THE THEORY OF TRACIAL VON NEUMANN ALGEBRAS DOES NOT HAVE A MODEL COMPANION

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Abstract. In this note, we show that the theory of tracial von Neumann algebras does not have a model companion. This will follow from the fact that the theory of any locally universal, McDuff II_1 factor does not have quantifier elimination. We also show how a positive solution to the Connes Embedding Problem implies that there can be no model-complete theory of II_1 factors.

1. Introduction

The model theoretic study of operator algebras is at a relatively young stage in its development (although many interesting results have already been proven, see [7], [8], [9]) and thus there are many foundational questions that need to be answered. In this note, we study the question that appears in the title: does the theory of tracial von Neumann algebras have a model companion? (Recall that a theory is said to be *model-complete* if every embedding between models of the theory is elementary and a model-complete theory T' is a *model companion* of a theory T if every model of T embeds into a model of T' and vice-versa.) We answer this question in the negative. Indeed, we prove that a locally universal, McDuff II_1 factor cannot have quantifier elimination. (See below for the definitions of *locally universal* and *McDuff*.) Since a model companion of the theory of tracial von Neumann algebras will have to be a model completion as well as the theory of a locally universal, McDuff II_1 factor, the result follows.

We then pose a weaker question: can there exist a model-complete theory of II_1 factors? Here, we show that a positive solution to the *Connes Embedding Problem* implies that the answer is once again negative.

Another motivation for this work came from considering independence relations in II_1 factors. Although all II_1 factors are unstable (see [7]), it is still possible that there are other reasonably well-behaved independence relations to consider. Indeed, the independence relation stemming from conditional expectation is a natural candidate. In the end of this note, we show how the failure of quantifier elimination seems to pose serious hurdles in showing that conditional expectation yields a strict independence relation in the sense of [1].

Goldbring's work was partially supported by NSF grant DMS-1007144.

We thank D. Shlyakhtenko for patiently explaining Brown's work when we posed the question to him of the existence of non-extendable embeddings of pairs $\mathcal{M} \subset \mathcal{N}$ into $\mathcal{R}^\omega$. (See the proof of Theorem 2.1 below.)

We recall that the class of tracial von Neumann algebras is universally axiomatizable in a suitable (continuous) signature, which we denote by $\mathcal{L}$ throughout this paper. (See [8, Proposition 3.3] for a list of universal axioms.)

Throughout, $\mathcal{R}$ denotes the hyperfinite II_1 factor. We recall that $\mathcal{R}$ embeds into any II_1 factor. We will say that a von Neumann algebra is $\mathcal{R}^\omega$-*embeddable* if it embeds into $\mathcal{R}^\mathcal{U}$ for some $\mathcal{U} \in \beta\mathbb{N}\backslash\mathbb{N}$. If M is $\mathcal{R}^\omega$ embeddable, then M embeds into $\mathcal{R}^\mathcal{U}$ for *all* $\mathcal{U} \in \beta\mathbb{N} \setminus \mathbb{N}$; see [8, Corollary 4.15]. For this reason, we fix $\mathcal{U} \in \beta\mathbb{N} \setminus \mathbb{N}$ throughout this note.

2. Model Companions

In the proof of our first theorem, we use the crossed product construction for von Neumann algebras; a good reference is [4, Chapter 4].

Theorem 2.1. $\mathrm{Th}(\mathcal{R})$ *does not have quantifier elimination.*

Proof. It is enough to find separable, $\mathcal{R}^\omega$-embeddable tracial von Neumann algebras $M \subset N$ and an embedding $\pi : M \to \mathcal{R}^\mathcal{U}$ that does not extend to an embedding $N \to \mathcal{R}^\mathcal{U}$. Indeed, if this is so, let N_1 be a separable model of $\mathrm{Th}(\mathcal{R})$ containing N. Then π does not extend to an embedding $N_1 \to \mathcal{R}^\mathcal{U}$; since $\mathcal{R}^\mathcal{U}$ is $\aleph_1$-saturated, this shows that $\mathrm{Th}(\mathcal{R})$ does not have QE.

In order to achieve the goal of the above paragraph, we claim that it is enough to find a countable discrete group Γ such that $L(\Gamma)$ is $\mathcal{R}^\omega$-embeddable, an embedding $\pi : L(\Gamma) \to \mathcal{R}^\mathcal{U}$, and $\alpha \in \mathrm{Aut}(L(\Gamma))$ such that there exists no unitary $u \in \mathcal{R}^\mathcal{U}$ satisfying $(\pi \circ \alpha)(x) = u\pi(x)u^*$ for all $x \in L(\Gamma)$. (We should remark that we are using the usual trace on $L(\Gamma)$ and that $\mathrm{Aut}(L(\Gamma))$ refers to the group of $*$-automorphisms preserving this trace.) First, we abuse notation and also use α to denote the homomorphism $\mathbb{Z} \to \mathrm{Aut}(L(\Gamma))$ which sends the generator of $\mathbb{Z}$ to the aforementioned α. Set $\mathcal{M} = L(\Gamma)$ and $\mathcal{N} = \mathcal{M} \rtimes_\alpha \mathbb{Z}$. Then N is a tracial von Neumann algebra. Moreover, we have that $\mathcal{N}$ is $\mathcal{R}^\omega$-embeddable if and only if $\mathcal{M}$ is—in fact, this is true for any crossed product algebra $\mathcal{M} \rtimes_\alpha G$ where G is amenable [2, Prop. 3.4(2)]. Now suppose, towards a contradiction, that π were to extend to an embedding $\widetilde{\pi} : \mathcal{N} \to \mathcal{R}^\mathcal{U}$. If $u \in L(\mathbb{Z}) \subset \mathcal{M} \rtimes_\alpha \mathbb{Z}$ is the generator of $\mathbb{Z}$, then setting $\tilde{u} = \widetilde{\pi}(u) \in \mathcal{R}^\mathcal{U}$, we would have that $\tilde{u}\pi(x)\tilde{u}^* = \pi(uxu^*) = \pi(\alpha(x))$ for all $x \in \mathcal{M}$, contradicting the fact that $\pi \circ \alpha$ is not unitarily conjugate to the embedding π in $\mathcal{R}^\mathcal{U}$.

An explicit construction of Γ, π and α as above has already appeared in the work of N. P. Brown [6]. Indeed, by [6, Corollary 6.11], we may choose $\Gamma = \mathrm{SL}(3, \mathbb{Z}) * \mathbb{Z}$ and $\alpha = \mathrm{id} * \theta$ for any nontrivial $\theta \in \mathrm{Aut}(L(\mathbb{Z}))$.

□

We say that a separable II_1 factor $\mathcal{S}$ is *locally universal* if every separable II_1 factor embeds into $\mathcal{S}^\mathcal{U}$. (By [8, Corollary 4.15], this notion is independent of $\mathcal{U}$.) In [9], it is shown that a locally universal II_1 factor exists. The *Connes Embedding Problems* (CEP) asks whether $\mathcal{R}$ is locally universal.

We say that a separable II_1 factor M is *McDuff* if $M \otimes \mathcal{R} \cong M$. For example, $\mathcal{R}$ is McDuff as is $M \otimes \mathcal{R}$ for any separable II_1 factor M. By examining Brown's argument in [6], we see that the only properties of $\mathcal{R}$ that are used (other than it being finite) is that $L(\Gamma)$ (for Γ as in the previous proof) is $\mathcal{R}^\omega$-embeddable and that $\mathcal{R}$ is McDuff. We thus have:

Theorem 2.2. *If $\mathcal{S}$ is a locally universal, McDuff II_1 factor, then* $\mathrm{Th}(\mathcal{S})$ *does not have QE.*

Let T_0 be the theory of tracial von Neumann algebras in the signature $\mathcal{L}$. Recall from the Introduction that T_0 is a universally axiomatizable theory. Let T_{II_1} be the $\mathcal{L}$-theory of II_1 factors, a $\forall\exists$-axiomatizable theory by [8, Proposition 3.4]. Moreover, since every tracial von Neumann algebra is contained in a II_1 factor, we see that $T_0 = (T_{\mathrm{II}_1})_\forall$. Thus, an existentially closed model of T_0 is a II_1 factor.

By [9, Proposition 3.9], there is a set Σ of $\forall\exists$-sentences in the language of tracial von Neumann algebras such that M is McDuff if and only if $M \models \Sigma$. Since every II_1 factor is contained in a McDuff II_1 factor (as $M \subseteq M \otimes \mathcal{R}$), it follows that an existentially closed II_1 factor is McDuff.

We can now prove our main result:

Theorem 2.3. *T_0 does not have a model companion.*

Proof. Suppose that T is a model companion for T_0. Since T_0 is univerally axiomatizable and has the amalgamation property (see [4, Chapter 4]), we have that T has QE.

Fix a separable model $\mathcal{S}$ of T. Then $\mathcal{S}$ is a locally universal II_1 factor. Indeed, given an arbitrary separable II_1 factor M, we have a separable model $\mathcal{S}_1 \models T$ containing M. Since $\mathcal{S}^\mathcal{U}$ is $\aleph_1$-saturated, we have that $\mathcal{S}_1$ embeds into $\mathcal{S}^\mathcal{U}$, yielding an embedding of M into $\mathcal{S}^\mathcal{U}$. Meanwhile, since T is the theory of existentially closed models of T_0, we see that $\mathcal{S}$ is McDuff. Thus, by Corollary 2.2, T does not have QE, a contradiction. □

3. Model Complete II_1 Factors

While we have proven that the theory of tracial von Neumann algebras does not have a model companion, at this point it is still possible that there is a model complete theory of II_1 factors. In this section, we show that a positive solution to the CEP implies that there is no model-complete theory of II_1 factors.

We begin by observing the following:

Lemma 3.1. *Every embedding $\mathcal{R} \to \mathcal{R}^\omega$ is elementary.*

Proof. This follows from the fact that every embedding $\mathcal{R} \to \mathcal{R}^\omega$ is unitarily equivalent to the diagonal embedding; see [10]. □

Remark. Lemma 3.1 implies that $\mathcal{R}$ is the unique prime model of its theory. Indeed, to show that $\mathcal{R}$ is a prime model of its theory, by Downward Löwenheim-Skolem (DLS), it is enough to show that whenever $M \equiv \mathcal{R}$ is separable, then $\mathcal{R}$ elementarily embeds into M. Since $\mathcal{R}^{\mathcal{U}}$ is $\aleph_1$-saturated, we have that M elementarily embeds into $\mathcal{R}^{\mathcal{U}}$. Composing an embedding $\mathcal{R} \to M$ with the elementary embedding $M \to \mathcal{R}^{\mathcal{U}}$ and applying Lemma 3.1, we see that the embedding $\mathcal{R} \to M$ is elementary.

Proposition 3.2. *Suppose that M is an $\mathcal{R}^{\omega}$-embeddable II_1 factor such that* $\mathrm{Th}(M)$ *is model-complete. Then $M \equiv \mathcal{R}$.*

Proof. Without loss of generality, we may assume that M is separable. Fix embeddings $i : \mathcal{R} \to M$ and $j : M \to \mathcal{R}^{\mathcal{U}}$. By Lemma 3.1, the composition $j \circ i : \mathcal{R} \to \mathcal{R}^{\mathcal{U}}$ is elementary. Notice now that we have elementary embedding $i^{\mathcal{U}} : \mathcal{R}^{\mathcal{U}} \to M^{\mathcal{U}}$ and $j^{\mathcal{U}} : M^{\mathcal{U}} \to (\mathcal{R}^{\mathcal{U}})^{\mathcal{U}}$, whose composition is the elementary embedding $(j \circ i)^{\mathcal{U}} = j^{\mathcal{U}} \circ i^{\mathcal{U}} : \mathcal{R}^{\mathcal{U}} \to (\mathcal{R}^{\mathcal{U}})^{\mathcal{U}}$. Iterate this procedure, and letting $\mathcal{R}_n$ (resp. M_n) denote the n^{th} iterated ultrapower of $\mathcal{R}$ (resp. M), we get a chain of embeddings

$$\mathcal{R}_0 = \mathcal{R} \hookrightarrow M_0 = M \hookrightarrow \mathcal{R}_1 \hookrightarrow M_1 \hookrightarrow \mathcal{R}_2 \hookrightarrow M_2 \hookrightarrow \cdots$$

such that all embeddings $\mathcal{R}_n \hookrightarrow \mathcal{R}_{n+1}$ are elementary (being iterated ultrapowers of the elementary embedding of $\mathcal{R}$ into $\mathcal{R}^{\mathcal{U}}$) and all embeddings $M_n \hookrightarrow M_{n+1}$ are elementary (as $\mathrm{Th}(M)$ is model complete). Let $M_\infty = \bigcup_{n<\omega} \mathcal{R}_n = \bigcup_{n<\omega} M_n$. Then $M \equiv M_\infty \equiv \mathcal{R}$. $\square$

Remark 3.3. Proposition 3.2 provides immediate examples of non-model complete theories of II_1 factors. Indeed, for $m \geq 2$, the von Neumann group algebra of the free group on m generators, $L(\mathbb{F}_m)$, is $\mathcal{R}^{\omega}$-embeddable but not elementarily equivalent to $\mathcal{R}$ (see [9, Subsection 3.2.2]), whence $\mathrm{Th}(L(\mathbb{F}_m))$ is not model-complete. It is an outstanding problem in operator algebras whether or not $L(\mathbb{F}_m) \cong L(\mathbb{F}_n)$ for all $m, n \geq 2$. A weaker, but still seemingly difficult, question is whether or not $L(\mathbb{F}_m) \equiv L(\mathbb{F}_n)$ for all $m, n \geq 2$. (An equivalent formulation of this question is whether or not there is $\mathcal{U} \in \beta\mathbb{N} \setminus \mathbb{N}$ such that $L(\mathbb{F}_m)^{\mathcal{U}} \cong L(\mathbb{F}_n)^{\mathcal{U}}$)?) Suppose this latter question has an affirmative answer. Then we see that the theory of free group von Neumann algebras is not model-complete, mirroring the corresponding fact that the theory of free groups is not model-complete. However, the natural embeddings $\mathbb{F}_m \to \mathbb{F}_n$, for $m < n$, are elementary. *Assuming $L(\mathbb{F}_m) \equiv L(\mathbb{F}_n)$, are the natural embeddings $L(\mathbb{F}_m) \to L(\mathbb{F}_n)$, for $m < n$, elementary?*

Remark 3.4. The property being exploited in the proof of Proposition 3.2 is that any embedding of $\mathcal{R}$ into a tracial von Neumann algebra with the same theory is elementary. The usual reason for this phenomenon is that the given theory is model complete, but, by Corollary 3.5 below, we know that, assuming a positive solution to the Connes Embedding Problem, this is not true for the theory of $\mathcal{R}$. In the realm of C*-algebras, all UHF algebras (direct limits of matrix algebras) and strongly self-absorbing algebras (D for which there is an isomorphism $\phi : D \to D \otimes D$ such that ϕ and $id_D \otimes 1$ are approximately unitarily equivalent) have the property that any embedding of them into a model of their theory is elementary. It would be interesting to know if the theory of any of these algebras is model complete.

Corollary 3.5. *Assume that the CEP has a positive solution. Then there is no model-complete theory of II_1 factors.*

Proof. Suppose that T is a model-complete theory of II_1 factors. By the positive solution to the CEP and Proposition 3.2, $T = \mathrm{Th}(\mathcal{R})$. Meanwhile, a positive solution to the CEP implies that $T_\forall = T_0$, whence T is a model companion for T_0, contradicting Theorem 2.3. □

4. Concluding Remarks

Theorem 2.1 presents a major hurdle in trying to understand the model theory of II_1 factors. In particular, it places a major roadblock in trying to understand potential independence relations in theories of II_1 factors. Indeed, although any II_1 factor is unstable (see [7]), one might wonder whether the natural notion of independence stemming from noncommutative probability theory might show that some II_1 factor is (real) rosy (see [1] for the definition of rosy theory). More precisely, fix some "large" II_1 factor M and consider the relation $\mathop{\smile\!\!\!\!\!\!|}$ on "small" subsets of M given by $A \mathop{\smile\!\!\!\!\!\!|}_C B$ if and only if, for all $a \in \langle AC \rangle$, $E_{\langle C \rangle}(a) = E_{\langle BC \rangle}(a)$. Here, $\langle * \rangle$ denotes the von Neumann subalgebra generated by $*$ and $E_{\langle * \rangle}$ is the conditional expectation (or orthogonal projection) map $E_{\langle * \rangle} : L^2M \to L^2\langle * \rangle$. In trying to verify some of the natural axioms for an independence relation (see [1]), one runs into trouble when trying to verify the extension axiom: If $B \subseteq C \subseteq D$ and $A \mathop{\smile\!\!\!\!\!\!|}_B C$, can we find A' realizing the same type as A over C such that $A' \mathop{\smile\!\!\!\!\!\!|}_B D$? If $M = \mathcal{R}^{\mathcal{U}}$ and "small" means "countable," then it seems quite likely that one could find an A' with the same *quantifier-free type* as A over C that is independent from D over B as quantifier-free types are determined by moments. Without quantifier-elimination, it seems quite difficult to prove the extension property for this purported notion of independence. (The question of whether or not the independence relation arising from conditional expectation yields a strict independence relation was also discussed in [5].)

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