Book Advertisement:

I. Farah, Combinatorial Set Theory and C^*–algebras, Springer, to appear.

(Draft available upon request.)

Strict order property and C^*–algebras

Ilijas Farah (joint work with I. Hirshberg and A. Vignati)

Isaacfest, September 23, 2017

Universal models

Definition

If $\mathcal{C}$ is a category of metric structures and κ is a cardinal, some $A \in \mathcal{C}$ is κ-universal if $\chi(A) = \kappa$ and [1]

$$(\forall B \in \mathcal{C})(\chi(B) \leq \kappa) \Rightarrow B \hookrightarrow A.$$

[1] Embeddings are not assumed to be elementary

Examples

1. Cantor: $(\mathbb{Q}, <)$ is an $\aleph_0$-universal linear ordering.
2. Hausdorff: The Continuum Hypothesis (CH) implies there exists a $2^{\aleph_0}$-universal linear ordering.

Examples

1. Cantor: $(\mathbb{Q}, <)$ is an $\aleph_0$-universal linear ordering.
2. Hausdorff: The Continuum Hypothesis (CH) implies there exists a $2^{\aleph_0}$-universal linear ordering.
3. There exists an $\aleph_0$-universal Boolean algebra.

Examples

1. Cantor: $(\mathbb{Q}, <)$ is an $\aleph_0$-universal linear ordering.
2. Hausdorff: The Continuum Hypothesis (CH) implies there exists a $2^{\aleph_0}$-universal linear ordering.
3. There exists an $\aleph_0$-universal Boolean algebra.
4. Parovičenko: CH implies that $\mathcal{P}(\mathbb{N})/\operatorname{Fin}$ is a $2^{\aleph_0}$-universal Boolean algebra.

Examples

1. Cantor: $(\mathbb{Q}, <)$ is an $\aleph_0$-universal linear ordering.
2. Hausdorff: The Continuum Hypothesis (CH) implies there exists a $2^{\aleph_0}$-universal linear ordering.
3. There exists an $\aleph_0$-universal Boolean algebra.
4. Parovičenko: CH implies that $\mathcal{P}(\mathbb{N})/\operatorname{Fin}$ is a $2^{\aleph_0}$-universal Boolean algebra.
5. $C([0,1])$ is an $\aleph_0$-universal Banach space.

Examples

1. Cantor: $(\mathbb{Q}, <)$ is an $\aleph_0$-universal linear ordering.
2. Hausdorff: The Continuum Hypothesis (CH) implies there exists a $2^{\aleph_0}$-universal linear ordering.
3. There exists an $\aleph_0$-universal Boolean algebra.
4. Parovičenko: CH implies that $\mathcal{P}(\mathbb{N})/\operatorname{Fin}$ is a $2^{\aleph_0}$-universal Boolean algebra.
5. $C([0,1])$ is an $\aleph_0$-universal Banach space.
6. Ozawa: There is no $\aleph_0$-universal II_1-factor.
7. Junge–Pisier: There is no $\aleph_0$-universal C^*–algebra.

Examples

1. Cantor: $(\mathbb{Q}, <)$ is an $\aleph_0$-universal linear ordering.
2. Hausdorff: The Continuum Hypothesis (CH) implies there exists a $2^{\aleph_0}$-universal linear ordering.
3. There exists an $\aleph_0$-universal Boolean algebra.
4. Parovičenko: CH implies that $\mathcal{P}(\mathbb{N})/\operatorname{Fin}$ is a $2^{\aleph_0}$-universal Boolean algebra.
5. $C([0,1])$ is an $\aleph_0$-universal Banach space.
6. Ozawa: There is no $\aleph_0$-universal II_1-factor.
7. Junge–Pisier: There is no $\aleph_0$-universal C^*–algebra.

Both results easily extend to any $\kappa < 2^{\aleph_0}$.

Examples

1. Cantor: $(\mathbb{Q}, <)$ is an $\aleph_0$-universal linear ordering.
2. Hausdorff: The Continuum Hypothesis (CH) implies there exists a $2^{\aleph_0}$-universal linear ordering.
3. There exists an $\aleph_0$-universal Boolean algebra.
4. Parovičenko: CH implies that $\mathcal{P}(\mathbb{N})/\operatorname{Fin}$ is a $2^{\aleph_0}$-universal Boolean algebra.
5. $C([0,1])$ is an $\aleph_0$-universal Banach space.
6. Ozawa: There is no $\aleph_0$-universal II_1-factor.
7. Junge–Pisier: There is no $\aleph_0$-universal C^*–algebra.

Both results easily extend to any $\kappa < 2^{\aleph_0}$.

8. Kirchberg: $\mathcal{O}_2$ is an $\aleph_0$-universal nuclear C^*–algebra.

Question (various)

Is there an $\aleph_1$-universal nuclear C^*–algebra?

Question (various)

Is there an $\aleph_1$-universal nuclear C^*–algebra?

Conjecture

The answer is independent from ZFC.

Question (various)

Is there an $\aleph_1$-universal nuclear C^*–algebra?

Conjecture

The answer is independent from ZFC.

Question

Suppose $\Phi\colon A \to B$, $\Psi\colon A \to C$ are injective *-homomorphisms between nuclear C^*–algebras. Is there a nuclear C^*–algebra D and injective *-homomorphisms $\Phi_1\colon B \to D$, $\Psi_1\colon C \to D$ such that the diagram commutes?

(This is what a model-theorist would call ‘amalgamation,’ but an operator-algebraist may find the terminology a bit misleading.)

Question (various)

Is there an $\aleph_1$-universal nuclear C^*–algebra?

Conjecture

The answer is independent from ZFC.

Question

Suppose $\Phi\colon A \to B$, $\Psi\colon A \to C$ are injective *-homomorphisms between nuclear C^*–algebras. Is there a nuclear C^*–algebra D and injective *-homomorphisms $\Phi_1\colon B \to D$, $\Psi_1\colon C \to D$ such that the diagram commutes?

(This is what a model-theorist would call ‘amalgamation,’ but an operator-algebraist may find the terminology a bit misleading.)
(We may assume $A \cong B \cong C \cong \mathcal{O}_2$.)

Question (various)

Is there an $\aleph_1$-universal nuclear C^*–algebra?

Conjecture

The answer is independent from ZFC.

Question

Suppose $\Phi\colon A \to B$, $\Psi\colon A \to C$ are injective *-homomorphisms between nuclear C^*–algebras. Is there a nuclear C^*–algebra D and injective *-homomorphisms $\Phi_1\colon B \to D$, $\Psi_1\colon C \to D$ such that the diagram commutes?

(This is what a model-theorist would call 'amalgamation,' but an operator-algebraist may find the terminology a bit misleading.)
(We may assume $A \cong B \cong C \cong \mathcal{O}_2$.)

Proposition

If Question has a positive answer then CH implies there exists an $\aleph_1$-universal nuclear C^*–algebra.

A candidate for an $\aleph_1$-universal nuclear C^*–algebra

Theorem

Every separable and unital inductive limit of copies of $\mathcal{O}_2$ is isomorphic to $\mathcal{O}_2$.

A candidate for an $\aleph_1$-universal nuclear C^*–algebra

Theorem

Every separable and unital inductive limit of copies of $\mathcal{O}_2$ is isomorphic to $\mathcal{O}_2$.

Proposition

There exists a unital inductive limit of copies of $\mathcal{O}_2$ of density character $\aleph_1$ that is not $\aleph_1$-universal.

Definition (Shelah)

Fix theory **T** and formula $\varphi(\bar{x}, \bar{y})$. Then φ has...

1. The Order Property (OP) if there exists $A \models \mathbf{T}$ and $\bar{x}_n$, for $n \in \mathbb{N}$, such that

$$\varphi(\bar{x}_m, \bar{x}_n) = \begin{cases} 1 & \text{if } m < n \\ 0 & \text{if } m \geq n. \end{cases}$$

Definition (Shelah)

Fix theory $\mathbf{T}$ and formula $\varphi(\bar{x}, \bar{y})$. Then φ has...

1. The Order Property (OP) if there exists $A \models \mathbf{T}$ and $\bar{x}_n$, for $n \in \mathbb{N}$, such that

$$\varphi(\bar{x}_m, \bar{x}_n) = \begin{cases} 1 & \text{if } m < n \\ 0 & \text{if } m \geq n. \end{cases}$$

2. The Strict Order Propery (strictOP) if it has the OP and defines a partial ordering in every model of $\mathbf{T}$.
3. The Independence Property (IP) if there exists $A \models \mathbf{T}$ such that for every n there are $\bar{a}_j$, $j < n$ in A such that

$$(\forall s \subseteq n)(\exists \bar{b} \in A)\varphi(\bar{a}_i, \bar{b}) = \begin{cases} 1 & \leftrightarrow i \in s \\ 0 & \leftrightarrow i \notin s \end{cases}$$

Definition

A theory has the OP/strictOP/IP if some formula has the OP/strictOP/IP.

Definition

A theory has the OP/strictOP/IP if some formula has the OP/strictOP/IP.

Theorem (Shelah)

if **T** *is a complete theory in a separable language then*

1. **T** *is stable if and only if it does not have the OP.*

Definition

A theory has the OP/strictOP/IP if some formula has the OP/strictOP/IP.

Theorem (Shelah)

if **T** *is a complete theory in a separable language then*

1. **T** *is stable if and only if it does not have the OP.*
2. *If* **T** *has the OP then it has strictOP or IP (or both).*

Definition

A theory has the OP/strictOP/IP if some formula has the OP/strictOP/IP.

Theorem (Shelah)

if **T** *is a complete theory in a separable language then*

1. **T** *is stable if and only if it does not have the OP.*
2. *If* **T** *has the OP then it has strictOP or IP (or both).*

Example

1. The theory of linear orders has the OP, but not the IP.

Definition

A theory has the OP/strictOP/IP if some formula has the OP/strictOP/IP.

Theorem (Shelah)

if **T** *is a complete theory in a separable language then*

1. **T** *is stable if and only if it does not have the OP.*
2. *If* **T** *has the OP then it has strictOP or IP (or both).*

Example

1. The theory of linear orders has the OP, but not the IP.
2. The theory of graphs has the OP, but not the strictOP.

Theorem (folklore–Shelah)

The assertion 'There is an $\aleph_1$-universal linear ordering' is independent from ZFC.

---

[2] Witnessed by a quantifier-free formula

Theorem (folklore–Shelah)

The assertion 'There is an $\aleph_1$-universal linear ordering' is independent from ZFC.

Proposition (Kojman–Shelah)

If there is no κ-universal linear ordering and **T** *has the strictOP,*[2] *then* **T** *has no κ-universal model.*

[2] Witnessed by a quantifier-free formula

Lemma (F.–Hart–Sherman)

(a) The theory of II_1 factors has the quantifier-free IP.
(b) The theory of abelian tracial vNa is stable.

Lemma (F.–Hart–Sherman)

(a) The theory of II_1 factors has the quantifier-free IP.
(b) The theory of abelian tracial vNa is stable.

Proof.

(a) $\varphi(x, y; x', y') = \|[x, y']\|_2$.

Take a tensor product of $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ and and $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. □

Lemma

The theory of C^*–algebras has the strictOP.

Proof.

Let $\varphi(x, y) = \|x^*x - x^*xy^*y\| + |1 - \|x^*x - y^*y\||$.
Find $f_n \colon [0,1] \to [0,1]$ such that $f_m(0) = 0$, $f_m(1) = 1$, and $f_m f_n = f_m$ if $m < n$. □

Lemma

The theory of C^*–algebras has the strictOP.

Proof.

Let $\varphi(x, y) = \|x^*x - x^*xy^*y\| + |1 - \|x^*x - y^*y\||$.
Find $f_n \colon [0,1] \to [0,1]$ such that $f_m(0) = 0$, $f_m(1) = 1$, and $f_m f_n = f_m$ if $m < n$. □

Lemma

Suppose A is a C^*–algebra and $X \subseteq A$ is a dense subset. For $\varepsilon < 1/100$, the transitive closure of the relation

$$x \preceq y \Leftrightarrow \varphi(x, y) < \varepsilon$$

is a partial ordering.

An application of OP

Theorem (F.–Katsura)

If κ *is an uncountable cardinal then there are* 2^κ *nonisomorphic UHF algebras elementarily equivalent to* M_{2^∞} *(and with the same* K*-theory as* M_{2^∞}*) in density* κ*.*

An application of OP

Theorem (F.–Katsura)

If κ is an uncountable cardinal then there are 2^κ nonisomorphic UHF algebras elementarily equivalent to M_{2^∞} (and with the same K-theory as M_{2^∞}) in density κ.

Proposition (≈F.–Katsura)

There exist $2^{\aleph_1}$ nonisomorphic unital inducitve limits of copies of $\mathcal{O}_2$ in density character $\mathcal{O}_2$.

An application of OP

Theorem (F.–Katsura)

If κ is an uncountable cardinal then there are 2^κ nonisomorphic UHF algebras elementarily equivalent to M_{2^∞} (and with the same K-theory as M_{2^∞}) in density κ.

Proposition (≈F.–Katsura)

There exist $2^{\aleph_1}$ nonisomorphic unital inducitve limits of copies of $\mathcal{O}_2$ in density character $\mathcal{O}_2$.

Theorem (Widom, F.–Katsura)

If κ is an uncountable cardinal then there are 2^κ nonisomorphic hyperfinite II_1-factors in density κ.

Theorem

1. *If there is no κ-universal linear ordering then there is no κ-universal C^*–algebra.*

Theorem

1. *If there is no κ-universal linear ordering then there is no κ-universal C^*–algebra.*
2. *If there is no κ-universal linear ordering then there is no κ-universal abelian C^*–algebra.*

Theorem

1. *If there is no κ-universal linear ordering then there is no κ-universal C^*–algebra.*
2. *If there is no κ-universal linear ordering then there is no κ-universal abelian C^*–algebra.*
3. *If there is no κ-universal linear ordering then there is no κ-universal nuclear C^*–algebra.*

Theorem

1. *If there is no κ-universal linear ordering then there is no κ-universal C^*–algebra.*
2. *If there is no κ-universal linear ordering then there is no κ-universal abelian C^*–algebra.*
3. *If there is no κ-universal linear ordering then there is no κ-universal nuclear C^*–algebra.*

Lemma ($\approx$ Schweitzer)

Every abelian C^*–algebra is isomorphic to a subalgebra of a simple, nuclear, C^*–algebra.

Corollary

If there is no κ-universal linear ordering then there is no κ-universal nuclear, simple, C^*–algebra.

Question

Does the theory of II_1 factors have the strictOP?