A MONAD MEASURE SPACE FOR LOGARITHMIC DENSITY

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Abstract. We provide a framework for proofs of structural theorems about sets with positive Banach logarithmic density. For example, we prove that if $A \subseteq \mathbb{N}$ has positive Banach logarithmic density, then A contains an approximate geometric progression of any length. We also prove that if $A, B \subseteq \mathbb{N}$ have positive Banach logarithmic density, then there are arbitrarily long intervals whose gaps on $A \cdot B$ are multiplicatively bounded, a multiplicative version Jin's sumset theorem. The main technical tool is the use of a quotient of a Loeb measure space with respect to a multiplicative cut.

1. Introduction

Szemeredi's theorem states that if $A \subseteq \mathbb{Z}$ has positive upper density, then A contains arbitrarily large arithmetic progressions. The main idea behind Furstenberg's proof of Szemeredi's theorem was to associate to the aforementioned set A a dynamical system (X, μ, T) and a measurable set $E \subseteq X$ with $\overline{d}(A) = \mu(E)$ satisfying, for any finite $F \subseteq \mathbb{Z}$:

$$\overline{d}\left(\bigcap_{i \in F}(A-i)\right) \geq \mu\left(\bigcap_{i \in F} T^{-i}(E)\right).$$

This association, now called the *Furstenberg correspondence principle* [2, Lemma 4.6], converted the task of proving Szemeredi's theorem into the task of proving a theorem of ergodic theory, now referred to as *Furstenberg's multiple recurrence theorem.* Furstenberg's correspondence principle holds for any countable amenable semigroup (with densities calculated with respect to particular Følner sequences) and there are many generalizations of Furstenberg's recurrence theorem. In short, Furstenberg's correspondence has led to a large collection of structural results in combinatorial number theory.

Nonstandard analysis provides an elegant way of establishing Furstenberg's original correspondence theorem. (For an introduction to nonstandard methods aimed specifically toward applications to combinatorial number theory see [13].) Indeed, one can consider the *hyperfinite interval* $[-N, N] \subseteq {}^*\mathbb{Z}$, equipped with its *Loeb measure* μ_L, which is the σ-additive measure obtained from the finitely-additive counting measure $\mu(A) := \mathrm{st}(\frac{|A|}{2N+1})$ defined on the algebra of hyperfinite subsets of $[-N, N]$ using the Caratheodory extension theorem. By the nonstandard characterization of upper density, there is an infinite $N \in {}^*\mathbb{N}$ for which

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$\overline{d}(A) = \mu_L({}^*A \cap [-N, N])$. Letting $T : [-N, N-1] \to [-N, N]$ be addition by 1 (which is easily seen to be measure preserving and defined on a measure 1 set), the dynamical system $([-N, N], \mu_L, T)$ and the measurable set $E := {}^*A \cap [-N, N]$ witness the conclusion of the Furstenberg correspondence principle.

In this paper, we consider a different kind of density, namely *logarithmic density* (see Section 2 for the precise definition) and seek to associate an appropriate measure space to sets of positive logarithmic density. Using the nonstandard characterization of logarithmic density, this is accomplished in the same manner as in the previous paragraph. However, this Loeb measure space contains a serious deficiency, namely the fact that multiplication is not measure preserving. The main result in this paper is that multiplication is measure-preserving on an appropriate quotient of the associated Loeb measure space (Theorem 2.24 below).

Initially, we had hoped to use this fact to deduce approximate geometric structure in sets of positive logarithmic density. Indeed, one can use Furstenberg's multiple recurrence theorem on the quotient space to obtain actual geometric structure in the quotient space, which, when pulled back to the original Loeb space and combined with the transfer principle, would yield approximate geometric structure in the original subset of the integers. While this process is valid and briefly explained in Section 3, in an upcoming paper we show that we can actually use the original Szemeredi theorem, combined with a "logarithmic change of coordinates," to more directly obtain the aforementioned approximate geometric structure and with better bounds on the nature of the approximation. Thus, we leave it as an open problem to find more sophisticated applications of the fact that multiplication on our quotient measure space is measure-preserving.

We then briefly discuss a family of densities on subsets of $\mathbb{N}$ for which the corresponding sets of positive measure in the quotient space contain arbitrarily long powers of arithmetic progressions.

In the next to last section, we show that the Lebesgue density theorem is valid in the aforementioned quotient measure space. In the last section, we use the Lebesgue density theorem to prove a multiplicative analog of a result of Jin [12], namely that if A and B both have positive Banach log density, then there are arbitrarily long intervals on which $A \cdot B$ has multiplicatively bounded gaps.

1.1. **A few words about nonstandard analysis.** For the reader unacquainted with nonstandard analysis, we briefly describe the main idea; a much more comprehensive introduction to nonstandard methods in number theory can be found in [13]. There is an ordered field extension ${}^*\mathbb{R}$ of the ordered field of real numbers which contains "ideal" elements such as *infinitesimal* elements and *infinite elements.* Moreover, one requires the *transfer principle* to hold: the field ${}^*\mathbb{R}$ "logically" behaves like $\mathbb{R}$ with respect to certain nice subsets of ${}^*\mathbb{R}$ called the *internal* sets. For example, any nonempty internal subset of ${}^*\mathbb{R}$ that is bounded above has a supremum. If $a, b \in {}^*\mathbb{R}$, we write $a \approx b$ if $|a - b|$ is infinitesimal. Every *finite* element $a \in {}^*\mathbb{R}$ (that is, $|a| \leq n$ for some $n \in \mathbb{N}$) is within an infinitesimal distance of a unique real number, called the *standard part* of a and denoted $\mathrm{st}(a)$.

Besides enlarging ${}^*\mathbb{R}$, one actually enlarges every infinite subset A of ${}^*\mathbb{R}$ to an internal set ${}^*A$ such that ${}^*A \cap \mathbb{R} = A$. In particular, $\mathbb{N}$ gets enlarged to ${}^*\mathbb{N}$. All "new" elements of ${}^*\mathbb{N}$ are infinite. A consequence of the transfer principle is that $\mathbb{N}$ is an *external* (that is, not internal) subset of ${}^*\mathbb{R}$. (If $\mathbb{N}$ were internal, then since it is bounded above in ${}^*\mathbb{R}$, it would have a maximum.) From this follows the so-called *overspill* principle: if $E \subseteq {}^*\mathbb{N}$ is internal and infinite, then $E \cap ({}^*\mathbb{N} \setminus \mathbb{N}) \neq \emptyset$.

Finally, there are certain internal subsets of ${}^*\mathbb{R}$ that logically behave like finite sets; such sets are called *hyperfinite*. If $A \subseteq {}^*\mathbb{R}$ is hyperfinite, then A has an *internal cardinality* $|A|$ which is an element of ${}^*\mathbb{N}$.

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2. Densities, cuts, and measures

2.1. Densities.

Convention 2.1. In this paper, $\mathbb{N}$ denotes the set of *positive* natural numbers.

For the convenience of the reader, we recall the following:

Definition 2.2. Suppose that $A \subseteq \mathbb{N}$. Then:

- The *upper density of* A is defined to be
$$\overline{d}(A) := \limsup_{n\to\infty} \frac{|A \cap [1,n]|}{n}.$$
- The *lower density of* A is defined to be
$$\underline{d}(A) := \liminf_{n\to\infty} \frac{|A \cap [1,n]|}{n}.$$

We also recall the definitions of *logarithmic densities*:

Definition 2.3. Suppose that $A \subseteq \mathbb{N}$. Then:

- The *upper logarithmic density of* A is defined to be
$$\overline{ld}(A) := \limsup_{n\to\infty} \frac{1}{\ln n} \sum_{x\in A\cap[1,n]} \frac{1}{x}.$$
- The *lower logarithmic density of* A is defined to be
$$\underline{ld}(A) := \liminf_{n\to\infty} \frac{1}{\ln n} \sum_{x\in A\cap[1,n]} \frac{1}{x}.$$

When dealing with logarithmic densities, it is useful to recall that, setting $H_n := \sum_{k=1}^{n} \frac{1}{k}$ (the so-called n^{th} *harmonic number*), we have $\lim_{n\to\infty}(H_n - \ln n) = \gamma$, the so-called *Euler-Mascheroni* constant. For example, it follows easily that $\overline{ld}(\mathbb{N}) = \underline{ld}(\mathbb{N}) = 1$.

The proof of the following lemma is straightforward.

Lemma 2.4. *Suppose that $A, B \subseteq \mathbb{N}$ and $n \in \mathbb{N}$.*

(1) $\overline{ld}(A+n) = \overline{ld}(A)$ *and* $\underline{ld}(A+n) = \underline{ld}(A)$.
(2) *If $A \triangle B$ is finite, then* $\overline{ld}(A) = \overline{ld}(B)$ *and* $\underline{ld}(A) = \underline{ld}(B)$.

The following fact is the content of [3, Lemma 2.1(e)(f)]:

Fact 2.5. *For $A \subseteq \mathbb{N}$, we have* $\underline{d}(A) \leqslant \underline{ld}(A) \leqslant \overline{ld}(A) \leqslant \overline{d}(A)$.

We would like to offer an alternative proof of the preceding fact. We will only prove that $\underline{d}(A) \leq \underline{ld}(A)$; the other inequality follows from the inequality for lower densities and the fact that $\overline{d}(A) = 1 - \underline{d}(\mathbb{N} \setminus A)$ and $\overline{ld}(A) = 1 - \underline{ld}(\mathbb{N} \setminus A)$. The heuristic behind our proof is simple: the logarithmic density of a set can only decrease if we "push the elements of the set to the right;" such a shift should leave the lower density fixed. Here are the specifics:

Set $f_A : \mathbb{N} \to \mathbb{R}$ to be defined by $f_A(n) := \sum_{x \in A \cap [1,n]} \frac{1}{x}$. Without loss of generality, we may assume $\underline{d}(A) > 0$. Take $\alpha < \underline{d}(A)$ and $H > \mathbb{N}$. It suffices to show that $\mathrm{st}(\frac{f_A(H)}{\ln H}) \geq \alpha$. Since two sets that differ by only a finite number of elements have the same lower density and the same lower logarithmic density, we can assume that $\inf_{n \geqslant 1} \frac{|A \cap [1,n]|}{n} > \alpha$.

Let $m := |{}^*A \cap [1, H]|$ and set

$$B = \left\{ \left\lfloor \frac{x}{\alpha} \right\rfloor + 1 : x \in [1, m] \right\} \cap [1, H].$$

Next observe that, for every $k \in [1, H]$, we have $\frac{|B \cap [1,k]|}{k} \leq \alpha$. (Without taking integer parts, B would be an arithmetic progression of real numbers, whence the densities are clearly bounded by α; by taking integer parts and then adding 1, if anything, we have reduced the densities.) Let $K := |B|$. Let $(a_n : n \leq m)$ and $(b_n : n \leq K)$ be the enumerations of $A \cap [1, H]$ and B in increasing order. Since $\alpha < \frac{|{}^*A \cap [1,k]|}{k}$ for each $k \in [1, H]$, it follows that $a_n \leq b_n$ for all $n \leq K$. We thus get that

$$f_A(H) = \sum_{n=1}^{m} \frac{1}{a_n} \geq \sum_{n=1}^{K} \frac{1}{a_n} \geq \sum_{n=1}^{K} \frac{1}{b_n} =: f_B(H).$$

Since $\frac{f_B(H)}{\ln H} \approx \frac{\alpha \ln(H)}{\ln H} = \alpha$, it follows that $\mathrm{st}(\frac{f_A(H)}{\ln H}) \geq \alpha$.

We also recall the following definition:

Definition 2.6. For $A \subseteq \mathbb{N}$, the *(upper) Banach density* of A is defined to be

$$\mathrm{BD}(A) := \lim_{n \to \infty} \sup_{k \geq 1} \frac{|A \cap [k, k+n]|}{n+1}.$$

Of course, for the preceding definition to be legitimate, one must prove that the limit involved always exists. This is a rather straightforward argument; it also follows immediately from Fekete's Lemma (see [10]).

We now want to define a Banach version of logarithmic density; to do so, we must show that the corresponding limit exists.

Lemma 2.7. *Suppose that $g : \mathbb{N} \to \mathbb{R}$ is a nondecreasing function satisfying, for all $j, n \in \mathbb{N}$, the inequality $g(n^j) \leq j g(n)$. Then $\lim_{n \to \infty} \frac{g(n)}{\ln n}$ exists and equals $\inf_{n \geq 1} \frac{g(n)}{\ln n}$.*

Proof. It is enough to show that, for every $n \in \mathbb{N}$ and $N \in {}^*\mathbb{N} \setminus \mathbb{N}$, we have $\mathrm{st}(\frac{g(N)}{\ln N}) \leq \frac{g(n)}{\ln n}$. Take $j \in {}^*\mathbb{N}$ such that $n^j \leq N < n^{j+1}$; note that $j > \mathbb{N}$. We conclude by observing that

$$\frac{g(N)}{\ln N} \leq \frac{(j+1) g(n)}{j \ln n} = (1 + \frac{1}{j}) \frac{g(n)}{\ln n} \approx \frac{g(n)}{\ln n}.$$

∎

Proposition 2.8. *For any $A \subseteq \mathbb{N}$, the limit*

$$\lim_{n\to\infty} \sup_{k\geq 1} \frac{1}{\ln n} \sum_{x\in A\cap[k,nk]} \frac{1}{x}$$

exists and equals

$$\inf_{n\geq 1} \sup_{k\geq 1} \frac{1}{\ln n} \sum_{x\in A\cap[k,nk]} \frac{1}{x}.$$

Proof. Define $g : \mathbb{N} \to \mathbb{R}$ by

$$g(n) \; = \; \sup_{k\geq 1} \left(\sum_{x\in[k,kn)\cap A} \frac{1}{x} \right).$$

Clearly g is nondecreasing, so, by Lemma 2.7, it suffices to show that $g(n^j) \leq jg(n)$ for all $j, n \in \mathbb{N}$. To see this, it suffices to observe that, for a fixed k, one has

$$\sum_{x\in[k,kn^j)\cap A} \frac{1}{x} \; = \; \sum_{s=1}^{j} \left(\sum_{x\in[kn^{s-1},kn^s)\cap A} \frac{1}{x} \right) \; \leq \; \sum_{s=1}^{j} g(n) \; = \; j \cdot g(n).$$

□

We are thus entitled to make the following:

Definition 2.9. For $A \subseteq \mathbb{N}$, the *(upper) Banach log density* of A is

$$\ell\mathrm{BD}(A) := \lim_{n\to\infty} \sup_{k\geq 1} \frac{1}{\ln n} \sum_{x\in A\cap[k,nk]} \frac{1}{x}.$$

Of course one could also define the lower Banach log density, but in this paper we only focus on the upper Banach log density.

The next proposition can be proven in a manner analogous to the corresponding statement for upper log density.

Proposition 2.10. *For any $A \subseteq \mathbb{N}$, we have $\ell\mathrm{BD}(A) \leq \mathrm{BD}(A)$.*

Finally, we will frequently make use of the following nonstandard formulation of Banach log density.

Proposition 2.11. *If $A \subseteq \mathbb{N}$, then $\ell\mathrm{BD}(A) \geq \alpha$ if and only if for every $N > \mathbb{N}$, there is $k \in {}^*\mathbb{N}$ such that*

$$\mathrm{st}\left(\frac{\sum_{x\in {}^*A\cap[k,Nk]} \frac{1}{x}}{\ln N} \right) \tag{2.1}$$

is at least α.

Equivalently, one can say that, for any $N > \mathbb{N}$, $\ell\mathrm{BD}(A)$ is the supremum over $k \in {}^*\mathbb{N}$ of the quantity in (2.1).

2.2. **Loeb measure spaces.** In this subsection, we fix $N > \mathbb{N}$. Motivated by the proposition at the end of the last subsection, we introduce a measure on internal subsets of intervals $[k, Nk]$. More precisely, for each internal set $A \subseteq [k, Nk]$, set

$$\nu(A) := \nu_{k,N}(A) = \operatorname{st}\left(\sum_{a\in A} \frac{1}{a \ln N}\right).$$

It is readily verified that ν is a finitely additive measure defined on the internal subsets of $[k, Nk]$, whence we obtain a Loeb measure space based on $[k, Nk]$, whose measure we continue to denote by $\nu = \nu_{k,N}$. By Proposition 2.11, for every $N > \mathbb{N}$, there is $k \in {}^*\mathbb{N}$ such that $\ell\mathrm{BD}(A) = \nu_{k,N}({}^*A \cap [k, Nk])$.

Recall that, for $n \in \mathbb{N}$, we set $H_n = \sum_{k=1}^{n} \frac{1}{k}$.

Proposition 2.12. *For any $k \leqslant a \leqslant b \leqslant Nk$, we have*

$$\nu([a, b]) = \operatorname{st}\left(\frac{\ln b - \ln a}{\ln N}\right).$$

In particular, $\nu([k, k\sqrt{N}]) = \nu([k\sqrt{N}, Nk]) = \frac{1}{2}$.

Proof. We assume that $a, b \in {}^*\mathbb{N} \setminus \mathbb{N}$; the other cases are similar and easier. We have

$$\frac{H_b - H_{a-1}}{\ln N} = \frac{(H_b - \ln b) + (\ln b - \ln a) + (\ln \frac{a}{a-1}) + (\ln(a-1) - H_{a-1})}{\ln N}.$$

Since $a, b > \mathbb{N}$, we have $H_{a-1} - \ln(a-1), H_b - \ln b \approx \gamma$. Also, $\ln \frac{a}{a-1} \approx 0$. It follows that

$$\nu([a, b]) = \operatorname{st}\left(\sum_{x=a}^{b} \frac{1}{x \ln N}\right) = \operatorname{st}\left(\frac{H_b - H_{a-1}}{\ln N}\right) = \operatorname{st}\left(\frac{\ln b - \ln a}{\ln N}\right).$$

$\square$

Corollary 2.13. *Suppose that $a, b, c \in {}^*\mathbb{N}$ are such that $a, b, ac, bc \in [k, Nk]$. Then $\nu([a, b]) = \nu([ac, bc])$.*

In contrast to the previous corollary, note that, under the same assumptions, $\nu(c\cdot [a, b]) \neq \nu([a, b])$ in general, that is, multiplication need not be measure preserving. Indeed,

$$\nu(c \cdot [a, b]) = \operatorname{st}\left(\sum_{x\in[a,b]} \frac{1}{cx \ln N}\right) = \operatorname{st}\left(\frac{1}{c}\sum_{x\in[a,b]} \frac{1}{x \ln N}\right).$$

We will shortly see that this problem vanishes when we pass to a certain quotient of the Loeb measure space.

In calculations pertaining to the aforementioned quotient space, it will become useful to know how to approximate the measures of certain internal subsets of $[k, Nk]$. First, let us establish some notation. We call an interval $[a, b] \subseteq [k, Nk]$ *big* if $\operatorname{st}(\frac{b}{a}) > 2$ (where, for the sake of this definition, the standard part of an infinite hyperreal is itself). Now suppose that $C \subseteq [k, Nk]$ is internal and we write $C = \bigsqcup_{i\in I}[a_i, b_i]$, where the intervals $[a_i, b_i]$ are the internal connected components of C, that is, they are the maximal intervals contained in C. (Note then that the set I and the sequences (a_i) and (b_i) are all internal.) We then say that C *has big* *components* if each connected component $[a_i, b_i]$ is big. In what follows, considering sets with big components will considerably simplify the computations.

For the proof of the next lemma, we will need to recall the following elementary estimates: suppose that $r, s \in \mathbb{N}$ are such that $2 \le r \le s$. Then:

$$\ln(s+1) - \ln(r) \le \sum_{i=r}^{s} \frac{1}{i} \le \ln(s) - \ln(r-1).$$

Lemma 2.14. *Suppose that $C = \bigsqcup_{i\in I}[a_i, b_i]$ has big components and that $C \subseteq {}^*\mathbb{N} \setminus \mathbb{N}$. Then $\nu(C) \approx \frac{1}{\ln N}\sum_{i\in I}(\ln(b_i) - \ln(a_i))$.*

Proof. Fix $i \in I$. Note then that

$$\ln(b_i) - \ln(a_i) \le \sum_{n\in[a_i,b_i]} \frac{1}{n} \le \ln(b_i) - \ln(a_i - 1).$$

It follows that

$$\frac{|(\ln(b_i) - \ln(a_i)) - \sum_{n\in[a_i,b_i]} \frac{1}{n}|}{\ln(b_i) - \ln(a_i)} \le \frac{\ln(a_i) - \ln(a_i - 1)}{\ln 2} \approx 0.$$

Fix $\epsilon > 0$. We then have

$$\begin{aligned}|(\sum_{i\in I}(\ln(b_i) - \ln(a_i))) - (\sum_{i\in I}\sum_{n\in[a_i,b_i]}\frac{1}{n})| &\le \sum_{i\in I}\epsilon\cdot(\ln(b_i) - \ln(a_i)) \\ &\le \epsilon\cdot\sum_{i\in I}\sum_{n=a_i+1}^{b_i}\frac{1}{n} \\ &\le \epsilon\cdot H_N.\end{aligned}$$

Therefore, we have

$$|\nu(C) - \frac{1}{\ln N}\sum_{i\in I}(\ln(b_i) - \ln(a_i))| \le 2\epsilon\cdot\frac{H_N}{\ln N} \approx 2\epsilon.$$

Since $\epsilon > 0$ was arbitrary, this yields the desired result. ☐

2.3. **Multiplicative cuts.** As mentioned in the previous section, we will soon pass to a certain quotient of the above Loeb measure spaces. In this regard, the following notion is central:

Definition 2.15. An infinite initial segment V of ${}^*\mathbb{N}$ is a *multiplicative cut* if $V\cdot V \subseteq V$.

Note the following obvious facts:

- multiplicative cuts are also additive cuts, that is, they are closed under addition;
- bounded multiplicative cuts must be external;
- $\mathbb{N}$ is the smallest multiplicative cut.

For $N \in {}^*\mathbb{N} \setminus \mathbb{N}$, we let

$$V_N = \bigcap_{n\in\mathbb{N}}[1, \lfloor N^{1/n}\rfloor]. \tag{2.2}$$

Then V_N is the largest multiplicative cut in $[1, N]$.

Definition 2.16. Suppose that U and V are infinite initial segments of ${}^*\mathbb{N} \cup \{0\}$ and ${}^*\mathbb{N}$ respectively. We set:

(1) $\ln V := \{x \in {}^*\mathbb{N} \cup \{0\} : \lfloor e^x \rfloor \in V\}$.
(2) $e^U = \bigcup_{x \in U} [1, \lfloor e^x \rfloor]$.

It is straightforward to verify the following facts:

(1) V is a multiplicative cut if and only if $\ln V$ is an additive cut.
(2) e^U is a multiplicative cut if and only if U is an additive cut.
(3) If U is an additive cut, then $\ln(e^U) = U$.
(4) If V is a multiplicative cut, then $e^{\ln V} = V$.

In the rest of this subsection, we fix $N \in {}^*\mathbb{N} \setminus \mathbb{N}$ and a multiplicative cut $V \subseteq [1, N]$.

Definition 2.17. For any $a, b \in {}^*\mathbb{N} \setminus \mathbb{N}$, we declare $a \sim_V b$ if and only if $|\lfloor \ln a \rfloor - \lfloor \ln b \rfloor| \in \ln V$.

Equivalently, if $a < b$, then $a \sim_V b$ if and only if $\lfloor \frac{b}{a} \rfloor \in V$. Note that $\sim_V$ is an equivalence relation on ${}^*\mathbb{N}$. For $a \in {}^*\mathbb{N}$, we set $[a]^V := \{x \in {}^*\mathbb{N} : a \sim_V x\}$. We also set $\varphi_V : {}^*\mathbb{N} \to {}^*\mathbb{N}/\sim_V$ to denote the quotient map, that is, $\varphi_V(a) := [a]^V$. If $V = \mathbb{N}$, we simply write φ instead of φ_V.

The proof of the following proposition is straightforward.

Proposition 2.18. *Fix $a \in {}^*\mathbb{N}$. Then:*

(1) *If $x, y \in [a]^V$ and $x < y$, then $[x, y] \subseteq [a]^V$.*
(2) $[a]^V = \bigcup_{x \in V} \left[\lfloor ax^{-1} \rfloor, ax\right]$.

It is straightforward to show that, if $[a]^V = [a']^V$ and $[b]^V = [b']^V$, then $[ab]^V = [a'b']^V$. (For instance, use that equality modulo an additive cut is a congruence relation with respect to addition on ${}^*\mathbb{N}$.) This allows us to set, for $a, b \in {}^*\mathbb{N}$, $[a]^V \cdot [b]^V := [ab]^V$. It is worth noting that this multiplication on equivalence classes satisfies cancellation: if $[a]^V \cdot [b]^V = [a]^V \cdot [c]^V$, then $[b]^V = [c]^V$.

We can also order equivalence classes by setting $[a]^V < [b]^V$ if and only if $a < b$ and $a \not\sim_V b$.

Proposition 2.19. $({}^*\mathbb{N}/\sim_V, <)$ *is a dense linear order.*

Proof. Suppose that $[a]^V < [b]^V$. Let $c := \lfloor \sqrt{ab} \rfloor$. It is readily verified (using that V is a multiplicative cut) that $[a]^V < [c]^V < [b]^V$. □

For any $k \in {}^*\mathbb{N}$, we set $\mathcal{H}_{k,N,V} := \varphi_V([k, Nk])$. Once again, to simplify notation, if $V = \mathbb{N}$, we simply drop the V and write $\mathcal{H}_{k,N}$ instead of $\mathcal{H}_{k,N,V}$. We will often abuse notation and write $\varphi_V : [k, Nk] \to \mathcal{H}_{k,N,V}$, that is, we will let φ_V also denote its restriction to $[k, Nk]$. Following traditional verbiage from the nonstandard analysis literature, one may refer to elements of $\mathcal{H}_{k,N,V}$ as *monads* (thus explaining the title of this article).

It is worth noting that if $a \in [k, Nk]$ is such that $ax > Nk$ or $\lfloor \frac{a}{x} \rfloor < k$ for some $x \in V$, then $[a]^V$ is not completely contained in $[k, Nk]$; for our purposes, the set of such exceptional a's will become negligible in a sense to be made precise shortly. In light of Proposition 2.11, the spaces $\mathcal{H}_{k,N,V}$ will prove important when studying Banach log density.

Remark 2.20. For each $a \in [k, Nk]$, set

$$\Phi(a) := \operatorname{st}\left(\frac{\ln a - \ln k}{\ln N}\right). \tag{2.3}$$

Then $\Phi : [k, Nk] \to [0, 1]$ is easily seen to be a surjection. Moreover, $\Phi(a) = \Phi(b)$ if and only if $a \sim_{V_N} b$, where V_N is defined as in (2.2). Hence, we obtain an order-preserving isomorphism $\Phi_\# : \mathcal{H}_{k,N,V_N} \to [0, 1]$ given by

$$\Phi_\#([a]^{V_N}) := \operatorname{st}\left(\frac{\ln a - \ln k}{\ln N}\right).$$

2.4. **Quotient measure spaces.** We are finally ready to describe the appropriate quotient of the above Loeb measure spaces. Once we have done this, we will show that certain natural operations on these measure spaces are measure-preserving (Theorems 2.23, 2.24, and 2.25 below).

Fix $N > \mathbb{N}$ and a multiplicative cut V contained in $[1, N]$. Via $\varphi : [k, Nk] \to \mathcal{H}_{k,N,V}$, the Loeb measure $\nu_{k,N}$ induces a measure $\mathfrak{m} = \mathfrak{m}_{k,N,V}$ on $\mathcal{H}_{k,N,V}$. More precisely, a set $E \subseteq \mathcal{H}_{k,N,V}$ is $\mathfrak{m}_{k,N,V}$-measurable if and only if $\varphi^{-1}(E)$ is $\nu_{k,N}$-measurable, in which case we set

$$\mathfrak{m}_{k,N,V}(E) := \nu_{k,N}(\varphi^{-1}(E)). \tag{2.4}$$

Of course, $\mathfrak{m}_{k,N,V}$ is a probability measure on $\mathcal{H}_{k,N,V}$. Since Loeb measures are complete, it follows that $\mathfrak{m}_{k,N,V}$ is also complete. As before, if $V = \mathbb{N}$, then we write $\mathfrak{m}_{k,N}$ instead of $\mathfrak{m}_{k,N,\mathbb{N}}$.

Example 2.21. *If $V = V_N$, then the order-preserving isomorphism $\Phi_\# : \mathcal{H}_{k,N,V_N} \to [0, 1]$ is also an isomorphism of measure spaces, where $[0, 1]$ is equipped with the usual Lebesgue measure.*

Proposition 2.22. *Suppose that $A \subseteq [k, Nk]$ is internal. Then $\varphi(A)$ is $\mathfrak{m}$-measurable.*

Proof. The proof is identical to that of [8, Proposition 6.3]. □

Recall that if $(X, \mathcal{B}, \mu)$ and $(Y, \mathcal{C}, \nu)$ are probability spaces, then $T : X \to Y$ is said to be *measure-preserving* if T is measurable and $\mu(T^{-1}(A)) = \nu(A)$ for all $A \in \mathcal{C}$. If, additionally, there is a measure-preserving map $U : Y \to X$ that is almost-everywhere an inverse to T, then we say that T is an *invertible measure-preserving* map; since any such U is unique up to a ν-null set, we may refer to it as T^{-1} and speak of "the" inverse to T.

Given $x := [a]^V$, we can define a map $T_x : \mathcal{H}_{k,N,V} \to \mathcal{H}_{ka,N,V}$ by $T_x(e) := xe$. The following two theorems are the main reason for considering quotient measure spaces.

Theorem 2.23. *For any $x := [a]^V$, we have $T_x : \mathcal{H}_{k,N,V} \to \mathcal{H}_{ka,N,V}$ is an invertible measure-preserving map.*

Proof. We will only show: if $E \subseteq \mathcal{H}_{k,N,V}$ is $\mathfrak{m}_{k,N,V}$-measurable, then $T_x(E)$ is $\mathfrak{m}_{ka,N,V}$-measurable and $\mathfrak{m}_{ka,N,V}(T_x(E)) = \mathfrak{m}_{k,N,V}(E)$. To finish the proof of the proposition, one would need to show that T_x is measurable and measure-preserving; the proof of this fact is similar to what we will actually show but is a bit messier.

Without loss of generality, we may suppose that $a \in {}^*\mathbb{N} \setminus \mathbb{N}$. Indeed, if $a \in \mathbb{N}$, then T_x is "essentially" the identity map on $\mathcal{H}_{k,N,V}$; see the discussion following the proof of the current proposition.

Without loss of generality, we may also assume that $\varphi_V^{-1}(E) \subseteq {}^*\mathbb{N} \setminus \mathbb{N}$. Fix (standard) $\epsilon > 0$. Since $\varphi_V^{-1}(E)$ is Loeb measurable, we can find internal sets $C, D \subseteq [k, Nk]$ with $C \subseteq \varphi_V^{-1}(E) \subseteq D$ and with $\nu_{k,N}(D \setminus C) < \epsilon$. Without loss of generality, we may assume that $D \subseteq {}^*\mathbb{N} \setminus \mathbb{N}$ and that both C and D have big components. Indeed, we can arrange that D has big components by deleting from D all of the components that are not big; note that the remaining set is internal and still contains $\varphi_V^{-1}(E)$. We can arrange that C has big components by prolonging each connected component to three times the right endpoint (and merging intervals where necessary); the resulting set is still internal, is still contained in $\varphi_V^{-1}(E)$, and is readily verified to have big components.

Decompose $C = \bigsqcup_{i\in I}[a_i, b_i]$ and $D = \bigsqcup_{j\in J}[c_j, d_j]$ into their connected components. Set $F := \bigsqcup_{i\in I}[aa_i, ab_i]$ and $G := \bigsqcup_{j\in J}[ac_j, ad_j]$.

Claim: $F \subseteq \varphi_V^{-1}(T_x(E)) \subseteq G$.
Proof of Claim: First suppose that $p \in F$. Fix $l \in [a_i, b_i]$ such that $al \leq p \leq a(l+1)$. Since $l \sim_V l+1$, we have $al \sim_V a(l+1)$, whence $[p]^V = [al]^V \in T_x(E)$. Now suppose that $p \in \varphi_V^{-1}(T_x(E))$, say $[p]^V = [a]^V \cdot [d]^V$ with $[d]^V \in E$. Take $y \in V$ such that $p \in [\lfloor \frac{ad}{y} \rfloor, ady]$. Since $a\lfloor \frac{d}{y} \rfloor \leq \lfloor \frac{ad}{y} \rfloor$, we have $p \in [a\lfloor \frac{d}{y} \rfloor, ady]$. Write $p = ak + r$ with $k \in [\lfloor \frac{d}{y} \rfloor, dy]$ and $0 \leq r < a$. Since $[\lfloor \frac{d}{y} \rfloor, dy] \subseteq \varphi_V^{-1}(E)$, we have $k \in [c_j, d_j]$ for some $j \in J$. Note that $d_j \notin \varphi_V^{-1}(E)$ as then $d_j + 1 \in \varphi_V^{-1}(E) \subseteq D$, a contradiction. Thus $p = ak + r \leq a(d_j - 1) + a = ad_j$, whence $p \in [ac_j, ad_j]$. This completes the proof of the claim.

Since F has big components and is contained in ${}^*\mathbb{N} \setminus \mathbb{N}$, by Lemma 2.14 we have that

$$\nu_{ka,N}(F) \approx \frac{1}{\ln N} \sum_{i\in I} (\ln(ab_i) - \ln(aa_i)) = \frac{1}{\ln N} (\ln(b_i) - \ln(a_i)) \approx \nu_{k,N}(C).$$

We conclude that $\nu_{ka,N}(G) = \nu_{k,N}(D)$. For the same reason, we have that $\nu_{ka,N}(G) = \nu_{k,N}(D)$.

It follows that $\nu_{ka,N}(G \setminus F) < \epsilon$. Since $\epsilon > 0$ was arbitrary, this shows that $\varphi_V^{-1}(T_x(E))$ is Loeb measurable. Moreover,

$$|\nu_{ka,N}(\varphi_V^{-1}(T_x(E))) - \nu_{k,N}(\varphi_V^{-1}(E))| \leq |\nu_{ka,N}(G) - \nu_{k,N}(D)| + 2\epsilon = 2\epsilon;$$

since $\epsilon > 0$ is arbitrary, we have $\nu_{ka,N}(\varphi^{-1}(T_x(E))) = \nu_{k,N}(\varphi^{-1}(E))$, that is, $\mathfrak{m}_{ka,N,V}(T_x(E)) = \mathfrak{m}_{k,N,V}(E)$. $\square$

Now suppose that $x := [a]^V$ is such that $a < V_N$, where V_N is as in Equation (2.2). Then the "inclusion" mapping $[k, Nk] \to [ka, Nka]$ (which is technically only defined on a ν-conull set) induces an invertible measure-preserving transformation $\mathcal{H}_{k,N,V} \to \mathcal{H}_{ka,N,V}$. In this way, we can identify the measure spaces $\mathcal{H}_{k,N,V}$ and $\mathcal{H}_{ka,N,V}$. Combining this identification and Theorem 2.23, we obtain the following:

Theorem 2.24. *For $x := [a]^V$ with $a < V_N$, the map $T_x : \mathcal{H}_{k,N,V} \to \mathcal{H}_{k,N,V}$ is an invertible measure-preserving transformation.*

For $u \in [1, N]$, set $u^{-1} := \lfloor \frac{N}{u} \rfloor$. Of course, this notion depends on N and occasionally we will want to make this dependence explicit, in which case we write $u^{-1,N}$.

The final goal of this subsection is to prove the following:

Theorem 2.25. *The map $\Upsilon = \Upsilon_{N,V} : \mathcal{H}_{1,N,V} \to \mathcal{H}_{1,N,V}$ given by $\Upsilon(\varphi_V(u)) := \varphi_V(u^{-1})$ is well-defined. Moreover, Υ is an invertible measure-preserving transformation satisfying $\Upsilon^{-1} = \Upsilon$.*

We break the proof of Theorem 2.25 up into a series of lemmas. We first prove that Υ is well-defined.

Lemma 2.26. *Suppose that $u, v \in [1, \frac{N}{2}]$ satisfy $u \sim_V v$. Then $u^{-1} \sim_V v^{-1}$.*

Proof. Without loss of generality, $u \leq v$. We must show that $\lfloor \frac{u^{-1}}{v^{-1}} \rfloor \in V$. Write $u^{-1} := \frac{N}{u} - \epsilon$ and $v^{-1} := \frac{N}{v} - \delta$, where $\epsilon, \delta \in [0, 1)$. Then:

$$\frac{u^{-1}}{v^{-1}} = \frac{N - \epsilon u}{N - \delta v} \cdot \frac{v}{u} \leq \frac{N}{N - v} \cdot \frac{v}{u} \leq 2 \cdot \frac{v}{u}.$$

□

We next prove that Υ is an involution.

Lemma 2.27. *Suppose that $x \in [1, \frac{N}{2}]$. Then $x \sim_V (x^{-1})^{-1}$.*

Proof. Since $x^{-1} \leq \frac{N}{x}$, we have $x \leq \frac{N}{x^{-1}}$, so $x \leq (x^{-1})^{-1}$. Write $(x^{-1})^{-1} = \frac{N}{x^{-1}} - \delta_1$ and $x^{-1} = \frac{N}{x} - \delta_2$, with $\delta_1, \delta_2 \in [0, 1)$. We then have:

$$\frac{(x^{-1})^{-1}}{x} = \frac{\frac{N}{\frac{N}{x} - \delta_2} - \delta_1}{x} = \frac{Nx - \delta_1 N + \delta_1 \delta_2 x}{x(N - \delta_2 x)} \leq \frac{N}{N - x} \leq 2.$$

□

Suppose that $A \subseteq [1, N]$ is internal and its decomposition into components is $A = \bigsqcup_{i \in I} [a_i, b_i]$. We say that A has *separated components* if, whenever $[a_i, b_i]$ and $[a_j, b_j]$ are adjacent components with $a_j > b_i$, we have $a_j > 2b_i$.

Lemma 2.28. *Suppose that A has separated components and is contained in $\bigcap_{k \in \mathbb{N}} [1, \frac{N}{k})$. Then, for any distinct $i, j \in I$, we have $[b_i^{-1}, a_i^{-1}] \cap [b_j^{-1}, a_j^{-1}] = \emptyset$.*

Proof. Without loss of generality, assume that $2b_i < a_j$. Suppose that $b_i^{-1} \leq x \leq a_i^{-1}$. Then $\frac{N}{b_i} - \epsilon \leq x \leq \frac{N}{a_i}$ for some $\epsilon \in [0, 1)$. We then have $a_i \leq \frac{N}{x} \leq \frac{N b_i}{N - b_i}$, so $a_i \leq x^{-1} \leq 2b_i$ since $\frac{b_i}{N} \approx 0$. If $b_j^{-1} \leq x \leq a_j^{-1}$, then we would have $a_j \leq x^{-1}$, contradicting $2b_i < a_j$. □

For internal $A \subseteq [1, N]$ with decomposition $A = \bigsqcup_{i \in I} [a_i, b_i]$, we set $A^{-1} = \bigsqcup_{i \in I} [b_i^{-1}, a_i^{-1}]$. If A has separated components and is contained in $\bigcap_{k \in \mathbb{N}} [1, \frac{N}{k})$, the preceding lemma tells us this definition of A^{-1} is also its decomposition into components.

Lemma 2.29. *Suppose that $A \subseteq [1, N]$ is internal, has big and separated components, and is contained in $({}^*\mathbb{N} \setminus \mathbb{N}) \cap \bigcap_{k \in \mathbb{N}} [1, \frac{N}{k})$. Then A^{-1} has big components and $\nu(A) = \nu(A^{-1})$.*

Proof. In order to show that A^{-1} has big components, it suffices to show that if $[a, b]$ is big and $\frac{b}{N}$ is infinitesimal, then $[b^{-1}, a^{-1}]$ is also big. Write $a^{-1} = \frac{N}{a} - \epsilon$ and $b^{-1} = \frac{N}{b} - \delta$. Then:

$$\frac{a^{-1}}{b^{-1}} = \frac{b}{a} \cdot \frac{N - \epsilon a}{N - \delta b} > \frac{b}{a} \cdot (1 - \frac{a}{N}).$$

The quantity on the right hand side of the display is appreciably larger than 2 since $\frac{b}{a}$ is appreciably larger than 2 and $\frac{a}{N}$ is infinitesimal.

We now must show that $\nu(A)=\nu(A^{-1})$. Decompose $A=\bigsqcup_{i\in I}[a_i,b_i]$ into its components; then $[b_i^{-1},a_i^{-1}]$ are the components of A^{-1}. By Lemma 2.14 (which applies to A^{-1} since $A\subseteq\bigcap_{k\in\mathbb{N}}[1,\frac{N}{k}))$, we know that

$$\nu(A)\approx\frac{1}{\ln N}\sum_{i\in I}(\ln(b_i)-\ln(a_i))$$

and

$$\nu(A^{-1})\approx\frac{1}{\ln N}\sum_{i\in I}(\ln(a_i^{-1})-\ln(b_i^{-1})).$$

For simplicity, set $\alpha_i:=\ln(b_i)-\ln(a_i)$ and $\beta_i:=\ln(a_i^{-1})-\ln(b_i^{-1})$. Fix $i\in I$ and write $a_i^{-1}=\frac{N}{a_i}-\epsilon$ and $b_i^{-1}=\frac{N}{b_i}-\delta$. Then $|\alpha_i-\beta_i|=|\ln(\frac{N-\epsilon a_i}{N-\delta b_i})|\approx 0$. Since A has big components, it follows that $\frac{|\alpha_i-\beta_i|}{\alpha_i}\approx 0$. It follows that

$$|\frac{\sum_{i\in I}\alpha_i}{\ln N}-\frac{\sum_{i\in I}\beta_i}{\ln N}|\leq\frac{\sum_{i\in I}|\alpha_i-\beta_i|}{\ln N}\leq\frac{\sum_{i\in I}|\alpha_i-\beta_i|}{\sum_{i\in I}\alpha_i}\approx 0.$$

Putting everything together, we get $\nu(A)=\nu(A^{-1})$. □

Lemma 2.30. *Suppose that $E\subseteq\mathcal{H}_{1,N,V}$ is $\mathfrak{m}_{1,N,V}$-measurable. Then $\Upsilon(E)$ is $\mathfrak{m}_{1,N,V}$-measurable and $\mathfrak{m}_{1,N,V}(\Upsilon(E))=\mathfrak{m}_{1,N,V}(E)$*

Proof. Fix $M\in{}^*\mathbb{N}\setminus\mathbb{N}$ such that $\nu([M,N/M])=1$. Without loss of generality, we may assume that $\varphi_V^{-1}(E)\subseteq[M,N/M]$. Fix $\epsilon>0$ and take internal sets $C\subseteq\varphi_V^{-1}(E)\subseteq D$ with $\nu_{1,N}(D\setminus C)<\epsilon$. Without loss of generality, $D\subseteq[M,N/M]$. Moreover, arguing as in the proof of Theorem 2.23, we may assume that both C and D have big and separated components. Decompose $C=\bigsqcup_{i\in I}[a_i,b_i]$ and $D=\bigsqcup_{j\in J}[c_j,d_j]$ into their measurable components.

Claim: $C^{-1}\subseteq\varphi_V^{-1}(\Upsilon(E))\subseteq D^{-1}$.
Proof of Claim: First suppose that $x\in[b_i^{-1},a_i^{-1}]$. Write $b^{-1}=\frac{N}{b}-\delta$ for some $\delta\in[0,1)$. Then

$$a\leq\frac{N}{x}\leq\frac{Nb}{N-\delta b}\leq\frac{Nb}{N-b}\leq 2b.$$

Since $b\sim_V 2b$, we have $\varphi_V(x^{-1})\in E$. Since $a_i\geq 2$, we have $x\leq a_i^{-1}\leq\frac{N}{2}$, so $x\sim_V(x^{-1})^{-1}\in\varphi_V^{-1}(\Upsilon(E))$ and thus $x\in\varphi_V^{-1}(\Upsilon(E))$. Now suppose that $x\in\varphi_V^{-1}(\Upsilon(E))$. Then $x\sim_V u^{-1}$ for some $u\in\varphi_V^{-1}(E)$. Choose $j\in J$ such that $u\in[c_j,d_j]$. Since $d_j+1\notin D$, we cannot have $u\sim_V d_j$. Now since $u,x^{-1}\in[1,\frac{N}{2}]$, we have $u\sim_V(u^{-1})^{-1}\sim_V x^{-1}$, whence $x^{-1}\leq d_j$. Note that $x^{-1}<d_j$, else we contradict $d_j+1\notin D$. It follows that $\frac{N}{x}\leq d_j$, so $\frac{N}{d_j}\leq x$, whence $d_j^{-1}\leq x$. Similarly, $u\not\sim_V c_j$, so $c_j\leq x^{-1}\leq\frac{N}{x}$. It follows that $x\leq\frac{N}{c_j}$, so $x\leq c_j^{-1}$. This completes the proof of the claim.

By Lemma 2.29, we have that $\nu(C^{-1})=\nu(C)$ and $\nu(D^{-1})=\nu(D)$. Once again, it follows that $\varphi_V^{-1}(\Upsilon(E))$ is measurable and has the same measure as E. □

Note that Lemmas 2.26, 2.27, and 2.30 together establish Theorem 2.25.

3. Geo-arithmetic progressions

In this short section, we indicate how our results from the previous section can be used to obtain approximate geometric structure in sets of positive Banach log density. As mentioned in the introduction, in an upcoming paper we show how stronger results can be deduced from Szemeredi's theorem and a logarithmic change of coordinates.

Let $x, a \in {}^*\mathbb{N}$. If $n \in \mathbb{N}$, we say that x is an *n-approximation* of a if $x/n < a < xn$. If every element $x \in X$ is an n-approximation of some $a \in A$, we say that X is an *n-approximate subset* of A.

For the convenience of the reader, we recall:

Fact 3.1 (Furstenberg's Recurrence Theorem). *Let $T : X \to X$ be a measure-preserving transformation on the probability space $(X, \mathcal{B}, \mu)$. Further suppose that $A \in \mathcal{B}$ satisfies $\mu(A) > 0$ and $l \in \mathbb{N}$ is given. Then there exists $n \in \mathbb{N}$ such that*

$$\mu(A \cap T^{-n}(A) \cap T^{-2n}(A) \cap \cdots \cap T^{-ln}(A)) > 0.$$

Theorem 3.2. *Let $A \subseteq \mathbb{N}$ be such that $\ell\mathrm{BD}(A) > 0$ and fix $l \in \mathbb{N}$. Then there exists $n \in \mathbb{N}$ such that, for any $m \in \mathbb{N}$, there exists a geometric progression $G = \{ar^i : i = 0, 1, \ldots, l-1\}$ with $a, r > m$ such that G is an n-approximate subset of A.*

Proof. Set $\alpha := \ell\mathrm{BD}(A)$. Take $k, N \in {}^*\mathbb{N}$ with $N > \mathbb{N}$ such that $\alpha = \nu_{k,N}({}^*A \cap [k, Nk])$. Let $E = \varphi({}^*A \cap [k, Nk]) \subseteq \mathcal{H}_{k,N}$. By Proposition 2.22, we have that E is $\mathfrak{m}_{k,N}$-measurable and that $\mathfrak{m}_{k,N}(E) \geqslant \alpha$. Fix $s \in {}^*\mathbb{N}$ with $\mathbb{N} < s < V_N$ and set $x := [s]^{\mathbb{N}}$. By Furstenberg's Recurrence Theorem applied to the transformation T_x on $\mathcal{H}_{k,N}$ (which is applicable by Theorem 2.24), we see that E contains a geometric progression $\{cq^i : i = 1, 2, \ldots, l\}$; here, $q = x^k$ for some $k \in \mathbb{N}$. Let $r := s^k$. Choose any $a \in \varphi^{-1}(cq)$. Then $a > \mathbb{N}$ and $\varphi(ar^{i-1}) = cq^i$. Let $n_i = \min\{j \in \mathbb{N} : [\lfloor \frac{ar^i}{j} \rfloor, ar^i j] \cap {}^*A \neq \emptyset\}$. Set $n = \max\{n_i : i = 0, 1, \ldots, l-1\}$. We now conclude that there exists an l-term geometric progression in $[k, Nk]$ with infinite ratio and infinite initial element such that every term in the progression is an n-approximation of some element in ${}^*A \cap [k, Nk]$. The theorem follows by the transfer principle. $\square$

We give two examples to show the necessity of some of the statements in the previous theorem. First, we show that we can only expect to get approximate arithmetic progressions in general.

Example 3.3. *Let A be the set of all square-free numbers. Then by Fact 2.5 we have $\ell\mathrm{BD}(A) \geqslant \overline{ld}(A) \geqslant \underline{d}(A) > 0$ but A does not contain any 3-term geometric progression.*

The next example shows that we really do need the Banach log density to be positive.

Example 3.4. *Let $\alpha < 1$. Fix a j such that $(j-1)/j > \alpha$. Let $u_0 = 2$, $u_{i+1} > (ju_i)^3$, and set*

$$A = \bigcup_{i=1}^{\infty} [u_i, ju_i].$$

Then:

(1) $\overline{d}(A) > \alpha$;

(2) $\ell\mathrm{BD}(A)=0$*; and*
(3) *for any* $n\in\mathbb{N}$*, there exists an* $m\in\mathbb{N}$ *such that there does not exist* 3*-term geometric progression* $G=\{a,ar,ar^2\}$ *with* $a,r>m$ *and* G *is an* n*-approximate subset of* A*.*

Proof. The verification of (1) is straightforward and left to the reader. For (2), fix $N>\mathbb{N}$ and $k\in{}^*\mathbb{N}$; it suffices to prove that

$$\frac{1}{\log N}\left(\sum_{x\in^* A\cap[k,Nk]}\frac{1}{x}\right)\approx 0.$$

Let $[u_{i+t},ju_{i+t}]$ for $t=0,1,\ldots,p-1$ enumerate those intervals completely contained in $[k,Nk]$. Note then that

$$\frac{1}{\log N}\left(\sum_{x\in^* A\cap[k,Nk]}\frac{1}{x}\right)\approx\frac{1}{\log N}\left(\sum_{t=0}^{p-1}\sum_{x\in^* A\cap[u_{i+t},ju_{i+t}]}\frac{1}{x}\right)\approx\frac{p\log j}{\log N}.$$

Since $u_{i+p-1}>u_i^{3^{p-1}}$ and $u_{i+p-1}/u_i\leq N$, we see that $3^{p-1}-1\leq\log N$, whence there is $C\in\mathbb{R}$ such that $p\leq C\log\log N$. It follows that

$$\frac{p\log j}{\log N}\leq\frac{C\log j\log\log N}{\log N}\approx 0,$$

finishing the proof of (2).

We now prove (3). Let $m=n^3j$. Let $a,r>m$ and $G=\{a,ar,ar^2\}$ be a 3-term geometric progression such that $u_{i_1}/n\leqslant a\leqslant ju_{i_1}n$ and $u_{i_2}/n\leqslant ar\leqslant ju_{i_2}n$. We show that ar^2 cannot be n-approximated by any element of A.

If $i_1=i_2$, then we get $\frac{u_{i_1}}{n}\leq a,ar\leq ju_{i_1}n$, whence $r\leq n^2j=m$, a contradiction. So we can assume that $i_2>i_1$. Then $\frac{u_{i_2}}{ju_{i_1}n^2}\leqslant r\leqslant\frac{ju_{i_2}n^2}{u_{i_1}}$.

Claim 1: $ju_{i_2}n\leq ar^2$.

Proof of Claim 1: Since $ju_{i_1}n>a>m=jn^3$, we have $u_{i_1}>n^2$. Hence $u_{i_2}>(ju_{i_1})^3>j^2u_{i_1}u_{i_1}^2>j^2u_{i_1}n^4$. Hence $ju_{i_2}n<u_{i_2}\frac{u_{i_2}}{ju_{i_1}n^3}$. Since $ar\geq u_{i_2}/n$ and $r\geq u_{i_2}/(ju_{i_1}n^2)$, we have that $ar^2\geq u_{i_2}^2/(ju_{i_1}n^3)$. This proves Claim 1.

Claim 2: $ar^2\leq u_{i_2+1}/n$.

Proof of Claim 2: Since $ar\leq ju_{i_2}n$ and $r\leq ju_{i_2}n^2/u_{i_1}$, we have that $ar^2\leq ju_{i_2}ju_{i_2}n^3/u_{i_1}$. Since $(ju_{i_2})^3<u_{i_2+1}$, we have that $ju_{i_2}\frac{ju_{i_2}n^3}{u_{i_1}}<u_{i_2+1}\frac{n^3}{ju_{i_2}u_{i_1}}$. Since $ju_{i_2}u_{i_1}>u_{i_1}^4>n^4$, we have that $u_{i_2+1}\frac{n^3}{ju_{i_2}u_{i_1}}<u_{i_2+1}/n$. This proves Claim 2.

From these two claims, it follows that ar^2 is not n-approximated by any element of A, whence G is not an n-approximate subset of A. □

4. Other densities

In this section, we introduce a family of densities on subsets of $\mathbb{N}$ for which the corresponding sets of positive measure in the quotient space contain arbitrarily long powers of arithmetic progressions. Since the properties of these densities have proofs analogous to the case of logarithmic density, we allow ourselves to just state the main definitions and results and omit almost all proofs.

The following family of densities might look a bit strange at first, but Proposition 4.6 below helps explain the definition.

Definition 4.1. For any positive integer m and any set $A \subseteq \mathbb{N}$ let

$$\mathrm{BD}_m(A) := \lim_{n\to\infty} \sup_{k\in\mathbb{N}} \frac{1}{mn} \sum_{x\in A\cap[k,(\lceil\sqrt[m]{k}\rceil+n)^m]} \frac{1}{x^{\frac{m-1}{m}}}.$$

Clearly, $\mathrm{BD}_1(A) = \mathrm{BD}(A)$.

Definition 4.2. Fix $m \in \mathbb{N}$, $N \in {}^*\mathbb{N} \setminus \mathbb{N}$, and $k \in {}^*\mathbb{N}$. Let $U \subseteq [1, N]$ be an additive cut (for example, $U = \mathbb{N}$). Let

$$I_{k,N,m} := [k, (\lceil\sqrt[m]{k}\rceil + N)^m].$$

For any $a, b \in I_{k,N,m}$, set $a \sim b$ if $|\sqrt[m]{a} - \sqrt[m]{b}| < u$ for some $u \in U$. Let

$$[a]_m := \{x \in I_{k,N,m} : x \sim a\}.$$

Clearly, if $x, y \in [a]$ and $x < y$, then $[x, y] \subseteq [a]$.

Proposition 4.3. *The relation $\sim$ is an equivalence relation.*

The monad $[a]$ is the set $(\lceil\sqrt[m]{a}\rceil \pm U)^m$ where

$$(\lceil\sqrt[m]{a}\rceil \pm U)^m := \left(\bigcup_{u\in U} \left[(\lceil\sqrt[m]{a}\rceil - u)^m, (\lceil\sqrt[m]{a}\rceil + u)^m\right]\right) \cap I_{k,N,m}.$$

Definition 4.4. Let m, N, k, U be the same as in Definition 4.2. Let

$$\mathcal{G}_{k,N,m} = \{[a] : a \in I_{k,N,m}\}.$$

Let $\varphi(a) = [a]$ be the quotient map from $I_{k,N,m}$ to $\mathcal{G}_{k,N,m}$.

For each internal set $A \subseteq [k, (\lceil\sqrt[m]{k}\rceil + N)^m]$, we set

$$\nu(A) := \mathrm{st}\left(\frac{1}{mN}\sum_{a\in A}\frac{1}{a^{\frac{m-1}{m}}}\right).$$

As before, we can extend ν to the σ-algebra generated by the internal sets.

Proposition 4.5. *Let $A \subseteq \mathbb{N}$ and $\alpha > 0$. Then $\mathrm{BD}_m(A) \geq \alpha$ if and only if there exists an $I_{k,N,m}$ such that $\nu({}^*A \cap I_{k,N,m}) \geq \alpha$.*

Proposition 4.6. *Let $[a, b] \subseteq I_{k,N,m}$. Then*

$$\nu([a, b]) = \mathrm{st}\left(\frac{\sqrt[m]{b} - \sqrt[m]{a}}{N}\right).$$

Furthermore, if $c \in {}^*\mathbb{N}$ is such that $(\lceil\sqrt[m]{b}\rceil + c)^m \in I_{k,N,m}$, then

$$\nu([(\lceil\sqrt[m]{a}\rceil + c)^m, (\lceil\sqrt[m]{b}\rceil + c)^m]) = \nu([a, b]).$$

Definition 4.7. For each set $E \subseteq \mathcal{G}_{k,N,m}$, we say that E is $\mathfrak{m}$-measurable if $\varphi^{-1}(E)$ is Loeb measurable, in which case we define the measure

$$\mathfrak{m}(E) = \nu(\varphi^{-1}(E)).$$

Theorem 4.8. *Let U_N denote the largest additive cut in $[1,N]$ and fix $U < c < U_N$. For each $[a] \in \mathcal{G}_{k,N,m}$ set*

$$T_c([a]) := [(\lceil \sqrt[m]{a} \rceil + c)^m].$$

Then T_c is an $\mathfrak{m}$-measure preserving transformation on $\mathcal{G}_{k,N,m}$.

Note that if $\mathfrak{m}(E) > 0$, then E contains arbitrarily long sequences of the form $[a], [(\lceil \sqrt[m]{a} \rceil + d)^m], [(\lceil \sqrt[m]{a} \rceil + 2d)^m], \ldots, [(\lceil \sqrt[m]{a} \rceil + ld)^m]$, i.e., E contains arbitrarily long m-th powers of arithmetic progressions. Thus, using the techniques of the previous section, if $A \subseteq \mathbb{N}$ satisfies $\mathrm{BD}_m(A) > 0$, then in A we can find approximations to arbitrarily long sequences of m-th powers of arithmetic progressions.

5. Lebesgue Density Theorem

In this section, we prove that a natural version of the Lebesgue density theorem holds for our quotient measure spaces. For the rest of this section, fix $N > \mathbb{N}$ and a multiplicative cut V contained in $[1,N]$. Suppose that $A \subseteq [k, Nk]$ is internal and set $X := \varphi_V(A)$. For $x \in \mathcal{H}_{k,N,V}$ and $r > V$, we write $\mathfrak{m}_{x,r,V}(X)$ to denote $\mathfrak{m}_{b,r,V}(X \cap [x, \varphi(r)x])$ for any $b \in \varphi_V^{-1}(\{x\})$; since $V \subseteq V_N$, we see, by the discussion preceding Theorem 2.24, that the definition of $\mathfrak{m}_{x,r,V}$ is independent of the choice of representative of $\varphi_V^{-1}(\{x\})$. We then set

$$\delta_+(x, X) = \liminf_{r > V} \mathfrak{m}_{x,r,V}(X),$$

or, equivalently, to clarify the meaning of lim inf in this setting:

$$\delta_+(x, X) = \sup_{s > V} \inf_{V < r < s} \mathfrak{m}_{x,r,V}(X).$$

One can define the notion of $\delta_-(x, X)$ in an analogous fashion. We say that $x \in \mathcal{H}_{k,N,V}$ is a *Lebesgue density point of X* if $\delta_+(x, X) = \delta_-(x, X) = 1$.

Here is the version of the Lebesgue Density Theorem in our setting. We model our proof after a proof of the classical Lebesgue density theorem given by Faure in [9].

Theorem 5.1. *Let A be an internal subset of $[k, Nk]$ and $X = \varphi_V(A)$. Then $\mathfrak{m}_{k,N,V}$-almost every point in X is a Lebesgue density point.*

Proof. We only show that almost every point x of X satisfies $\delta_+(x, X) = 1$; the proof for δ_- is exactly the same. Fix n and set $X_n := \{x \in X \ : \ \delta_+(x, X) < \frac{n}{n+1}\}$. It suffices to show that $\mathfrak{m}^*_{k,N,V}(X_n) = 0$. (Here, $\mathfrak{m}^*_{k,N,V}$ denotes the outer measure.) Fix $\epsilon > 0$. Take internal sets $C \subseteq D \subseteq [k, Nk]$ such that $C \subseteq \varphi_V^{-1}(X) \subseteq D \subseteq {}^*\mathbb{N} \setminus \mathbb{N}$ and $\nu(D \setminus C) < \epsilon$. (In this proof, we write ν for $\nu_{k,N}$.) Fix $D' \subseteq D$ internal such that $\varphi_V^{-1}(X_n) \subseteq D'$ and such that $\nu(D') < \nu^*(\varphi_V^{-1}(X_n)) + \epsilon$. We now set

$$C' := \{a \in C \ : \ (\exists b \geq 2)([a, ba] \subseteq D' \text{ and } \frac{1}{\ln b} \sum_{x \in C \cap [a, ba]} \frac{1}{x} < \frac{n+1}{n+2})\}.$$

Note that C' is internal and $C' \subseteq C \cap D'$.

We first claim that $\varphi_V^{-1}(X_n) \cap C \subseteq C'$. Fix $a \in \varphi_V^{-1}(X_n) \cap C$. Since $[a]^V \subseteq \varphi_V^{-1}(X_n) \subseteq D'$, there is $c > V$ such that $[a, ca] \subseteq D'$. Since $\delta_+(\varphi_V(a), X) < \frac{n}{n+1}$, there is $V < b < c$ such that $\nu(\varphi_V^{-1}(X)) < \frac{n+1}{n+2}$. It follows that

$$\frac{1}{\ln b} \sum_{x \in C \cap [a, ba]} \frac{1}{x} \approx \nu_{a,b}(C) \leq \nu_{a,b}(\varphi_V^{-1}(X)) < \frac{n+1}{n+2},$$

whence we conclude that $a \in C'$.

Since $\varphi_V^{-1}(X_n) \subseteq C' \cup (D \setminus C)$, we get $\nu^*(\varphi_V^{-1}(X_n)) \leq \nu(C') + \epsilon$, so $\nu(D') - \nu(C') \leq \nu(D') - \nu^*(\varphi^{-1}(X_n)) + \epsilon < 2\epsilon$.

Without loss of generality, we may suppose that D' has big components. Decompose $D' := \bigsqcup_i [a_i, b_i]$ into its components. We now claim that $\frac{1}{\ln(b_i) - \ln(a_i)} \sum_{x \in C' \cap [a_i, b_i]} \frac{1}{x} \leq \frac{n+1}{n+2}$ for each i. Fix i and let $e_i \in [a_i+2, b_i+1]$ be maximal such that $\frac{1}{\ln(e_i - 1) - \ln(a_i)} \sum_{x \in C' \cap [a_i, e_i - 1]} \frac{1}{x} \leq \frac{n+1}{n+2}$. We want to show that $e_i = b_i + 1$. Suppose, towards a contradiction, that $e_i \leq b_i$. First suppose that $e_i \in C'$. Take $b \geq 2$ such that $[e_i, be_i] \subseteq D'$ and $\frac{1}{\ln b} \sum_{x \in C \cap [e_i, be_i]} \frac{1}{x} \leq \frac{n+1}{n+2}$. Then

$$\begin{aligned} \sum_{x \in C' \cap [a_i, be_i]} \frac{1}{x} &= \sum_{x \in C' \cap [a_i, e_i - 1]} \frac{1}{x} + \sum_{x \in C' \cap [e_i, be_i]} \frac{1}{x} \\ &\leq \frac{n+1}{n+2}((\ln(e_i - 1) - \ln(a_i)) + \ln b) \\ &\leq \frac{n+1}{n+2}(\ln(be_i) - \ln(a_i)). \end{aligned}$$

Since $[e_i, be_i] \subseteq D'$, we have $be_i \leq b_i$, so $be_i + 1 \leq b_i + 1$ contradicts the maximality of e_i. We now suppose that $e_i \notin C'$. Then

$$\begin{aligned} \sum_{x \in C' \cap [a_i, e_i]} \frac{1}{x} &= \sum_{x \in C' \cap [a_i, e_i - 1]} \frac{1}{x} \\ &\leq \frac{n+1}{n+2}(\ln(e_i - 1) - \ln(a_i)) \\ &\leq \frac{n+1}{n+2}(\ln(e_i) - \ln(a_i)). \end{aligned}$$

Thus $e_i + 1$ also works, contradicting the choice of e_i.

We now can calculate:

$$\begin{aligned} \nu(D') &\leq \nu(C') + 2\epsilon \\ &\approx \frac{1}{\ln N} \sum_{x \in C' \cap [k, Nk]} \frac{1}{x} + 2\epsilon \\ &= \frac{1}{\ln N} \sum_i \sum_{x \in C' \cap [a_i, b_i]} \frac{1}{x} + 2\epsilon \\ &\leq \frac{1}{\ln N} \sum_i \frac{n+1}{n+2}(\ln(b_i) - \ln(a_i)) + 2\epsilon \\ &\approx \frac{n+1}{n+2} \cdot \nu(D') + 2\epsilon. \end{aligned}$$

The last step used that D' has big components and is contained in ${}^*\mathbb{N} \setminus \mathbb{N}$.

We now conclude that $\nu^*(\varphi^{-1}(X_n)) \leq \nu(D') \leq 2(n+2)\epsilon$. Since ϵ was arbitrary (but n is fixed), we get that $\nu^*(\varphi^{-1}(X_n)) = 0$, so $\mathfrak{m}^*_{k,N,V}(X_n) = 0$, as desired. $\square$

6. Productset phenomenon

In this final section, we use the Lebesgue Density Theorem for multiplicative cuts to obtain a multiplicative analog of Jin's sumset result from [12]. First a lemma:

Lemma 6.1. *Suppose that A is an internal subset of $[j, Nj]$ and B is an internal subset of $[k, Nk]$. Set $X = \varphi_V(A)$ and $Y = \varphi_V(B)$. Suppose that $\mathfrak{m}_{j,N,V}(X) > 0$ and $\mathfrak{m}_{k,N,V}(Y) > 0$. Then XY contains a non-empty interval in $\mathcal{H}_{jk,N^2,V}$.*

Proof. Let $x \in X$ and $y \in Y$ be Lebesgue density points of X and Y respectively. Then there exists $r > V$ such that

$$\mathfrak{m}_{x,r,V}(X \cap [x, xr]) > \frac{2}{3}$$

and

$$\mathfrak{m}_{\frac{y}{r},r,V}(Y \cap [\frac{y}{r}, y]) > \frac{2}{3}.$$

Here, and in the rest of this proof, $\frac{y}{r}$ denotes $\varphi_V(\lfloor \frac{a}{r} \rfloor)$ for any $a \in \varphi_V^{-1}(\{y\})$. We now set

$$E_X := \{u \in \varphi_V([1, r]) : ux \in X\}$$

and

$$E_Y := \left\{v \in \varphi_V([1, r]) : \frac{y}{v} \in Y\right\}.$$

Note that

$$T_x(E_X) = X \cap [x, xr], \text{ and } T_{\frac{y}{r}}(\Upsilon_r(E_Y)) = Y \cap [r^{-1}y, y].$$

By Proposition 2.23 and Lemma 2.30, we have that $\mathfrak{m}_{1,r,V}(E_X) > 2/3$ and $\mathfrak{m}_{1,r,V}(E_Y) > 2/3$.

In order to finish the proof of the theorem, we show that $xys \in XY$ for any s satisfying $V < s < r^{1/3}$. Towards this end, consider the set $E'_X := \{u \in \varphi_V([1, r]) : usx \in X\}$. Then

$$E_X \cap [s, r] \subset T_s\left(E'_X \cap [1, \frac{r}{s}]\right)$$

so that

$$\begin{aligned}\mathfrak{m}_{1,r,V}(E'_X) &\geq \mathfrak{m}_{1,r,V}(T_s\left(E'_X \cap [1, \frac{r}{s}]\right) \\ &\geq \mathfrak{m}_{1,r,V}(E_X \cap [s, r]) \\ &> 2/3 - \mathfrak{m}_{1,r,V}([1, s]) \\ &> 1/3.\end{aligned}$$

Since $\mathfrak{m}_{1,r,V}(E'_X) + \mathfrak{m}_{1,r,V}(E_Y) > 1$, there exists $u_0 \in E'_X \cap E_Y$. Then u_0sx is in X and $\frac{y}{u_0}$ is in Y. Thus $sxy \in XY$, as desired. □

We now obtain a multiplicative analog of the main result of [12]:

Theorem 6.2. *Suppose that $A, B \subseteq \mathbb{N}$ satisfy $\ell\mathrm{BD}(A), \ell\mathrm{BD}(B) > 0$. Then there exists $m \in \mathbb{N}$ such that for all $n \in \mathbb{N}$, there is $x \in \mathbb{N}$ such that, for every $[u, mu] \subseteq [x, nx]$, we have $[u, mu] \cap (A \cdot B) \neq \emptyset$.*

Proof. We work with the cut $V = \mathbb{N}$. Fix $N > \mathbb{N}$; by Proposition 2.11, there exists $j, k \in {}^*\mathbb{N}$ such that $\nu_{j,N}({}^*A \cap [j, jN]) > 0$ and $\nu_{k,N}({}^*B \cap [k, kN]) > 0$. Let $X := {}^*A \cap [j, jN] \subseteq \mathcal{H}_{j,N}$ and $Y := {}^*B \cap [k, kN] \subseteq \mathcal{H}_{k,N}$. By Lemma 6.1, XY contains a nonempty interval in $\mathcal{H}_{jk,N^2}$, say $\varphi([a, b])$ with $\frac{b}{a} > \mathbb{N}$.

Let $\{c_i : i \leq M\}$ enumerate ${}^*(A \cdot B) \cap [a, b]$ in increasing order and let $m := \max_{i<M}\left\{\left\lceil \frac{c_{i+1}}{c_i} \right\rceil\right\}$. Then $m \in \mathbb{N}$, else X would not contain the entire interval $\varphi([a, b])$. We claim that this m is as desired. Indeed, given any $n \in \mathbb{N}$, we have $b \geq na$ and for any interval $[u, mu] \subseteq [a, na]$ we have $[u, mu] \cap {}^*(A \cdot B) \neq \emptyset$, whence we obtain the existence of the desired $x \in \mathbb{N}$ by transfer. □

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