
(Draft available upon request.)
Strict order property and $\mathbb{C}^*$–algebras

Ilijas Farah (joint work with I. Hirshberg and A. Vignati)

Isaacfest, September 23, 2017
Definition
If \( \mathcal{C} \) is a category of metric structures and \( \kappa \) is a cardinal, some \( A \in \mathcal{C} \) is \( \kappa \)-universal if \( \chi(A) = \kappa \) and ¹

\[
(\forall B \in \mathcal{C})(\chi(B) \leq \kappa) \Rightarrow B \hookrightarrow A.
\]

¹ Embeddings are not assumed to be elementary
1. Cantor: \((\mathbb{Q}, <)\) is an \(\aleph_0\)-universal linear ordering.

2. Hausdorff: The Continuum Hypothesis (CH) implies there exists a \(2^{\aleph_0}\)-universal linear ordering.
Examples

1. Cantor: \((\mathbb{Q}, <)\) is an \(\aleph_0\)-universal linear ordering.

2. Hausdorff: The Continuum Hypothesis (CH) implies there exists a \(2^{\aleph_0}\)-universal linear ordering.

3. There exists an \(\aleph_0\)-universal Boolean algebra.

4. Paroviˇcenko: CH implies that \(P(\mathbb{N})/\text{Fin}\) is a \(2^{\aleph_0}\)-universal Boolean algebra.

5. \(C([0,1])\) is an \(\aleph_0\)-universal Banach space.

6. Ozawa: There is no \(\aleph_0\)-universal II\(_1\)-factor.

7. Junge–Pisier: There is no \(\aleph_0\)-universal C\(^*\)-algebra.

Both results easily extend to any \(\kappa < 2^{\aleph_0}\).

8. Kirchberg: \(O_2\) is an \(\aleph_0\)-universal nuclear C\(^*\)-algebra.
Examples

1. Cantor: \((\mathbb{Q}, <)\) is an \(\aleph_0\)-universal linear ordering.
2. Hausdorff: The Continuum Hypothesis (CH) implies there exists a \(2^{\aleph_0}\)-universal linear ordering.
3. There exists an \(\aleph_0\)-universal Boolean algebra.
4. Parovičenko: CH implies that \(\mathcal{P}(\mathbb{N})/\text{Fin}\) is a \(2^{\aleph_0}\)-universal Boolean algebra.
Examples

1. Cantor: \((\mathbb{Q}, <)\) is an \(\aleph_0\)-universal linear ordering.
2. Hausdorff: The Continuum Hypothesis (CH) implies there exists a \(2^{\aleph_0}\)-universal linear ordering.
3. There exists an \(\aleph_0\)-universal Boolean algebra.
4. Parovičenko: CH implies that \(\mathcal{P}(\mathbb{N})/\text{Fin}\) is a \(2^{\aleph_0}\)-universal Boolean algebra.
5. \(C([0, 1])\) is an \(\aleph_0\)-universal Banach space.
Examples

1. Cantor: \((\mathbb{Q}, <)\) is an \(\aleph_0\)-universal linear ordering.

2. Hausdorff: The Continuum Hypothesis (CH) implies there exists a \(2^{\aleph_0}\)-universal linear ordering.

3. There exists an \(\aleph_0\)-universal Boolean algebra.

4. Parovičenko: CH implies that \(\mathcal{P}(\mathbb{N})/\text{Fin}\) is a \(2^{\aleph_0}\)-universal Boolean algebra.

5. \(C([0, 1])\) is an \(\aleph_0\)-universal Banach space.

6. Ozawa: There is no \(\aleph_0\)-universal \(\text{II}_1\)-factor.

7. Junge–Pisier: There is no \(\aleph_0\)-universal \(C^*\)-algebra.
Examples

1. Cantor: \((\mathbb{Q}, <)\) is an \(\aleph_0\)-universal linear ordering.

2. Hausdorff: The Continuum Hypothesis (CH) implies there exists a \(2^{\aleph_0}\)-universal linear ordering.

3. There exists an \(\aleph_0\)-universal Boolean algebra.

4. Parovičenko: CH implies that \(\mathcal{P}(\mathbb{N})/\text{Fin}\) is a \(2^{\aleph_0}\)-universal Boolean algebra.

5. \(C([0, 1])\) is an \(\aleph_0\)-universal Banach space.

6. Ozawa: There is no \(\aleph_0\)-universal II\(_1\)-factor.

7. Junge–Pisier: There is no \(\aleph_0\)-universal \(C^*\)-algebra.

Both results easily extend to any \(\kappa < 2^{\aleph_0}\).
Examples

1. Cantor: $(\mathbb{Q}, <)$ is an $\aleph_0$-universal linear ordering.
2. Hausdorff: The Continuum Hypothesis (CH) implies there exists a $2^{\aleph_0}$-universal linear ordering.
3. There exists an $\aleph_0$-universal Boolean algebra.
4. Parovičenko: CH implies that $\mathcal{P}(\mathbb{N})/\text{Fin}$ is a $2^{\aleph_0}$-universal Boolean algebra.
5. $C([0, 1])$ is an $\aleph_0$-universal Banach space.
6. Ozawa: There is no $\aleph_0$-universal $\text{II}_1$-factor.
7. Junge–Pisier: There is no $\aleph_0$-universal $\mathcal{C}^*$–algebra.

Both results easily extend to any $\kappa < 2^{\aleph_0}$.
8. Kirchberg: $\mathcal{O}_2$ is an $\aleph_0$-universal nuclear $\mathcal{C}^*$–algebra.
Question (various)

Is there an $\aleph_1$-universal nuclear $C^*$-algebra?

Conjecture

The answer is independent from ZFC.

Question

Suppose $\Phi: A \to B$, $\Psi: A \to C$ are injective $\ast$-homomorphisms between nuclear $C^*$-algebras. Is there a nuclear $C^*$-algebra $D$ and injective $\ast$-homomorphisms $\Phi_1: B \to D$, $\Psi_1: C \to D$ such that the diagram commutes?

(This is what a model-theorist would call 'amalgamation,' but an operator-algebraist may find the terminology a bit misleading.)

We may assume $A \cong B \cong C \cong O_2$.

Proposition

If Question has a positive answer then CH implies there exists an $\aleph_1$-universal nuclear $C^*$-algebra.
Question (various)

Is there an $\aleph_1$-universal nuclear C*-algebra?

Conjecture

The answer is independent from ZFC.
Question (various)

Is there an $\aleph_1$-universal nuclear $C^*$–algebra?

Conjecture

The answer is independent from ZFC.

Question

Suppose $\Phi: A \to B$, $\Psi: A \to C$ are injective $^*$-homomorphisms between nuclear $C^*$–algebras. Is there a nuclear $C^*$–algebra $D$ and injective $^*$-homomorphisms $\Phi_1: B \to D$, $\Psi_1: C \to D$ such that the diagram commutes?

(This is what a model-theorist would call ‘amalgamation,’ but an operator-algebraist may find the terminology a bit misleading.)
Question (various)

*Is there an $\aleph_1$-universal nuclear $C^*$-algebra?*

Conjecture

*The answer is independent from ZFC.*

Question

*Suppose $\Phi: A \to B$, $\Psi: A \to C$ are injective $^*$-homomorphisms between nuclear $C^*$-algebras. Is there a nuclear $C^*$-algebra $D$ and injective $^*$-homomorphisms $\Phi_1: B \to D$, $\Psi_1: C \to D$ such that the diagram commutes?*

(This is what a model-theorist would call ‘amalgamation,’ but an operator-algebraist may find the terminology a bit misleading.)

*(We may assume $A \cong B \cong C \cong O_2$.)*
Question (various)

Is there an $\mathbb{N}_1$-universal nuclear $C^*$–algebra?

Conjecture

The answer is independent from ZFC.

Question

Suppose $\Phi: A \to B$, $\Psi: A \to C$ are injective $^*$-homomorphisms between nuclear $C^*$–algebras. Is there a nuclear $C^*$–algebra $D$ and injective $^*$-homomorphisms $\Phi_1: B \to D$, $\Psi_1: C \to D$ such that the diagram commutes?

(This is what a model-theorist would call ‘amalgamation,’ but an operator-algebraist may find the terminology a bit misleading.)

(We may assume $A \cong B \cong C \cong \mathcal{O}_2$.)

Proposition

If Question has a positive answer then CH implies there exists an $\mathbb{N}_1$-universal nuclear $C^*$–algebra.
A candidate for an $\aleph_1$-universal nuclear C*-algebra

**Theorem**

*Every separable and unital inductive limit of copies of $O_2$ is isomorphic to $O_2$.***
A candidate for an $\aleph_1$-universal nuclear $C^*$–algebra

Theorem
Every separable and unital inductive limit of copies of $O_2$ is isomorphic to $O_2$.

Proposition
There exists a unital inductive limit of copies of $O_2$ of density character $\aleph_1$ that is not $\aleph_1$-universal.
**Definition (Shelah)**

Fix theory $\mathbf{T}$ and formula $\varphi(\bar{x}, \bar{y})$. Then $\varphi$ has...

1. The Order Property (OP) if there exists $A \models \mathbf{T}$ and $\bar{x}_n$, for $n \in \mathbb{N}$, such that

$$
\varphi(\bar{x}_m, \bar{x}_n) = \begin{cases} 
1 & \text{if } m < n \\
0 & \text{if } m \geq n.
\end{cases}
$$
Definition (Shelah)

Fix theory $\mathbf{T}$ and formula $\varphi(\bar{x}, \bar{y})$. Then $\varphi$ has...

1. The Order Property (OP) if there exists $A \models \mathbf{T}$ and $\bar{x}_n$, for $n \in \mathbb{N}$, such that

$$
\varphi(\bar{x}_m, \bar{x}_n) = \begin{cases} 
1 & \text{if } m < n \\
0 & \text{if } m \geq n.
\end{cases}
$$

2. The Strict Order Property (strictOP) if it has the OP and defines a partial ordering in every model of $\mathbf{T}$.

3. The Independence Property (IP) if there exists $A \models \mathbf{T}$ such that for every $n$ there are $\bar{a}_j, j < n$ in $A$ such that

$$
(\forall s \subseteq n)(\exists \bar{b} \in A)\varphi(\bar{a}_i, \bar{b}) = \begin{cases} 
1 & \iff i \in s \\
0 & \iff i \notin s
\end{cases}
$$
Definition
A theory has the OP/strictOP/IP if some formula has the OP/strictOP/IP.

Theorem (Shelah)
if $T$ is a complete theory in a separable language then
1. $T$ is stable if and only if it does not have the OP.
2. If $T$ has the OP then it has strictOP or IP (or both).

Example
1. The theory of linear orders has the OP, but not the IP.
2. The theory of graphs has the OP, but not the strictOP.
Definition
A theory has the OP/strictOP/IP if some formula has the OP/strictOP/IP.

Theorem (Shelah)
if $T$ is a complete theory in a separable language then
1. $T$ is stable if and only if it does not have the OP.
Definition
A theory has the OP/strictOP/IP if some formula has the OP/strictOP/IP.

Theorem (Shelah)
if $T$ is a complete theory in a separable language then
1. $T$ is stable if and only if it does not have the OP.
2. If $T$ has the OP then it has strictOP or IP (or both).

Example
1. The theory of linear orders has the OP, but not the IP.
2. The theory of graphs has the OP, but not the strictOP.
Definition
A theory has the OP/strictOP/IP if some formula has the OP/strictOP/IP.

Theorem (Shelah)
if $T$ is a complete theory in a separable language then
1. $T$ is stable if and only if it does not have the OP.
2. If $T$ has the OP then it has strictOP or IP (or both).

Example
1. The theory of linear orders has the OP, but not the IP.
Definition
A theory has the OP/strictOP/IP if some formula has the OP/strictOP/IP.

Theorem (Shelah)
if $T$ is a complete theory in a separable language then
1. $T$ is stable if and only if it does not have the OP.
2. If $T$ has the OP then it has strictOP or IP (or both).

Example
1. The theory of linear orders has the OP, but not the IP.
2. The theory of graphs has the OP, but not the strictOP.
Theorem (folklore–Shelah)

The assertion ‘There is an $\aleph_1$-universal linear ordering’ is independent from ZFC.

$^2$Witnessed by a quantifier-free formula
Theorem (folklore–Shelah)

*The assertion ‘There is an \( \aleph_1 \)-universal linear ordering’ is independent from ZFC.*

Proposition (Kojman–Shelah)

*If there is no \( \kappa \)-universal linear ordering and \( T \) has the strictOP,\(^2\) then \( T \) has no \( \kappa \)-universal model.*

\(^2\)Witnessed by a quantifier-free formula
Lemma (F.–Hart–Sherman)

(a) The theory of II₁ factors has the quantifier-free IP.
(b) The theory of abelian tracial vNa is stable.
Lemma (F.–Hart–Sherman)

(a) The theory of II$_1$ factors has the quantifier-free IP.
(b) The theory of abelian tracial vNa is stable.

Proof.
(a) $\varphi(x, y; x', y') = \|[x, y']\|_2$.

Take a tensor product of $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.
Lemma

The theory of C*-algebras has the strictOP.

Proof.
Let $\varphi(x, y) = \|x^*x - x^*xy^*y\| + |1 - \|x^*x - y^*y\||$.
Find $f_n: [0, 1] \to [0, 1]$ such that $f_m(0) = 0$, $f_m(1) = 1$, and $f_mf_n = f_m$ if $m < n$. 

\qed
Lemma

*The theory of \( C^* \)-algebras has the strict OP.*

Proof.

Let \( \varphi(x, y) = \|x^*x - x^*xy^*y\| + |1 - \|x^*x - y^*y\||. \)

Find \( f_n : [0, 1] \rightarrow [0, 1] \) such that \( f_m(0) = 0, f_m(1) = 1, \) and \( f_mf_n = f_m \) if \( m < n. \)

Lemma

*Suppose \( A \) is a \( C^* \)-algebra and \( X \subseteq A \) is a dense subset. For \( \varepsilon < 1/100, \) the transitive closure of the relation* \( x \preceq y \iff \varphi(x, y) < \varepsilon \)

*is a partial ordering.*
An application of OP

Theorem (F.–Katsura)

If $\kappa$ is an uncountable cardinal then there are $2^{\kappa}$ nonisomorphic UHF algebras elementarily equivalent to $M_{2^\infty}$ (and with the same $K$-theory as $M_{2^\infty}$) in density $\kappa$. 
An application of OP

Theorem (F.–Katsura)

If $\kappa$ is an uncountable cardinal then there are $2^\kappa$ nonisomorphic UHF algebras elementarily equivalent to $M_{2^\infty}$ (and with the same $K$-theory as $M_{2^\infty}$) in density $\kappa$.

Proposition ($\approx$ F.–Katsura)

There exist $2^{\aleph_1}$ nonisomorphic unital inducitve limits of copies of $O_2$ in density character $O_2$. 

An application of OP

Theorem (F.–Katsura)

If $\kappa$ is an uncountable cardinal then there are $2^{\kappa}$ nonisomorphic UHF algebras elementarily equivalent to $M_{2^{\infty}}$ (and with the same $K$-theory as $M_{2^{\infty}}$) in density $\kappa$.

Proposition ($\approx$ F.–Katsura)

There exist $2^{\aleph_1}$ nonisomorphic unital inductive limits of copies of $\mathcal{O}_2$ in density character $\mathcal{O}_2$.

Theorem (Widom, F.–Katsura)

If $\kappa$ is an uncountable cardinal then there are $2^{\kappa}$ nonisomorphic hyperfinite $II_1$-factors in density $\kappa$. 
Theorem

1. If there is no $\kappa$-universal linear ordering then there is no $\kappa$-universal C$^*$–algebra.

Lemma (≈ Schweitzer)

Every abelian C$^*$–algebra is isomorphic to a subalgebra of a simple, nuclear, C$^*$–algebra.

Corollary

If there is no $\kappa$-universal linear ordering then there is no $\kappa$-universal nuclear, simple, C$^*$–algebra.
Theorem

1. *If there is no* $\kappa$-*universal linear ordering then there is no* $\kappa$-*universal* $C^*$-*algebra.*

2. *If there is no* $\kappa$-*universal linear ordering then there is no* $\kappa$-*universal abelian* $C^*$-*algebra.*
Theorem

1. If there is no $\kappa$-universal linear ordering then there is no $\kappa$-universal C*-algebra.
2. If there is no $\kappa$-universal linear ordering then there is no $\kappa$-universal abelian C*-algebra.
3. If there is no $\kappa$-universal linear ordering then there is no $\kappa$-universal nuclear C*-algebra.
Theorem

1. If there is no $\kappa$-universal linear ordering then there is no $\kappa$-universal $C^*$–algebra.

2. If there is no $\kappa$-universal linear ordering then there is no $\kappa$-universal abelian $C^*$–algebra.

3. If there is no $\kappa$-universal linear ordering then there is no $\kappa$-universal nuclear $C^*$–algebra.

Lemma ($\approx$ Schweitzer)

Every abelian $C^*$–algebra is isomorphic to a subalgebra of a simple, nuclear, $C^*$–algebra.

Corollary

If there is no $\kappa$-universal linear ordering then there is no $\kappa$-universal nuclear, simple, $C^*$–algebra.
Question

Does the theory of $\text{II}_1$ factors have the strict OP?