}\}.$$

Lemma

Let $a, b \in A$ be normal. Then $qftp(a) = qftp(b)$ if and only if $\sigma(a) = \sigma(b)$.

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Lemma

If A has quantifier elimination then all non-trivial projections in A have the same type.

Proof.

- If p, q are non-trivial projections then $\sigma(p) = \sigma(q) = \{0, 1\}$.
- Therefore $qftp(p) = qftp(q)$.
- Quantifier elimination then implies $tp(p) = tp(q)$.

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Theorem

$M_3(\mathbb{C})$ does not have quantifier elimination.

Proof.

- Let $p = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ and $q = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.
- p is a minimal projection, q is a non-minimal projection
- QE would imply $tp(p) = tp(q)$, contradiction.

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The infinite-dimensional case

Lemma (E.-Farah-Kirchberg-Vignati)

If T is a theory of infinite-dimensional non-commutative C-algebras, and T has quantifier elimination, then every model of T is purely infinite and simple.*

Quantifier elimination and $\mathcal{O}_2$

$\mathcal{O}_2$ has several model-theoretic properties related to quantifier elimination. For example:

- (Cuntz): $\mathcal{O}_2$ is purely infinite and simple.
- If $B \equiv \mathcal{O}_2$, then every embedding of $\mathcal{O}_2$ into B is elementary.
- (E.-Farah-Kirchberg-Vignati): If a and b are normal elements of $\mathcal{O}_2$, then the following are equivalent:
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Lemma (E.-Farah-Kirchberg-Vignati)

Suppose that T is a theory of infinite-dimensional, non-commutative C-algebras, and that T has quantifier elimination. Then $\mathcal{O}_2$ embeds into every model of T.*

Proof.

- Suppose $A \models T$ is separable.
- We know that A is purely infinite and simple, so 1 is Murray-von Neumann equivalent to a non-trivial projection.
- Pick $s \in A$ such that $s^*s = 1$ and $p = ss^* < 1$.
- $tp(1-p) = tp(p)$ by QE.
- Therefore there is $t \in A$ such that $t^*t = 1$ and $tt^* = 1 - p$.
- s, t generate a unital copy of $\mathcal{O}_2$.

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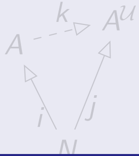
Theorem (E.-Farah-Kirchberg-Vignati)

There is no theory T of infinite-dimensional, non-commutative C-algebras such that T has quantifier elimination.*

Proof outline.

- Suppose T was such a theory. Pick a separable $A \models T$ and a non-principal ultrafilter $\mathcal{U}$ on $\mathbb{N}$.
- QE implies: Whenever N is finitely generated, and $i : N \to A$ and $j : N \to A^{\mathcal{U}}$ are embeddings, there is an embedding $k : A \to A^{\mathcal{U}}$ so that $j = k \circ i$.



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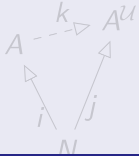

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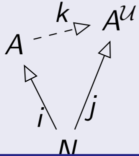

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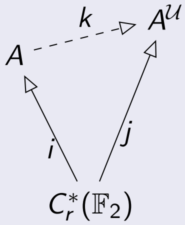

- We will use $N = C_r^*(\mathbb{F}_2)$.

Proof outline (con't).

- $C_r^*(\mathbb{F}_2)$ is exact, so embeds in $\mathcal{O}_2$ (Kirchberg-Phillips).
- QE implies $\mathcal{O}_2$ embeds in A.
- Let $i : C_r^*(\mathbb{F}_2) \to A$ be the composition of these embeddings.

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- There is an embedding $h : C_r^*(\mathbb{F}_2) \to \prod_{\mathcal{U}} M_n(\mathbb{C})$ (Haagerup-Thorbjørnsen).
- Each $M_n(\mathbb{C})$ embeds into $\mathcal{O}_2$. Taking ultraproducts get an embedding of $\prod_{\mathcal{U}} M_n(\mathbb{C})$ into $\mathcal{O}_2^{\mathcal{U}}$.
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- We show that there is no embedding $k : A \to A^{\mathcal{U}}$ such that $j = k \circ i$. In fact, we can't even get $k : \mathcal{O}_2 \to A^{\mathcal{U}}$. Suppose we could.
- $\mathcal{O}_2$ is nuclear, so by Choi-Effros we get a c.p.c. lift of k
  - $\psi : \mathcal{O}_2 \to \ell_\infty(A)$ such that $k = \pi \circ \psi$, where $\pi : \ell_\infty(A) \to A^{\mathcal{U}}$ is the quotient map.
- There is a c.p.c. map $\theta : \ell_\infty(A) \to \prod_{n \in \mathbb{N}} M_n(\mathbb{C})$, induced by conditional expectations.
- Then $\theta \circ \psi : C_r^*(\mathbb{F}_2) \to \prod_{n \in \mathbb{N}} M_n(\mathbb{C})$ is a c.p.c. lift of h.

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- Algebras admitting embeddings into $\prod_{\mathcal{U}} M_n(\mathbb{C})$ with c.p.c. lifts are quasidiagonal. So $C_r^*(\mathbb{F}_2)$ is quasidiagonal.
- For a group G, if $C_r^*(G)$ is quasidiagonal then G is amenable (Rosenberg). But $\mathbb{F}_2$ is not amenable.

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The story so far...

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Goldbring-Sinclair have a more elementary proof that the theory of C*-algebras does not have a model companion, but it does not give information about QE.

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The theory of non-commutative C-algebras does not have a model companion.*

Goldbring-Sinclair have a more elementary proof that the theory of C*-algebras does not have a model companion, but it does not give information about QE.

The infinite-dimensional commutative case

Positive results

Some positive results obtained by translating classical results about the model theory of Boolean algebras, via Stone and Gelfand-Naimark dualities:

Theorem

The algebras that are existentially closed amongst all commutative C-algebras are exactly those of the form $C(X)$ for X a compact 0-dimensional space without isolated points.*

Positive results

Corollary

The theory of $C(\text{Cantor set})$ is the model companion of the theory of commutative C-algebras.*

Corollary

The theory of $C(\text{Cantor set})$ has quantifier elimination.

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The theory of C(Cantor set) is the model companion of the theory of commutative C-algebras.*

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Continua

Lemma (E.-Amador-Farah-Hart-Kawach-Kim-Vignati)

If there is a theory T of infinite-dimensional commutative C-algebras with quantifier elimination other than the theory of $C(\text{Cantor set})$, then every model of T must be of the form $C(X)$ for X a hereditarily indecomposable continuum.*

Lemma

There is a theory T_{conn} such that $M \models T_{conn}$ if and only if $M \cong C(X)$ where X is an infinite continuum.

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Model companions

Suppose T is a theory whose class of models is closed under direct limits.

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The **model companion** of T is the theory T^* whose models are precisely the existentially closed models of T.

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If X and Y are infinite continua, then $Th_\forall(X) = Th_\forall(Y)$.

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Theorem (E.-Goldbring-Vignati)

No extension of T_{conn} has quantifier elimination.

Proof.

- If T^* an extension of T_{conn} with QE, then by K.P. Hart's lemma T^* is the model companion of T_{conn}.
- The models of T_{conn} are closed under substructure and direct limit, so T^* having QE is equivalent to T_{conn} having amalgamation.
- T_{conn} does not have amalgamation.

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Question

What do the existentially closed models of T_{conn} look like?

Theorem (E.-Goldbring-Vignati)

$C(\mathbb{P})$ is an existentially closed model of T_{conn}.

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The following are equivalent:

- *Every embedding between models of the theory of $C(\mathbb{P})$ is elementary.*
- *There is a continuum X such that every embedding between models of the theory of $C(X)$ is elementary.*
- *The theory of continua has a model companion.*
- *The model companion of the theory of continua is $Th(C(\mathbb{P}))$.*

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Thank you!