Tutorial on **von Neumann algebras**

Adrian Ioana

University of California, San Diego

Model Theory and Operator Algebras
UC Irvine
September 20-22, 2017

The algebra of bounded linear operators

- H separable Hilbert space, e.g. $\ell^2(\mathbb{N})$, $L^2([0,1], \mathbf{Leb})$.
- $\mathbb{B}(H)$ algebra of **bounded** linear operators $T : H \to H$, i.e. such that $\|T\| = \sup\{\|T\xi\| \mid \|\xi\| = 1\} < \infty$.

The algebra of bounded linear operators

- H separable Hilbert space, e.g. $\ell^2(\mathbb{N})$, $L^2([0,1],\textbf{Leb})$.
- $\mathbb{B}(H)$ algebra of **bounded** linear operators $T : H \to H$, i.e. such that $\|T\| = \sup\{\|T\xi\| \mid \|\xi\| = 1\} < \infty$.
- the **adjoint** $T^* \in \mathbb{B}(H)$ given by $\langle T\xi, \eta\rangle = \langle \xi, T^*\eta\rangle$, for all $\xi, \eta \in H$.

The algebra of bounded linear operators

- H separable Hilbert space, e.g. $\ell^2(\mathbb{N})$, $L^2([0,1], \textbf{Leb})$.
- $\mathbb{B}(H)$ algebra of **bounded** linear operators $T : H \to H$, i.e. such that $\|T\| = \sup\{\|T\xi\| \mid \|\xi\| = 1\} < \infty$.
- the **adjoint** $T^* \in \mathbb{B}(H)$ given by $\langle T\xi, \eta\rangle = \langle \xi, T^*\eta\rangle$, for all $\xi, \eta \in H$.
- the **spectrum** of T is $\sigma(T) = \{\lambda \in \mathbb{C} | \ T - \lambda\,\mathbf{I} \text{ not invertible}\}$.
- **Fact:** $\sigma(T)$ is a compact non-empty subset of $\mathbb{C}$.

The algebra of bounded linear operators

- H separable Hilbert space, e.g. $\ell^2(\mathbb{N})$, $L^2([0,1], \mathbf{Leb})$.
- $\mathbb{B}(H)$ algebra of **bounded** linear operators $T : H \to H$,
  i.e. such that $\|T\| = \sup\{\|T\xi\| \mid \|\xi\| = 1\} < \infty$.
- the **adjoint** $T^* \in \mathbb{B}(H)$ given by $\langle T\xi, \eta\rangle = \langle \xi, T^*\eta\rangle$, for all $\xi, \eta \in H$.
- the **spectrum** of T is $\sigma(T) = \{\lambda \in \mathbb{C} \mid T - \lambda\, \mathbf{I} \text{ not invertible}\}$.
- **Fact:** $\sigma(T)$ is a compact non-empty subset of $\mathbb{C}$.

Definition

An operator $T \in \mathbb{B}(H)$ is called

1. **self-adjoint** if $T = T^*$.
2. **positive** if $\langle T\xi, \xi\rangle \geq 0$, for all $\xi \in H$. (positive $\Rightarrow$ self-adjoint)

The algebra of bounded linear operators

- H separable Hilbert space, e.g. $\ell^2(\mathbb{N})$, $L^2([0,1],\mathbf{Leb})$.
- $\mathbb{B}(H)$ algebra of **bounded** linear operators $T : H \to H$, i.e. such that $\|T\| = \sup\{\|T\xi\| \mid \|\xi\| = 1\} < \infty$.
- the **adjoint** $T^* \in \mathbb{B}(H)$ given by $\langle T\xi, \eta\rangle = \langle \xi, T^*\eta\rangle$, for all $\xi, \eta \in H$.
- the **spectrum** of T is $\sigma(T) = \{\lambda \in \mathbb{C} |\ T - \lambda\,\mathbf{I}$ not invertible$\}$.
- **Fact:** $\sigma(T)$ is a compact non-empty subset of $\mathbb{C}$.

Definition

An operator $T \in \mathbb{B}(H)$ is called

1. **self-adjoint** if $T = T^*$.
2. **positive** if $\langle T\xi, \xi\rangle \geq 0$, for all $\xi \in H$. (positive $\Rightarrow$ self-adjoint)
3. a **projection** if $T = T^* = T^2 \Leftrightarrow T$ is the orthogonal projection onto a closed subspace $K \subset H$.

The algebra of bounded linear operators

- H separable Hilbert space, e.g. $\ell^2(\mathbb{N})$, $L^2([0,1], \mathbf{Leb})$.
- $\mathbb{B}(H)$ algebra of **bounded** linear operators $T : H \to H$, i.e. such that $\|T\| = \sup\{\|T\xi\| \mid \|\xi\| = 1\} < \infty$.
- the **adjoint** $T^* \in \mathbb{B}(H)$ given by $\langle T\xi, \eta\rangle = \langle \xi, T^*\eta\rangle$, for all $\xi, \eta \in H$.
- the **spectrum** of T is $\sigma(T) = \{\lambda \in \mathbb{C} | \ T - \lambda \mathbf{I} \text{ not invertible}\}$.
- **Fact:** $\sigma(T)$ is a compact non-empty subset of $\mathbb{C}$.

Definition

An operator $T \in \mathbb{B}(H)$ is called

1. **self-adjoint** if $T = T^*$.
2. **positive** if $\langle T\xi, \xi\rangle \geq 0$, for all $\xi \in H$. (positive $\Rightarrow$ self-adjoint)
3. a **projection** if $T = T^* = T^2 \Leftrightarrow T$ is the orthogonal projection onto a closed subspace $K \subset H$.
4. a **unitary** if $TT^* = T^*T = I$.

Topologies on $\mathbb{B}(H)$

- **norm topology**: $T_i \to T \iff \|T_i - T\| \to 0$.

Topologies on $\mathbb{B}(H)$

- **norm topology**: $T_i \to T \iff \|T_i - T\| \to 0$.
- **strong operator topology** (SOT): $T_i \to T \iff \|T_i\xi - T\xi\| \to 0$, for all $\xi \in H$.
- **weak operator topology** (WOT): $T_i \to T \iff \langle T_i\xi, \eta\rangle \to \langle T\xi, \eta\rangle$, for all $\xi, \eta \in H$.

Topologies on $\mathbb{B}(H)$

- **norm topology**: $T_i \to T \iff \|T_i - T\| \to 0$.
- **strong operator topology** (SOT): $T_i \to T \iff \|T_i\xi - T\xi\| \to 0$, for all $\xi \in H$.
- **weak operator topology** (WOT): $T_i \to T \iff \langle T_i\xi, \eta\rangle \to \langle T\xi, \eta\rangle$, for all $\xi, \eta \in H$.

Exercise 1: Suppose that $T_i \to T$ (WOT) and T_i, T are all projections (or all unitaries). Prove that $T_i \to T$ (SOT).

Topologies on $\mathbb{B}(H)$

- **norm topology**: $T_i \to T \iff \|T_i - T\| \to 0$.
- **strong operator topology** (SOT): $T_i \to T \iff \|T_i\xi - T\xi\| \to 0$, for all $\xi \in H$.
- **weak operator topology** (WOT): $T_i \to T \iff \langle T_i\xi, \eta\rangle \to \langle T\xi, \eta\rangle$, for all $\xi, \eta \in H$.

Exercise 1: Suppose that $T_i \to T$ (WOT) and T_i, T are all projections (or all unitaries). Prove that $T_i \to T$ (SOT).

Definition

A subalgebra $A \subset \mathbb{B}(H)$ is called a $*$-**algebra** if $T^* \in A$, for all $T \in A$.

Topologies on $\mathbb{B}(H)$

- **norm topology**: $T_i \to T \iff \|T_i - T\| \to 0$.
- **strong operator topology** (SOT): $T_i \to T \iff \|T_i\xi - T\xi\| \to 0$, for all $\xi \in H$.
- **weak operator topology** (WOT): $T_i \to T \iff \langle T_i\xi, \eta\rangle \to \langle T\xi, \eta\rangle$, for all $\xi, \eta \in H$.

Exercise 1: Suppose that $T_i \to T$ (WOT) and T_i, T are all projections (or all unitaries). Prove that $T_i \to T$ (SOT).

Definition

A subalgebra $A \subset \mathbb{B}(H)$ is called a $*$-**algebra** if $T^* \in A$, for all $T \in A$.
A $*$-subalgebra $A \subset \mathbb{B}(H)$ is called a

1. **C*-algebra** if it is closed in the norm topology.

Topologies on $\mathbb{B}(H)$

- **norm topology**: $T_i \to T \iff \|T_i - T\| \to 0$.
- **strong operator topology** (SOT): $T_i \to T \iff \|T_i\xi - T\xi\| \to 0$, for all $\xi \in H$.
- **weak operator topology** (WOT): $T_i \to T \iff \langle T_i\xi, \eta\rangle \to \langle T\xi, \eta\rangle$, for all $\xi, \eta \in H$.

Exercise 1: Suppose that $T_i \to T$ (WOT) and T_i, T are all projections (or all unitaries). Prove that $T_i \to T$ (SOT).

Definition

A subalgebra $A \subset \mathbb{B}(H)$ is called a $*$-**algebra** if $T^* \in A$, for all $T \in A$.
A $*$-subalgebra $A \subset \mathbb{B}(H)$ is called a

1. **C*-algebra** if it is closed in the norm topology.
2. **von Neumann algebra** if it is WOT-closed.

Topologies on $\mathbb{B}(H)$

- **norm topology**: $T_i \to T \iff \|T_i - T\| \to 0$.
- **strong operator topology** (SOT): $T_i \to T \iff \|T_i\xi - T\xi\| \to 0$, for all $\xi \in H$.
- **weak operator topology** (WOT): $T_i \to T \iff \langle T_i\xi, \eta\rangle \to \langle T\xi, \eta\rangle$, for all $\xi, \eta \in H$.

Exercise 1: Suppose that $T_i \to T$ (WOT) and T_i, T are all projections (or all unitaries). Prove that $T_i \to T$ (SOT).

Definition

A subalgebra $A \subset \mathbb{B}(H)$ is called a $*$-**algebra** if $T^* \in A$, for all $T \in A$.
A $*$-subalgebra $A \subset \mathbb{B}(H)$ is called a

1. **C*-algebra** if it is closed in the norm topology.
2. **von Neumann algebra** if it is WOT-closed.

Terminology. A homomorphism $\pi : A \to B$ between two $*$-algebras is called a $*$-**homomorphism** if $\pi(T^*) = \pi(T)^*$, for all $T \in A$.
A bijective $*$-homomorphism $\pi : A \to B$ is called a $*$-**isomorphism.**

Borel functional calculus

Let $T \in \mathbb{B}(H)$ be an operator which is **normal**: $T^*T = TT^*$.

Borel functional calculus

Let $T \in \mathbb{B}(H)$ be an operator which is **normal**: $T^*T = TT^*$.
Let $B(\sigma(T))$ denote the $*$-algebra of bounded Borel functions $f : \sigma(T) \to \mathbb{C}$, with the $*$-operation given by $f^*(z) = \overline{f(z)}$.

Borel functional calculus

Let $T \in \mathbb{B}(H)$ be an operator which is **normal**: $T^*T = TT^*$.
Let $B(\sigma(T))$ denote the $*$-algebra of bounded Borel functions $f : \sigma(T) \to \mathbb{C}$, with the $*$-operation given by $f^*(z) = \overline{f(z)}$.

Theorem: $\exists$ a $*$-homomorphism $\pi : B(\sigma(T)) \to \mathbb{B}(H)$ such that

Borel functional calculus

Let $T \in \mathbb{B}(H)$ be an operator which is **normal**: $T^*T = TT^*$.
Let $B(\sigma(T))$ denote the $*$-algebra of bounded Borel functions $f : \sigma(T) \to \mathbb{C}$, with the $*$-operation given by $f^*(z) = \overline{f(z)}$.

Theorem: $\exists$ a $*$-homomorphism $\pi : B(\sigma(T)) \to \mathbb{B}(H)$ such that

- $\pi(z) = T$ (and thus $\pi(\bar{z}) = T^*$).

Borel functional calculus

Let $T \in \mathbb{B}(H)$ be an operator which is **normal**: $T^*T = TT^*$.
Let $B(\sigma(T))$ denote the $*$-algebra of bounded Borel functions $f : \sigma(T) \to \mathbb{C}$, with the $*$-operation given by $f^*(z) = \overline{f(z)}$.

Theorem: $\exists$ a $*$-homomorphism $\pi : B(\sigma(T)) \to \mathbb{B}(H)$ such that

- $\pi(z) = T$ (and thus $\pi(\bar{z}) = T^*$).
- $\|\pi(f)\| \leq \|f\|_\infty$, for all $f \in B(\sigma(T))$, with equality if $f \in C(\sigma(T))$.

Borel functional calculus

Let $T \in \mathbb{B}(H)$ be an operator which is **normal**: $T^*T = TT^*$.
Let $B(\sigma(T))$ denote the $*$-algebra of bounded Borel functions $f : \sigma(T) \to \mathbb{C}$, with the $*$-operation given by $f^*(z) = \overline{f(z)}$.

Theorem: $\exists$ a $*$-homomorphism $\pi : B(\sigma(T)) \to \mathbb{B}(H)$ such that

- $\pi(z) = T$ (and thus $\pi(\bar{z}) = T^*$).
- $\|\pi(f)\| \leq \|f\|_\infty$, for all $f \in B(\sigma(T))$, with equality if $f \in C(\sigma(T))$.
- If $f_i \in B(\sigma(T))$ is a net such that $\sup_i \|f_i\|_\infty < \infty$ and $f_i(z) \to f(z)$, for all $z \in \sigma(T)$, then $\pi(f_i) \to \pi(f)$ (SOT).

Borel functional calculus

Let $T \in \mathbb{B}(H)$ be an operator which is **normal**: $T^*T = TT^*$.
Let $B(\sigma(T))$ denote the $*$-algebra of bounded Borel functions $f : \sigma(T) \to \mathbb{C}$, with the $*$-operation given by $f^*(z) = \overline{f(z)}$.

Theorem: $\exists$ a $*$-homomorphism $\pi : B(\sigma(T)) \to \mathbb{B}(H)$ such that

- $\pi(z) = T$ (and thus $\pi(\bar{z}) = T^*$).
- $\|\pi(f)\| \leq \|f\|_\infty$, for all $f \in B(\sigma(T))$, with equality if $f \in C(\sigma(T))$.
- If $f_i \in B(\sigma(T))$ is a net such that $\sup_i \|f_i\|_\infty < \infty$ and $f_i(z) \to f(z)$, for all $z \in \sigma(T)$, then $\pi(f_i) \to \pi(f)$ (SOT).

Notation. $f(T) := \pi(f)$, for all $f \in B(\sigma(T))$.

Borel functional calculus

Let $T \in \mathbb{B}(H)$ be an operator which is **normal**: $T^*T = TT^*$.
Let $B(\sigma(T))$ denote the $*$-algebra of bounded Borel functions $f : \sigma(T) \to \mathbb{C}$, with the $*$-operation given by $f^*(z) = \overline{f(z)}$.

Theorem: $\exists$ a $*$-homomorphism $\pi : B(\sigma(T)) \to \mathbb{B}(H)$ such that

- $\pi(z) = T$ (and thus $\pi(\bar{z}) = T^*$).
- $\|\pi(f)\| \leq \|f\|_\infty$, for all $f \in B(\sigma(T))$, with equality if $f \in C(\sigma(T))$.
- If $f_i \in B(\sigma(T))$ is a net such that $\sup_i \|f_i\|_\infty < \infty$ and $f_i(z) \to f(z)$, for all $z \in \sigma(T)$, then $\pi(f_i) \to \pi(f)$ (SOT).

Notation. $f(T) := \pi(f)$, for all $f \in B(\sigma(T))$.

Consequences. Assume M is a von Neumann algebra containing T.

Borel functional calculus

Let $T \in \mathbb{B}(H)$ be an operator which is **normal**: $T^*T = TT^*$.
Let $B(\sigma(T))$ denote the $*$-algebra of bounded Borel functions $f : \sigma(T) \to \mathbb{C}$, with the $*$-operation given by $f^*(z) = \overline{f(z)}$.

Theorem: $\exists$ a $*$-homomorphism $\pi : B(\sigma(T)) \to \mathbb{B}(H)$ such that

- $\pi(z) = T$ (and thus $\pi(\bar{z}) = T^*$).
- $\|\pi(f)\| \leq \|f\|_\infty$, for all $f \in B(\sigma(T))$, with equality if $f \in C(\sigma(T))$.
- If $f_i \in B(\sigma(T))$ is a net such that $\sup_i \|f_i\|_\infty < \infty$ and $f_i(z) \to f(z)$, for all $z \in \sigma(T)$, then $\pi(f_i) \to \pi(f)$ (SOT).

Notation. $f(T) := \pi(f)$, for all $f \in B(\sigma(T))$.

Consequences. Assume M is a von Neumann algebra containing T.
1) $f(T) \in M$, for all $f \in B(\sigma(T))$.

Borel functional calculus

Let $T \in \mathbb{B}(H)$ be an operator which is **normal**: $T^*T = TT^*$.
Let $B(\sigma(T))$ denote the $*$-algebra of bounded Borel functions $f : \sigma(T) \to \mathbb{C}$, with the $*$-operation given by $f^*(z) = \overline{f(z)}$.

Theorem: $\exists$ a $*$-homomorphism $\pi : B(\sigma(T)) \to \mathbb{B}(H)$ such that

- $\pi(z) = T$ (and thus $\pi(\bar{z}) = T^*$).
- $\|\pi(f)\| \leq \|f\|_\infty$, for all $f \in B(\sigma(T))$, with equality if $f \in C(\sigma(T))$.
- If $f_i \in B(\sigma(T))$ is a net such that $\sup_i \|f_i\|_\infty < \infty$ and $f_i(z) \to f(z)$, for all $z \in \sigma(T)$, then $\pi(f_i) \to \pi(f)$ (SOT).

Notation. $f(T) := \pi(f)$, for all $f \in B(\sigma(T))$.

Consequences. Assume M is a von Neumann algebra containing T.
1) $f(T) \in M$, for all $f \in B(\sigma(T))$.
2) $1_\Delta(T) \in M$ is a projection, for any Borel set $\Delta \subset \sigma(T)$.

Borel functional calculus

Let $T \in \mathbb{B}(H)$ be an operator which is **normal**: $T^*T = TT^*$.
Let $B(\sigma(T))$ denote the $*$-algebra of bounded Borel functions $f : \sigma(T) \to \mathbb{C}$, with the $*$-operation given by $f^*(z) = \overline{f(z)}$.

Theorem: $\exists$ a $*$-homomorphism $\pi : B(\sigma(T)) \to \mathbb{B}(H)$ such that

- $\pi(z) = T$ (and thus $\pi(\bar{z}) = T^*$).
- $\|\pi(f)\| \leq \|f\|_\infty$, for all $f \in B(\sigma(T))$, with equality if $f \in C(\sigma(T))$.
- If $f_i \in B(\sigma(T))$ is a net such that $\sup_i \|f_i\|_\infty < \infty$ and $f_i(z) \to f(z)$, for all $z \in \sigma(T)$, then $\pi(f_i) \to \pi(f)$ (SOT).

Notation. $f(T) := \pi(f)$, for all $f \in B(\sigma(T))$.

Consequences. Assume M is a von Neumann algebra containing T.
1) $f(T) \in M$, for all $f \in B(\sigma(T))$.
2) $1_\Delta(T) \in M$ is a projection, for any Borel set $\Delta \subset \sigma(T)$.
3) T is a norm limit of linear combinations of projections in M.

Borel functional calculus

Let $T \in \mathbb{B}(H)$ be an operator which is **normal**: $T^*T = TT^*$.
Let $B(\sigma(T))$ denote the $*$-algebra of bounded Borel functions $f : \sigma(T) \to \mathbb{C}$, with the $*$-operation given by $f^*(z) = \overline{f(z)}$.

Theorem: $\exists$ a $*$-homomorphism $\pi : B(\sigma(T)) \to \mathbb{B}(H)$ such that

- $\pi(z) = T$ (and thus $\pi(\bar{z}) = T^*$).
- $\|\pi(f)\| \leq \|f\|_\infty$, for all $f \in B(\sigma(T))$, with equality if $f \in C(\sigma(T))$.
- If $f_i \in B(\sigma(T))$ is a net such that $\sup_i \|f_i\|_\infty < \infty$ and $f_i(z) \to f(z)$, for all $z \in \sigma(T)$, then $\pi(f_i) \to \pi(f)$ (SOT).

Notation. $f(T) := \pi(f)$, for all $f \in B(\sigma(T))$.

Consequences. Assume M is a von Neumann algebra containing T.
1) $f(T) \in M$, for all $f \in B(\sigma(T))$.
2) $1_\Delta(T) \in M$ is a projection, for any Borel set $\Delta \subset \sigma(T)$.
3) T is a norm limit of linear combinations of projections in M.
4) If T is positive, then there exists $S \in M$ such that $T = S^*S$.
(since $\sigma(T) \subset [0, \infty)$, we can take $S = T^{1/2}$).

Examples. 1) $\mathbb{B}(H)$ is a von Neumann algebra.
2) The algebra $\mathbb{K}(H)$ of all compact operators on H is a C*-algebra.

Examples. 1) $\mathbb{B}(H)$ is a von Neumann algebra.
2) The algebra $\mathbb{K}(H)$ of all compact operators on H is a C*-algebra.

Exercise 2. Show that $\mathbb{K}(H)$ is not a von Neumann algebra if $\dim H = \infty$.

von Neumann's bicommutant theorem

Examples. 1) $\mathbb{B}(H)$ is a von Neumann algebra.
2) The algebra $\mathbb{K}(H)$ of all compact operators on H is a C*-algebra.

Exercise 2. Show that $\mathbb{K}(H)$ is not a von Neumann algebra if $\dim H = \infty$.

Exercise 3. Let $S \subset \mathbb{B}(H)$ be a set such that $x^* \in S$, for all $x \in S$.

von Neumann's bicommutant theorem

Examples. 1) $\mathbb{B}(H)$ is a von Neumann algebra.
2) The algebra $\mathbb{K}(H)$ of all compact operators on H is a C*-algebra.

Exercise 2. Show that $\mathbb{K}(H)$ is not a von Neumann algebra if $\dim H = \infty$.

Exercise 3. Let $S \subset \mathbb{B}(H)$ be a set such that $x^* \in S$, for all $x \in S$.
Show that $S' = \{y \in \mathbb{B}(H) \mid xy = yx, \forall\, x \in S\}$ (the commutant of S) is a von Neumann algebra.

Examples. 1) $\mathbb{B}(H)$ is a von Neumann algebra.
2) The algebra $\mathbb{K}(H)$ of all compact operators on H is a C*-algebra.

Exercise 2. Show that $\mathbb{K}(H)$ is not a von Neumann algebra if $\dim H = \infty$.

Exercise 3. Let $S \subset \mathbb{B}(H)$ be a set such that $x^* \in S$, for all $x \in S$. Show that $S' = \{y \in \mathbb{B}(H) \mid xy = yx, \forall\, x \in S\}$ (the commutant of S) is a von Neumann algebra.

Notation. If M is a von Neumann algebra, its **center** is $\mathcal{Z}(M) = M \cap M'$.

Examples. 1) $\mathbb{B}(H)$ is a von Neumann algebra.
2) The algebra $\mathbb{K}(H)$ of all compact operators on H is a C*-algebra.

Exercise 2. Show that $\mathbb{K}(H)$ is not a von Neumann algebra if $\dim H = \infty$.

Exercise 3. Let $S \subset \mathbb{B}(H)$ be a set such that $x^* \in S$, for all $x \in S$.
Show that $S' = \{y \in \mathbb{B}(H) \mid xy = yx, \forall\, x \in S\}$ (the commutant of S) is a von Neumann algebra.

Notation. If M is a von Neumann algebra, its **center** is $\mathcal{Z}(M) = M \cap M'$.

Theorem (von Neumann)

If $M \subset \mathbb{B}(H)$ is a unital ($1 \in M$) $*$-algebra, then **TFAE**:

1. M is WOT-closed.
2. M is SOT-closed.
3. $M = M'' := (M')'$.

von Neumann's bicommutant theorem

Examples. 1) $\mathbb{B}(H)$ is a von Neumann algebra.
2) The algebra $\mathbb{K}(H)$ of all compact operators on H is a C*-algebra.

Exercise 2. Show that $\mathbb{K}(H)$ is not a von Neumann algebra if $\dim H = \infty$.

Exercise 3. Let $S \subset \mathbb{B}(H)$ be a set such that $x^* \in S$, for all $x \in S$.
Show that $S' = \{y \in \mathbb{B}(H) \mid xy = yx, \forall\, x \in S\}$ (the commutant of S) is a von Neumann algebra.

Notation. If M is a von Neumann algebra, its **center** is $\mathcal{Z}(M) = M \cap M'$.

Theorem (von Neumann)

If $M \subset \mathbb{B}(H)$ is a unital ($1 \in M$) $*$-algebra, then **TFAE**:

1. M is WOT-closed.
2. M is SOT-closed.
3. $M = M'' := (M')'$.

(3) $\Rightarrow$ (1) $\Rightarrow$ (2), so it remains to prove that (2) $\Rightarrow$ (3).

Proof of the bicommutant theorem, I

Assumption: $M \subset \mathbb{B}(H)$ is an SOT-closed unital $*$-algebra.
Goal: $M = M''$. To this end, fix $x \in M''$.

Proof of the bicommutant theorem, I

Assumption: $M \subset \mathbb{B}(H)$ is an SOT-closed unital $*$-algebra.
Goal: $M = M''$. To this end, fix $x \in M''$.

In order to show that $x \in M$, it suffices to argue that $x \in \overline{M}^{SOT}$:
$\forall \xi_1, ..., \xi_n \in H$ and $\varepsilon > 0$, $\exists y \in M$ s.t. $\|x\xi_i - y\xi_i\| < \varepsilon, \forall i = 1, ..., n$.

Proof of the bicommutant theorem, I

Assumption: $M \subset \mathbb{B}(H)$ is an SOT-closed unital $*$-algebra.
Goal: $M = M''$. To this end, fix $x \in M''$.

In order to show that $x \in M$, it suffices to argue that $x \in \overline{M}^{SOT}$:
$\forall \xi_1, ..., \xi_n \in H$ and $\varepsilon > 0$, $\exists y \in M$ s.t. $\|x\xi_i - y\xi_i\| < \varepsilon, \forall i = 1, ..., n$.

First, we assume that $n = 1$ and prove the following:

Proof of the bicommutant theorem, I

Assumption: $M \subset \mathbb{B}(H)$ is an SOT-closed unital $*$-algebra.
Goal: $M = M''$. To this end, fix $x \in M''$.

In order to show that $x \in M$, it suffices to argue that $x \in \overline{M}^{SOT}$:
$\forall \xi_1, ..., \xi_n \in H$ and $\varepsilon > 0$, $\exists y \in M$ s.t. $\|x\xi_i - y\xi_i\| < \varepsilon, \forall i = 1, ..., n$.

First, we assume that $n = 1$ and prove the following:

Claim

Let $K \subset H$ be an M-**invariant** closed subspace. ($zK \subset K$, for all $z \in M$)

Proof of the bicommutant theorem, I

Assumption: $M \subset \mathbb{B}(H)$ is an SOT-closed unital $*$-algebra.
Goal: $M = M''$. To this end, fix $x \in M''$.

In order to show that $x \in M$, it suffices to argue that $x \in \overline{M}^{SOT}$:
$\forall \xi_1, ..., \xi_n \in H$ and $\varepsilon > 0$, $\exists y \in M$ s.t. $\|x\xi_i - y\xi_i\| < \varepsilon, \forall i = 1, ..., n$.

First, we assume that $n = 1$ and prove the following:

Claim

Let $K \subset H$ be an M-**invariant** closed subspace. ($zK \subset K$, for all $z \in M$)
If p is the orthogonal projection onto K, then $p \in M'$.

Proof of the bicommutant theorem, I

Assumption: $M \subset \mathbb{B}(H)$ is an SOT-closed unital $*$-algebra.
Goal: $M = M''$. To this end, fix $x \in M''$.

In order to show that $x \in M$, it suffices to argue that $x \in \overline{M}^{SOT}$:
$\forall \xi_1, ..., \xi_n \in H$ and $\varepsilon > 0$, $\exists y \in M$ s.t. $\|x\xi_i - y\xi_i\| < \varepsilon, \forall i = 1, ..., n$.

First, we assume that $n = 1$ and prove the following:

Claim

Let $K \subset H$ be an M-**invariant** closed subspace. ($zK \subset K$, for all $z \in M$)
If p is the orthogonal projection onto K, then $p \in M'$.

Proof of the claim. If $z \in M$, then $zpH = zK \subset K$.

Proof of the bicommutant theorem, I

Assumption: $M \subset \mathbb{B}(H)$ is an SOT-closed unital $*$-algebra.
Goal: $M = M''$. To this end, fix $x \in M''$.

In order to show that $x \in M$, it suffices to argue that $x \in \overline{M}^{SOT}$:
$\forall \xi_1, ..., \xi_n \in H$ and $\varepsilon > 0$, $\exists y \in M$ s.t. $\|x\xi_i - y\xi_i\| < \varepsilon, \forall i = 1, ..., n$.

First, we assume that $n = 1$ and prove the following:

Claim

Let $K \subset H$ be an M-**invariant** closed subspace. ($zK \subset K$, for all $z \in M$)
If p is the orthogonal projection onto K, then $p \in M'$.

Proof of the claim. If $z \in M$, then $zpH = zK \subset K$.

Hence $(1-p)zpH \subset (1-p)K = \{0\}$ and thus $(1-p)zp = 0$ **(a)**.

Proof of the bicommutant theorem, I

Assumption: $M \subset \mathbb{B}(H)$ is an SOT-closed unital $*$-algebra.
Goal: $M = M''$. To this end, fix $x \in M''$.

In order to show that $x \in M$, it suffices to argue that $x \in \overline{M}^{SOT}$:
$\forall \xi_1, ..., \xi_n \in H$ and $\varepsilon > 0$, $\exists y \in M$ s.t. $\|x\xi_i - y\xi_i\| < \varepsilon, \forall i = 1, ..., n$.

First, we assume that $n = 1$ and prove the following:

Claim

Let $K \subset H$ be an M-**invariant** closed subspace. ($zK \subset K$, for all $z \in M$)
If p is the orthogonal projection onto K, then $p \in M'$.

Proof of the claim. If $z \in M$, then $zpH = zK \subset K$.

Hence $(1-p)zpH \subset (1-p)K = \{0\}$ and thus $(1-p)zp = 0$ **(a)**.

Since $z^* \in M \Rightarrow (1-p)z^*p = 0 \Rightarrow pz(1-p) = 0$ **(b)**.

Proof of the bicommutant theorem, I

Assumption: $M \subset \mathbb{B}(H)$ is an SOT-closed unital $*$-algebra.
Goal: $M = M''$. To this end, fix $x \in M''$.

In order to show that $x \in M$, it suffices to argue that $x \in \overline{M}^{SOT}$:
$\forall \xi_1, \ldots, \xi_n \in H$ and $\varepsilon > 0$, $\exists y \in M$ s.t. $\|x\xi_i - y\xi_i\| < \varepsilon, \forall i = 1, \ldots, n$.

First, we assume that $n = 1$ and prove the following:

Claim

Let $K \subset H$ be an M-**invariant** closed subspace. ($zK \subset K$, for all $z \in M$)
If p is the orthogonal projection onto K, then $p \in M'$.

Proof of the claim. If $z \in M$, then $zpH = zK \subset K$.

Hence $(1-p)zpH \subset (1-p)K = \{0\}$ and thus $(1-p)zp = 0$ **(a)**.

Since $z^* \in M \Rightarrow (1-p)z^*p = 0 \Rightarrow pz(1-p) = 0$ **(b)**.

(a) and **(b)** $\Rightarrow pz - zp = pz(1-p) - (1-p)zp = 0$, proving the claim.

Proof of the bicommutant theorem, II

Goal: If $M \subset \mathbb{B}(H)$ is a unital $*$-algebra and $x \in M''$, then
$\forall \xi_1, ..., \xi_n \in H$ and $\varepsilon > 0$, $\exists y \in M$ s.t. $\|x\xi_i - y\xi_i\| < \varepsilon, \forall i = 1, ..., n$.

Proof of the bicommutant theorem, II

Goal: If $M \subset \mathbb{B}(H)$ is a unital $*$-algebra and $x \in M''$, then $\forall \xi_1, ..., \xi_n \in H$ and $\varepsilon > 0$, $\exists y \in M$ s.t. $\|x\xi_i - y\xi_i\| < \varepsilon, \forall i = 1, ..., n$.

In the present case $n = 1$, let p be the orthogonal projection onto

$$\overline{M\xi_1} := \overline{\{y\xi_1 \mid y \in M\}}.$$

Proof of the bicommutant theorem, II

Goal: If $M \subset \mathbb{B}(H)$ is a unital $*$-algebra and $x \in M''$, then $\forall \xi_1, ..., \xi_n \in H$ and $\varepsilon > 0$, $\exists y \in M$ s.t. $\|x\xi_i - y\xi_i\| < \varepsilon, \forall i = 1, ..., n$.

In the present case $n = 1$, let p be the orthogonal projection onto

$$\overline{M\xi_1} := \overline{\{y\xi_1 \mid y \in M\}}.$$

Since $\overline{M\xi_1} \subset H$ is a closed M-invariant subspace, the claim implies $p \in M'$.

Proof of the bicommutant theorem, II

Goal: If $M \subset \mathbb{B}(H)$ is a unital $*$-algebra and $x \in M''$, then
$\forall \xi_1, \ldots, \xi_n \in H$ and $\varepsilon > 0$, $\exists y \in M$ s.t. $\|x\xi_i - y\xi_i\| < \varepsilon, \forall i = 1, \ldots, n$.

In the present case $n = 1$, let p be the orthogonal projection onto

$$\overline{M\xi_1} := \overline{\{y\xi_1 \mid y \in M\}}.$$

Since $\overline{M\xi_1} \subset H$ is a closed M-invariant subspace, the claim implies $p \in M'$.

Since $x \in (M')'$, we have $xp = px$, and thus $x\xi_1 = xp\xi_1 = p(x\xi_1) \in \overline{M\xi_1}$.

Proof of the bicommutant theorem, II

Goal: If $M \subset \mathbb{B}(H)$ is a unital $*$-algebra and $x \in M''$, then
$\forall \xi_1, ..., \xi_n \in H$ and $\varepsilon > 0$, $\exists y \in M$ s.t. $\|x\xi_i - y\xi_i\| < \varepsilon, \forall i = 1, ..., n$.

In the present case $n = 1$, let p be the orthogonal projection onto

$$\overline{M\xi_1} := \overline{\{y\xi_1 \mid y \in M\}}.$$

Since $\overline{M\xi_1} \subset H$ is a closed M-invariant subspace, the claim implies $p \in M'$.

Since $x \in (M')'$, we have $xp = px$, and thus $x\xi_1 = xp\xi_1 = p(x\xi_1) \in \overline{M\xi_1}$.

This implies that $\|x\xi_1 - y\xi_1\| < \varepsilon$, for some $y \in M$, and finishes the proof in the case $n = 1$.

Proof of the bicommutant theorem, III

Goal: If $M \subset \mathbb{B}(H)$ is a unital $*$-algebra and $x \in M''$, then
$\forall \xi_1, ..., \xi_n \in H$ and $\varepsilon > 0$, $\exists y \in M$ s.t. $\|x\xi_i - y\xi_i\| < \varepsilon, \forall i = 1, ..., n$.

Since we have verified the case $n = 1$, we may assume $n \geq 2$.

Proof of the bicommutant theorem, III

Goal: If $M \subset \mathbb{B}(H)$ is a unital $*$-algebra and $x \in M''$, then
$\forall \xi_1, ..., \xi_n \in H$ and $\varepsilon > 0$, $\exists y \in M$ s.t. $\|x\xi_i - y\xi_i\| < \varepsilon, \forall i = 1, ..., n$.

Since we have verified the case $n = 1$, we may assume $n \geq 2$.
We will use a "matrix trick". We define a diagonal $*$-homomorphism $\pi : \mathbb{B}(H) \to \mathbb{B}(H^n) \equiv \mathbb{M}_n(\mathbb{B}(H))$ by letting

$$\pi(y) = \begin{pmatrix} y & 0 & .. & 0 \\ 0 & y & ... & 0 \\ ... & ... & ... & ... \\ 0 & 0 & ... & y \end{pmatrix}$$

Proof of the bicommutant theorem, III

Goal: If $M \subset \mathbb{B}(H)$ is a unital $*$-algebra and $x \in M''$, then $\forall \xi_1, ..., \xi_n \in H$ and $\varepsilon > 0$, $\exists y \in M$ s.t. $\|x\xi_i - y\xi_i\| < \varepsilon, \forall i = 1, ..., n.$

Since we have verified the case $n = 1$, we may assume $n \geq 2$.
We will use a "matrix trick". We define a diagonal $*$-homomorphism $\pi : \mathbb{B}(H) \to \mathbb{B}(H^n) \equiv \mathbb{M}_n(\mathbb{B}(H))$ by letting

$$\pi(y) = \begin{pmatrix} y & 0 & .. & 0 \\ 0 & y & ... & 0 \\ ... & ... & ... & ... \\ 0 & 0 & ... & y \end{pmatrix}$$

Equivalently: $\pi(y)(\eta_1 \oplus ... \oplus \eta_n) = y\eta_1 \oplus ... \oplus y\eta_n.$

Proof of the bicommutant theorem, III

Goal: If $M \subset \mathbb{B}(H)$ is a unital $*$-algebra and $x \in M''$, then $\forall \xi_1, ..., \xi_n \in H$ and $\varepsilon > 0$, $\exists y \in M$ s.t. $\|x\xi_i - y\xi_i\| < \varepsilon, \forall i = 1, ..., n$.

Since we have verified the case $n = 1$, we may assume $n \geq 2$.
We will use a "matrix trick". We define a diagonal $*$-homomorphism $\pi : \mathbb{B}(H) \to \mathbb{B}(H^n) \equiv \mathbb{M}_n(\mathbb{B}(H))$ by letting

$$\pi(y) = \begin{pmatrix} y & 0 & .. & 0 \\ 0 & y & ... & 0 \\ ... & ... & ... & ... \\ 0 & 0 & ... & y \end{pmatrix}$$

Equivalently: $\pi(y)(\eta_1 \oplus ... \oplus \eta_n) = y\eta_1 \oplus ... \oplus y\eta_n$.

Exercise 4. Show that $\pi(M'') \subset \mathbb{M}_n(M')'$ and $\pi(M)' \subset \mathbb{M}_n(M')$.

Proof of the bicommutant theorem, III

Goal: If $M \subset \mathbb{B}(H)$ is a unital $*$-algebra and $x \in M''$, then
$\forall \xi_1, ..., \xi_n \in H$ and $\varepsilon > 0$, $\exists y \in M$ s.t. $\|x\xi_i - y\xi_i\| < \varepsilon, \forall i = 1, ..., n.$

Since we have verified the case $n = 1$, we may assume $n \geq 2$.
We will use a "matrix trick". We define a diagonal $*$-homomorphism $\pi : \mathbb{B}(H) \to \mathbb{B}(H^n) \equiv \mathbb{M}_n(\mathbb{B}(H))$ by letting

$$\pi(y) = \begin{pmatrix} y & 0 & .. & 0 \\ 0 & y & ... & 0 \\ ... & ... & ... & ... \\ 0 & 0 & ... & y \end{pmatrix}$$

Equivalently: $\pi(y)(\eta_1 \oplus ... \oplus \eta_n) = y\eta_1 \oplus ... \oplus y\eta_n$.

Exercise 4. Show that $\pi(M'') \subset \mathbb{M}_n(M')'$ and $\pi(M)' \subset \mathbb{M}_n(M')$.

This implies that $\pi(M'') \subset (\pi(M)')'$, hence $\pi(x) \in \pi(M)''$.

Proof of the bicommutant theorem, III

Goal: If $M \subset \mathbb{B}(H)$ is a unital $*$-algebra and $x \in M''$, then $\forall \xi_1, ..., \xi_n \in H$ and $\varepsilon > 0$, $\exists y \in M$ s.t. $\|x\xi_i - y\xi_i\| < \varepsilon, \forall i = 1, ..., n$.

Since we have verified the case $n = 1$, we may assume $n \geq 2$.
We will use a "matrix trick". We define a diagonal $*$-homomorphism $\pi : \mathbb{B}(H) \to \mathbb{B}(H^n) \equiv \mathbb{M}_n(\mathbb{B}(H))$ by letting

$$\pi(y) = \begin{pmatrix} y & 0 & .. & 0 \\ 0 & y & ... & 0 \\ ... & ... & ... & ... \\ 0 & 0 & ... & y \end{pmatrix}$$

Equivalently: $\pi(y)(\eta_1 \oplus ... \oplus \eta_n) = y\eta_1 \oplus ... \oplus y\eta_n$.

Exercise 4. Show that $\pi(M'') \subset \mathbb{M}_n(M')'$ and $\pi(M)' \subset \mathbb{M}_n(M')$.

This implies that $\pi(M'') \subset (\pi(M)')'$, hence $\pi(x) \in \pi(M)''$.
If $\xi = \xi_1 \oplus ... \oplus \xi_n \in H^n$, the case $n = 1 \Rightarrow \exists y \in M$ s.t. $\|\pi(x)\xi - \pi(y)\xi\| < \varepsilon$, or equivalently $\left(\sum_{i=1}^n \|x\xi_i - y\xi_i\|^2\right)^{1/2} < \varepsilon$.

L^∞-algebras as von Neumann algebras

Proposition

Let (X, μ) be a **standard** probability space (X is a Polish space, μ is a Borel measure on X with $\mu(X) = 1$).

L^∞-algebras as von Neumann algebras

Proposition

Let (X, μ) be a **standard** probability space (X is a Polish space, μ is a Borel measure on X with $\mu(X) = 1$).
Define a $*$-homomorphism $\pi : L^\infty(X, \mu) \to \mathbb{B}(L^2(X, \mu))$ by $\pi(f)(\xi) = f\xi$, for all $f \in L^\infty(X)$ and $\xi \in L^2(X)$.

L^∞-algebras as von Neumann algebras

Proposition

Let (X, μ) be a **standard** probability space (X is a Polish space, μ is a Borel measure on X with $\mu(X) = 1$).
Define a $*$-homomorphism $\pi : L^\infty(X, \mu) \to \mathbb{B}(L^2(X, \mu))$ by $\pi(f)(\xi) = f\xi$, for all $f \in L^\infty(X)$ and $\xi \in L^2(X)$.

Then $\pi(L^\infty(X))' = \pi(L^\infty(X))$.

L^∞-algebras as von Neumann algebras

Proposition

Let (X, μ) be a **standard** probability space (X is a Polish space, μ is a Borel measure on X with $\mu(X) = 1$).
Define a $*$-homomorphism $\pi : L^\infty(X, \mu) \to \mathbb{B}(L^2(X, \mu))$ by $\pi(f)(\xi) = f\xi$, for all $f \in L^\infty(X)$ and $\xi \in L^2(X)$.

Then $\pi(L^\infty(X))' = \pi(L^\infty(X))$.Therefore, $\pi(L^\infty(X)) \subset \mathbb{B}(L^2(X))$ is a maximal abelian von Neumann subalgebra.

L^∞-algebras as von Neumann algebras

Proposition

Let (X, μ) be a **standard** probability space (X is a Polish space, μ is a Borel measure on X with $\mu(X) = 1$).
Define a $*$-homomorphism $\pi : L^\infty(X, \mu) \to \mathbb{B}(L^2(X, \mu))$ by $\pi(f)(\xi) = f\xi$, for all $f \in L^\infty(X)$ and $\xi \in L^2(X)$.

Then $\pi(L^\infty(X))' = \pi(L^\infty(X))$.Therefore, $\pi(L^\infty(X)) \subset \mathbb{B}(L^2(X))$ is a maximal abelian von Neumann subalgebra.

Proof. Let $T \in \pi(L^\infty(X))'$ and put $g = T(\mathbf{1}_X) \in L^2(X)$.

L^∞-algebras as von Neumann algebras

Proposition

Let (X, μ) be a **standard** probability space (X is a Polish space, μ is a Borel measure on X with $\mu(X) = 1$).
Define a $*$-homomorphism $\pi : L^\infty(X, \mu) \to \mathbb{B}(L^2(X, \mu))$ by $\pi(f)(\xi) = f\xi$, for all $f \in L^\infty(X)$ and $\xi \in L^2(X)$.

Then $\pi(L^\infty(X))' = \pi(L^\infty(X))$.Therefore, $\pi(L^\infty(X)) \subset \mathbb{B}(L^2(X))$ is a maximal abelian von Neumann subalgebra.

Proof. Let $T \in \pi(L^\infty(X))'$ and put $g = T(\mathbf{1}_X) \in L^2(X)$.

Then $fg = \pi(f)T(\mathbf{1}) = T\pi(f)(\mathbf{1}) = T(f)$, for all $f \in L^\infty(X)$.

L^∞-algebras as von Neumann algebras

Proposition

Let (X, μ) be a **standard** probability space (X is a Polish space, μ is a Borel measure on X with $\mu(X) = 1$).
Define a $*$-homomorphism $\pi : L^\infty(X, \mu) \to \mathbb{B}(L^2(X, \mu))$ by $\pi(f)(\xi) = f\xi$, for all $f \in L^\infty(X)$ and $\xi \in L^2(X)$.

Then $\pi(L^\infty(X))' = \pi(L^\infty(X))$.Therefore, $\pi(L^\infty(X)) \subset \mathbb{B}(L^2(X))$ is a maximal abelian von Neumann subalgebra.

Proof. Let $T \in \pi(L^\infty(X))'$ and put $g = T(\mathbf{1}_X) \in L^2(X)$.

Then $fg = \pi(f)T(\mathbf{1}) = T\pi(f)(\mathbf{1}) = T(f)$, for all $f \in L^\infty(X)$.
Since $L^\infty(X) \subset L^2(X)$ is dense, it suffices to show that $g \in L^\infty(X)$.

L^∞-algebras as von Neumann algebras

Proposition

Let (X, μ) be a **standard** probability space (X is a Polish space, μ is a Borel measure on X with $\mu(X) = 1$).
Define a $*$-homomorphism $\pi : L^\infty(X, \mu) \to \mathbb{B}(L^2(X, \mu))$ by $\pi(f)(\xi) = f\xi$, for all $f \in L^\infty(X)$ and $\xi \in L^2(X)$.

Then $\pi(L^\infty(X))' = \pi(L^\infty(X))$.Therefore, $\pi(L^\infty(X)) \subset \mathbb{B}(L^2(X))$ is a maximal abelian von Neumann subalgebra.

Proof. Let $T \in \pi(L^\infty(X))'$ and put $g = T(\mathbf{1}_X) \in L^2(X)$.

Then $fg = \pi(f)T(\mathbf{1}) = T\pi(f)(\mathbf{1}) = T(f)$, for all $f \in L^\infty(X)$.
Since $L^\infty(X) \subset L^2(X)$ is dense, it suffices to show that $g \in L^\infty(X)$.

For $n \geq 1$, let $f_n = \mathbf{1}_{\{x \in X \mid |g(x)| \geq \|T\| + 1/n\}}$. Then $|f_n g| \geq (\|T\| + 1/n)|f_n|$,

L^∞-algebras as von Neumann algebras

Proposition

Let (X, μ) be a **standard** probability space (X is a Polish space, μ is a Borel measure on X with $\mu(X) = 1$).
Define a $*$-homomorphism $\pi : L^\infty(X, \mu) \to \mathbb{B}(L^2(X, \mu))$ by $\pi(f)(\xi) = f\xi$, for all $f \in L^\infty(X)$ and $\xi \in L^2(X)$.

Then $\pi(L^\infty(X))' = \pi(L^\infty(X))$.Therefore, $\pi(L^\infty(X)) \subset \mathbb{B}(L^2(X))$ is a maximal abelian von Neumann subalgebra.

Proof. Let $T \in \pi(L^\infty(X))'$ and put $g = T(\mathbf{1}_X) \in L^2(X)$.

Then $fg = \pi(f)T(\mathbf{1}) = T\pi(f)(\mathbf{1}) = T(f)$, for all $f \in L^\infty(X)$.
Since $L^\infty(X) \subset L^2(X)$ is dense, it suffices to show that $g \in L^\infty(X)$.

For $n \geq 1$, let $f_n = \mathbf{1}_{\{x \in X| \ |g(x)| \geq \|T\| + 1/n\}}$. Then $|f_n g| \geq (\|T\| + 1/n)|f_n|$, hence $\|T\| \|f_n\|_2 \geq \|T(f_n)\|_2 = \|f_n g\|_2 \geq (\|T\| + 1/n)\|f_n\|_2$, so $f_n = 0$, a.e.

L^∞-algebras as von Neumann algebras

Proposition

Let (X,μ) be a **standard** probability space (X is a Polish space, μ is a Borel measure on X with $\mu(X)=1$).
Define a $*$-homomorphism $\pi : L^\infty(X,\mu) \to \mathbb{B}(L^2(X,\mu))$ by $\pi(f)(\xi) = f\xi$, for all $f \in L^\infty(X)$ and $\xi \in L^2(X)$.

Then $\pi(L^\infty(X))' = \pi(L^\infty(X))$.Therefore, $\pi(L^\infty(X)) \subset \mathbb{B}(L^2(X))$ is a maximal abelian von Neumann subalgebra.

Proof. Let $T \in \pi(L^\infty(X))'$ and put $g = T(\mathbf{1}_X) \in L^2(X)$.

Then $fg = \pi(f)T(\mathbf{1}) = T\pi(f)(\mathbf{1}) = T(f)$, for all $f \in L^\infty(X)$.
Since $L^\infty(X) \subset L^2(X)$ is dense, it suffices to show that $g \in L^\infty(X)$.

For $n \geq 1$, let $f_n = \mathbf{1}_{\{x\in X|\ |g(x)|\geq\|T\|+1/n\}}$. Then $|f_n g| \geq (\|T\|+1/n)|f_n|$, hence $\|T\|\|f_n\|_2 \geq \|T(f_n)\|_2 = \|f_n g\|_2 \geq (\|T\|+1/n)\|f_n\|_2$, so $f_n = 0$, a.e.

Thus, $\mu(\{x \in X|\ |g(x)| \geq \|T\| + 1/n\}) = 0$ for all $n \geq 1 \Rightarrow \|g\|_\infty \leq \|T\|$.

States and traces

Definition. Let A be a unital C*-algebra. A unital linear functional $\varphi : A \to \mathbb{C}$ is called a **state** if is positive: $\varphi(x) \geq 0$, for all $x \in A, x \geq 0$.

States and traces

Definition. Let A be a unital C*-algebra. A unital linear functional $\varphi : A \to \mathbb{C}$ is called a **state** if is positive: $\varphi(x) \geq 0$, for all $x \in A, x \geq 0$.
Example. $\varphi(T) = \langle T(\xi), \xi \rangle$ is a state on $\mathbb{B}(H)$, for $\xi \in H$, $\|\xi\| = 1$.

Definition. Let A be a unital C*-algebra. A unital linear functional $\varphi : A \to \mathbb{C}$ is called a **state** if is positive: $\varphi(x) \geq 0$, for all $x \in A, x \geq 0$.
Example. $\varphi(T) = \langle T(\xi), \xi \rangle$ is a state on $\mathbb{B}(H)$, for $\xi \in H$, $\|\xi\| = 1$.

Exercise 5. Show that a) $|\varphi(y^*x)|^2 \leq \varphi(x^*x)\varphi(y^*y)$, for all $x, y \in A$.
b) φ is bounded and $\|\varphi\| = \varphi(1)$.

States and traces

Definition. Let A be a unital C*-algebra. A unital linear functional $\varphi : A \to \mathbb{C}$ is called a **state** if is positive: $\varphi(x) \geq 0$, for all $x \in A, x \geq 0$.
Example. $\varphi(T) = \langle T(\xi), \xi \rangle$ is a state on $\mathbb{B}(H)$, for $\xi \in H$, $\|\xi\| = 1$.

Exercise 5. Show that a) $|\varphi(y^*x)|^2 \leq \varphi(x^*x)\varphi(y^*y)$, for all $x, y \in A$.
b) φ is bounded and $\|\varphi\| = \varphi(1)$.

Definition

A von Neumann algebra $M \subset \mathbb{B}(H)$ is called **tracial** if there exists a state $\varphi : M \to \mathbb{C}$ (called a **trace**) which is:

1. faithful if $x \geq 0$ and $\varphi(x) = 0$, then $x = 0$.

States and traces

Definition. Let A be a unital C*-algebra. A unital linear functional $\varphi : A \to \mathbb{C}$ is called a **state** if is positive: $\varphi(x) \geq 0$, for all $x \in A, x \geq 0$.
Example. $\varphi(T) = \langle T(\xi), \xi \rangle$ is a state on $\mathbb{B}(H)$, for $\xi \in H$, $\|\xi\| = 1$.

Exercise 5. Show that a) $|\varphi(y^*x)|^2 \leq \varphi(x^*x)\varphi(y^*y)$, for all $x, y \in A$.
b) φ is bounded and $\|\varphi\| = \varphi(1)$.

Definition

A von Neumann algebra $M \subset \mathbb{B}(H)$ is called **tracial** if there exists a state $\varphi : M \to \mathbb{C}$ (called a **trace**) which is:

1. faithful if $x \geq 0$ and $\varphi(x) = 0$, then $x = 0$.
2. normal (or completely additive) $\varphi(\sum_{i \in I} p_i) = \sum_{i \in I} \varphi(p_i)$, for any collection of mutually orthogonal projections $\{p_i\}_{i \in I}$ in M.

Definition. Let A be a unital C*-algebra. A unital linear functional $\varphi : A \to \mathbb{C}$ is called a **state** if is positive: $\varphi(x) \geq 0$, for all $x \in A, x \geq 0$.
Example. $\varphi(T) = \langle T(\xi), \xi \rangle$ is a state on $\mathbb{B}(H)$, for $\xi \in H$, $\|\xi\| = 1$.

Exercise 5. Show that a) $|\varphi(y^*x)|^2 \leq \varphi(x^*x)\varphi(y^*y)$, for all $x, y \in A$.
b) φ is bounded and $\|\varphi\| = \varphi(1)$.

Definition

A von Neumann algebra $M \subset \mathbb{B}(H)$ is called **tracial** if there exists a state $\varphi : M \to \mathbb{C}$ (called a **trace**) which is:

1. faithful if $x \geq 0$ and $\varphi(x) = 0$, then $x = 0$.
2. normal (or completely additive) $\varphi(\sum_{i \in I} p_i) = \sum_{i \in I} \varphi(p_i)$, for any collection of mutually orthogonal projections $\{p_i\}_{i \in I}$ in M.
3. tracial $\varphi(xy) = \varphi(yx)$, for all $x, y \in M$.

States and traces

Definition. Let A be a unital C*-algebra. A unital linear functional $\varphi : A \to \mathbb{C}$ is called a **state** if is positive: $\varphi(x) \geq 0$, for all $x \in A, x \geq 0$.
Example. $\varphi(T) = \langle T(\xi), \xi \rangle$ is a state on $\mathbb{B}(H)$, for $\xi \in H$, $\|\xi\| = 1$.

Exercise 5. Show that a) $|\varphi(y^*x)|^2 \leq \varphi(x^*x)\varphi(y^*y)$, for all $x, y \in A$.
b) φ is bounded and $\|\varphi\| = \varphi(1)$.

Definition

A von Neumann algebra $M \subset \mathbb{B}(H)$ is called **tracial** if there exists a state $\varphi : M \to \mathbb{C}$ (called a **trace**) which is:

1. faithful if $x \geq 0$ and $\varphi(x) = 0$, then $x = 0$.
2. normal (or completely additive) $\varphi(\sum_{i \in I} p_i) = \sum_{i \in I} \varphi(p_i)$, for any collection of mutually orthogonal projections $\{p_i\}_{i \in I}$ in M.
3. tracial $\varphi(xy) = \varphi(yx)$, for all $x, y \in M$.

Examples. 1) $L^\infty(X, \mu)$ with $\varphi(f) = \int_X f \, \mathrm{d}\mu$.
2) $\mathbb{M}_n(\mathbb{C})$ with the usual normalized trace $\varphi([a_{i,j}]) = (a_{1,1} + ... + a_{n,n})/n$.
3) $\mathbb{B}(H)$ is not tracial if $\dim H = \infty$.

Equivalence of projections

Definition

Let $M \subset \mathbb{B}(H)$ be a von Neumann algebra and $p, q \in M$ be projections.

- p is equivalent to q ($p \sim q$) if $\exists v \in M$ such that $v^*v = p, vv^* = q$. (v is an isometry between pH and qH, called a **partial isometry**).

Equivalence of projections

Definition

Let $M \subset \mathbb{B}(H)$ be a von Neumann algebra and $p, q \in M$ be projections.

- p is equivalent to q ($p \sim q$) if $\exists v \in M$ such that $v^*v = p, vv^* = q$. (v is an isometry between pH and qH, called a **partial isometry**).
- p is dominated by q ($p \preceq q$) if $\exists q' \in M$ such that $q' \leq q$ and $p \sim q$

Equivalence of projections

Definition

Let $M \subset \mathbb{B}(H)$ be a von Neumann algebra and $p, q \in M$ be projections.

- p is equivalent to q ($p \sim q$) if $\exists v \in M$ such that $v^*v = p, vv^* = q$. (v is an isometry between pH and qH, called a **partial isometry**).
- p is dominated by q ($p \preceq q$) if $\exists q' \in M$ such that $q' \leq q$ and $p \sim q$

Lemma 1. If $pMq \neq \{0\} \Rightarrow \exists p', q' \in M$ s.t. $p' \leq p, q' \leq q$, and $p' \sim q'$.

Equivalence of projections

Definition

Let $M \subset \mathbb{B}(H)$ be a von Neumann algebra and $p, q \in M$ be projections.

- p is equivalent to q ($p \sim q$) if $\exists v \in M$ such that $v^*v = p, vv^* = q$. (v is an isometry between pH and qH, called a **partial isometry**).
- p is dominated by q ($p \preceq q$) if $\exists q' \in M$ such that $q' \leq q$ and $p \sim q$

Lemma 1. If $pMq \neq \{0\} \Rightarrow \exists p', q' \in M$ s.t. $p' \leq p, q' \leq q$, and $p' \sim q'$. [If $x \in M$ satisfies $pxq \neq 0$, then polar decomposition gives a partial isometry $v \in M$ s.t. $x = v|x|$. One checks that $vv^* \leq p$ and $v^*v \leq q$.]

Equivalence of projections

Definition

Let $M \subset \mathbb{B}(H)$ be a von Neumann algebra and $p, q \in M$ be projections.

- p is equivalent to q ($p \sim q$) if $\exists v \in M$ such that $v^*v = p, vv^* = q$. (v is an isometry between pH and qH, called a **partial isometry**).
- p is dominated by q ($p \preceq q$) if $\exists q' \in M$ such that $q' \leq q$ and $p \sim q$

Lemma 1. If $pMq \neq \{0\} \Rightarrow \exists p', q' \in M$ s.t. $p' \leq p, q' \leq q$, and $p' \sim q'$. [If $x \in M$ satisfies $pxq \neq 0$, then polar decomposition gives a partial isometry $v \in M$ s.t. $x = v|x|$. One checks that $vv^* \leq p$ and $v^*v \leq q$.]

Lemma 2. $\exists$ a projection $z \in \mathcal{Z}(M)$ s.t. $pz \preceq qz$ and $q(1-z) \preceq p(1-z)$.

Equivalence of projections

Definition

Let $M \subset \mathbb{B}(H)$ be a von Neumann algebra and $p, q \in M$ be projections.

- p is equivalent to q ($p \sim q$) if $\exists v \in M$ such that $v^*v = p, vv^* = q$. (v is an isometry between pH and qH, called a **partial isometry**).
- p is dominated by q ($p \preceq q$) if $\exists q' \in M$ such that $q' \leq q$ and $p \sim q$

Lemma 1. If $pMq \neq \{0\} \Rightarrow \exists p', q' \in M$ s.t. $p' \leq p, q' \leq q$, and $p' \sim q'$. [If $x \in M$ satisfies $pxq \neq 0$, then polar decomposition gives a partial isometry $v \in M$ s.t. $x = v|x|$. One checks that $vv^* \leq p$ and $v^*v \leq q$.]

Lemma 2. $\exists$ a projection $z \in \mathcal{Z}(M)$ s.t. $pz \preceq qz$ and $q(1-z) \preceq p(1-z)$.
Proof sketch. Let (p', q') be a maximal pair of projections such that $p', q' \in M, p' \leq p, q' \leq q$ and $p' \sim q'$.

Equivalence of projections

Definition

Let $M \subset \mathbb{B}(H)$ be a von Neumann algebra and $p, q \in M$ be projections.

- p is equivalent to q ($p \sim q$) if $\exists v \in M$ such that $v^*v = p, vv^* = q$. (v is an isometry between pH and qH, called a **partial isometry**).
- p is dominated by q ($p \preceq q$) if $\exists q' \in M$ such that $q' \leq q$ and $p \sim q$

Lemma 1. If $pMq \neq \{0\} \Rightarrow \exists p', q' \in M$ s.t. $p' \leq p, q' \leq q$, and $p' \sim q'$. [If $x \in M$ satisfies $pxq \neq 0$, then polar decomposition gives a partial isometry $v \in M$ s.t. $x = v|x|$. One checks that $vv^* \leq p$ and $v^*v \leq q$.]

Lemma 2. $\exists$ a projection $z \in \mathcal{Z}(M)$ s.t. $pz \preceq qz$ and $q(1-z) \preceq p(1-z)$.
Proof sketch. Let (p', q') be a maximal pair of projections such that $p', q' \in M, p' \leq p, q' \leq q$ and $p' \sim q'$. Since $p - p'$ and $q - q'$ have no equivalent subprojections, Lemma 1 $\Rightarrow (p - p')M(q - q') = \{0\}$.

Equivalence of projections

Definition

Let $M \subset \mathbb{B}(H)$ be a von Neumann algebra and $p, q \in M$ be projections.

- p is equivalent to q ($p \sim q$) if $\exists v \in M$ such that $v^*v = p, vv^* = q$. (v is an isometry between pH and qH, called a **partial isometry**).
- p is dominated by q ($p \preceq q$) if $\exists q' \in M$ such that $q' \leq q$ and $p \sim q$

Lemma 1. If $pMq \neq \{0\} \Rightarrow \exists p', q' \in M$ s.t. $p' \leq p, q' \leq q$, and $p' \sim q'$. [If $x \in M$ satisfies $pxq \neq 0$, then polar decomposition gives a partial isometry $v \in M$ s.t. $x = v|x|$. One checks that $vv^* \leq p$ and $v^*v \leq q$.]

Lemma 2. $\exists$ a projection $z \in \mathcal{Z}(M)$ s.t. $pz \preceq qz$ and $q(1-z) \preceq p(1-z)$.
Proof sketch. Let (p', q') be a maximal pair of projections such that $p', q' \in M, p' \leq p, q' \leq q$ and $p' \sim q'$. Since $p - p'$ and $q - q'$ have no equivalent subprojections, Lemma 1 $\Rightarrow (p-p')M(q-q') = \{0\}$.
Let z be the ortho. proj. onto $\overline{M(q-q')H}$. Then $z \in \mathcal{Z}(M)$.

Equivalence of projections

Definition

Let $M \subset \mathbb{B}(H)$ be a von Neumann algebra and $p, q \in M$ be projections.

- p is equivalent to q ($p \sim q$) if $\exists v \in M$ such that $v^*v = p, vv^* = q$. (v is an isometry between pH and qH, called a **partial isometry**).
- p is dominated by q ($p \preceq q$) if $\exists q' \in M$ such that $q' \leq q$ and $p \sim q$

Lemma 1. If $pMq \neq \{0\} \Rightarrow \exists p', q' \in M$ s.t. $p' \leq p, q' \leq q$, and $p' \sim q'$. [If $x \in M$ satisfies $pxq \neq 0$, then polar decomposition gives a partial isometry $v \in M$ s.t. $x = v|x|$. One checks that $vv^* \leq p$ and $v^*v \leq q$.]

Lemma 2. $\exists$ a projection $z \in \mathcal{Z}(M)$ s.t. $pz \preceq qz$ and $q(1-z) \preceq p(1-z)$.
Proof sketch. Let (p', q') be a maximal pair of projections such that $p', q' \in M, p' \leq p, q' \leq q$ and $p' \sim q'$. Since $p - p'$ and $q - q'$ have no equivalent subprojections, Lemma 1 $\Rightarrow (p - p')M(q - q') = \{0\}$.
Let z be the ortho. proj. onto $\overline{M(q - q')H}$. Then $z \in \mathcal{Z}(M)$.
Since $(p - p')z = 0 \Rightarrow pz = p'z \sim q'z \leq qz \Rightarrow pz \preceq qz$.

Equivalence of projections

Definition

Let $M \subset \mathbb{B}(H)$ be a von Neumann algebra and $p, q \in M$ be projections.

- p is equivalent to q ($p \sim q$) if $\exists v \in M$ such that $v^*v = p, vv^* = q$. (v is an isometry between pH and qH, called a **partial isometry**).
- p is dominated by q ($p \preceq q$) if $\exists q' \in M$ such that $q' \leq q$ and $p \sim q$

Lemma 1. If $pMq \neq \{0\} \Rightarrow \exists p', q' \in M$ s.t. $p' \leq p, q' \leq q$, and $p' \sim q'$. [If $x \in M$ satisfies $pxq \neq 0$, then polar decomposition gives a partial isometry $v \in M$ s.t. $x = v|x|$. One checks that $vv^* \leq p$ and $v^*v \leq q$.]

Lemma 2. $\exists$ a projection $z \in \mathcal{Z}(M)$ s.t. $pz \preceq qz$ and $q(1-z) \preceq p(1-z)$.
Proof sketch. Let (p', q') be a maximal pair of projections such that $p', q' \in M, p' \leq p, q' \leq q$ and $p' \sim q'$. Since $p - p'$ and $q - q'$ have no equivalent subprojections, Lemma 1 $\Rightarrow (p - p')M(q - q') = \{0\}$.
Let z be the ortho. proj. onto $\overline{M(q - q')H}$. Then $z \in \mathcal{Z}(M)$.
Since $(p - p')z = 0 \Rightarrow pz = p'z \sim q'z \leq qz \Rightarrow pz \preceq qz$.
Since $(q - q') \leq z \Rightarrow (q - q')(1 - z) = 0 \Rightarrow q(1 - z) \preceq p(1 - z)$.

Classification into types for factors

A von Neumann algebra M is called a factor if $\mathcal{Z}(M) = M \cap M' = \mathbb{C}1$.

Classification into types for factors

A von Neumann algebra M is called a factor if $\mathcal{Z}(M) = M \cap M' = \mathbb{C}1$.

Corollary. If M is a factor, then $\forall p, q$ projections, either $p \preceq q$ or $q \preceq p$.

Classification into types for factors

A von Neumann algebra M is called a factor if $\mathcal{Z}(M) = M \cap M' = \mathbb{C}1$.
Corollary. If M is a factor, then $\forall p, q$ projections, either $p \preceq q$ or $q \preceq p$.

Note: If H is separable, any vNa M is an integral $M = \int_X M_x \, \mathrm{d}\mu(x)$ of factors $\{M_x\}_{x \in X}$ "indexed" over a probability space (X, μ).

Definition. A projection p in a von Neumann algebra M is

- minimal if $q \in M$, $q \leq p \Rightarrow q \in \{0, p\}$.
- finite if $q \in M, q \leq p$, $q \sim p \Rightarrow q = p$.

Classification into types for factors

A von Neumann algebra M is called a factor if $\mathcal{Z}(M) = M \cap M' = \mathbb{C}1$.
Corollary. If M is a factor, then $\forall p, q$ projections, either $p \preceq q$ or $q \preceq p$.

Note: If H is separable, any vNa M is an integral $M = \int_X M_x \, \mathrm{d}\mu(x)$ of factors $\{M_x\}_{x \in X}$ "indexed" over a probability space (X, μ).

Definition. A projection p in a von Neumann algebra M is

- minimal if $q \in M$, $q \leq p \Rightarrow q \in \{0, p\}$.
- finite if $q \in M, q \leq p$, $q \sim p \Rightarrow q = p$.

Remark. Any minimal projection is finite. If $M = \mathbb{B}(H)$, then a projection p is minimal iff $\dim(pH) \in \{0, 1\}$ and finite iff $\dim(pH) < \infty$.

Classification into types for factors

A von Neumann algebra M is called a factor if $\mathcal{Z}(M) = M \cap M' = \mathbb{C}1$.
Corollary. If M is a factor, then $\forall p, q$ projections, either $p \preceq q$ or $q \preceq p$.

Note: If H is separable, any vNa M is an integral $M = \int_X M_x \, \mathrm{d}\mu(x)$ of factors $\{M_x\}_{x \in X}$ "indexed" over a probability space (X, μ).

Definition. A projection p in a von Neumann algebra M is

- minimal if $q \in M$, $q \leq p \Rightarrow q \in \{0, p\}$.
- finite if $q \in M, q \leq p$, $q \sim p \Rightarrow q = p$.

Remark. Any minimal projection is finite. If $M = \mathbb{B}(H)$, then a projection p is minimal iff $\dim(pH) \in \{0, 1\}$ and finite iff $\dim(pH) < \infty$.

A factor $M \subset \mathbb{B}(H)$ is called of

1. **type I** if it has a non-zero minimal projection.
2. **type II** if it has a non-zero finite projection, but not minimal proj. If $\mathbf{1} \in M$ is finite, M is of **type II**$_1$; otherwise, M is of **type II**$_\infty$.
3. **type III** if it contains no non-zero finite projection.

Examples of factors

- $\mathbb{B}(K)$ is a factor of type I, for any Hilbert space K.
- Any factor M of type I is isomorphic to $\mathbb{B}(K)$, for a Hilbert space K.

Examples of factors

- $\mathbb{B}(K)$ is a factor of type I, for any Hilbert space K.
- Any factor M of type I is isomorphic to $\mathbb{B}(K)$, for a Hilbert space K.
  Proof. Let $p \in M$ be a minimal projection. Let $\{p_i\}_{i \in I}$ be a maximal family of mutually orthogonal proj. in M which are equivalent to p.

Examples of factors

- $\mathbb{B}(K)$ is a factor of type I, for any Hilbert space K.
- Any factor M of type I is isomorphic to $\mathbb{B}(K)$, for a Hilbert space K.
  Proof. Let $p \in M$ be a minimal projection. Let $\{p_i\}_{i \in I}$ be a maximal family of mutually orthogonal proj. in M which are equivalent to p. Put $q = 1 - \sum_{i \in I} p_i$. Then $p \preceq q$ or $q \preceq p$. Maximality $\Rightarrow q \preceq p$.

Examples of factors

- $\mathbb{B}(K)$ is a factor of type I, for any Hilbert space K.
- Any factor M of type I is isomorphic to $\mathbb{B}(K)$, for a Hilbert space K.
  Proof. Let $p \in M$ be a minimal projection. Let $\{p_i\}_{i\in I}$ be a maximal family of mutually orthogonal proj. in M which are equivalent to p. Put $q = 1 - \sum_{i\in I} p_i$. Then $p \preceq q$ or $q \preceq p$. Maximality $\Rightarrow q \preceq p$. Since p is minimal $\Rightarrow q = 0$ or $q = p$ (impossible by maximality).

Examples of factors

- $\mathbb{B}(K)$ is a factor of type I, for any Hilbert space K.
- Any factor M of type I is isomorphic to $\mathbb{B}(K)$, for a Hilbert space K.
  Proof. Let $p \in M$ be a minimal projection. Let $\{p_i\}_{i \in I}$ be a maximal family of mutually orthogonal proj. in M which are equivalent to p. Put $q = 1 - \sum_{i \in I} p_i$. Then $p \preceq q$ or $q \preceq p$. Maximality $\Rightarrow q \preceq p$. Since p is minimal $\Rightarrow q = 0$ or $q = p$ (impossible by maximality). Thus, $q = 0$ and hence $\sum_{i \in I} p_i = 1$. **(a)**

Examples of factors

- $\mathbb{B}(K)$ is a factor of type I, for any Hilbert space K.
- Any factor M of type I is isomorphic to $\mathbb{B}(K)$, for a Hilbert space K.
  Proof. Let $p \in M$ be a minimal projection. Let $\{p_i\}_{i \in I}$ be a maximal family of mutually orthogonal proj. in M which are equivalent to p.
  Put $q = 1 - \sum_{i \in I} p_i$. Then $p \preceq q$ or $q \preceq p$. Maximality $\Rightarrow$ $q \preceq p$.
  Since p is minimal $\Rightarrow$ $q = 0$ or $q = p$ (impossible by maximality).
  Thus, $q = 0$ and hence $\sum_{i \in I} p_i = 1$. **(a)**
  Since p is minimal $\Rightarrow$ $pMp = \mathbb{C}p$. **(b)**

Examples of factors

- $\mathbb{B}(K)$ is a factor of type I, for any Hilbert space K.
- Any factor M of type I is isomorphic to $\mathbb{B}(K)$, for a Hilbert space K.
  Proof. Let $p \in M$ be a minimal projection. Let $\{p_i\}_{i\in I}$ be a maximal family of mutually orthogonal proj. in M which are equivalent to p.
  Put $q = 1 - \sum_{i\in I} p_i$. Then $p \preceq q$ or $q \preceq p$. Maximality $\Rightarrow q \preceq p$.
  Since p is minimal $\Rightarrow q = 0$ or $q = p$ (impossible by maximality).
  Thus, $q = 0$ and hence $\sum_{i\in I} p_i = 1$. **(a)**
  Since p is minimal $\Rightarrow pMp = \mathbb{C}p$. **(b)**
  Combining **(a)** and **(b)** $\Rightarrow M \cong \mathbb{B}(\ell^2 I)$.

Examples of factors

- $\mathbb{B}(K)$ is a factor of type I, for any Hilbert space K.
- Any factor M of type I is isomorphic to $\mathbb{B}(K)$, for a Hilbert space K.
  Proof. Let $p \in M$ be a minimal projection. Let $\{p_i\}_{i\in I}$ be a maximal family of mutually orthogonal proj. in M which are equivalent to p.
  Put $q = 1 - \sum_{i\in I} p_i$. Then $p \preceq q$ or $q \preceq p$. Maximality $\Rightarrow q \preceq p$.
  Since p is minimal $\Rightarrow q = 0$ or $q = p$ (impossible by maximality).
  Thus, $q = 0$ and hence $\sum_{i\in I} p_i = 1$. **(a)**
  Since p is minimal $\Rightarrow pMp = \mathbb{C}p$. **(b)**
  Combining **(a)** and **(b)** $\Rightarrow M \cong \mathbb{B}(\ell^2 I)$.

  Let Γ an infinite countable group, (X, μ) non-atomic measure space.

Examples of factors

- $\mathbb{B}(K)$ is a factor of type I, for any Hilbert space K.
- Any factor M of type I is isomorphic to $\mathbb{B}(K)$, for a Hilbert space K.
  Proof. Let $p \in M$ be a minimal projection. Let $\{p_i\}_{i \in I}$ be a maximal family of mutually orthogonal proj. in M which are equivalent to p.
  Put $q = 1 - \sum_{i \in I} p_i$. Then $p \preceq q$ or $q \preceq p$. Maximality $\Rightarrow q \preceq p$.
  Since p is minimal $\Rightarrow q = 0$ or $q = p$ (impossible by maximality).
  Thus, $q = 0$ and hence $\sum_{i \in I} p_i = 1$. **(a)**
  Since p is minimal $\Rightarrow pMp = \mathbb{C}p$. **(b)**
  Combining **(a)** and **(b)** $\Rightarrow M \cong \mathbb{B}(\ell^2 I)$.

  Let Γ an infinite countable group, (X, μ) non-atomic measure space.
- Γ infinite conjugacy class $\rightsquigarrow$ II_1 factor $L(\Gamma)$.

Examples of factors

- $\mathbb{B}(K)$ is a factor of type I, for any Hilbert space K.
- Any factor M of type I is isomorphic to $\mathbb{B}(K)$, for a Hilbert space K.
  Proof. Let $p \in M$ be a minimal projection. Let $\{p_i\}_{i \in I}$ be a maximal family of mutually orthogonal proj. in M which are equivalent to p.
  Put $q = 1 - \sum_{i \in I} p_i$. Then $p \preceq q$ or $q \preceq p$. Maximality $\Rightarrow q \preceq p$.
  Since p is minimal $\Rightarrow q = 0$ or $q = p$ (impossible by maximality).
  Thus, $q = 0$ and hence $\sum_{i \in I} p_i = 1$. **(a)**
  Since p is minimal $\Rightarrow pMp = \mathbb{C}p$. **(b)**
  Combining **(a)** and **(b)** $\Rightarrow M \cong \mathbb{B}(\ell^2 I)$.

  Let Γ an infinite countable group, (X, μ) non-atomic measure space.
- Γ infinite conjugacy class $\rightsquigarrow$ II_1 factor $L(\Gamma)$.
- $\Gamma \curvearrowright (X, \mu)$ non-singular free ergodic action $\rightsquigarrow$ factor $L^\infty(X) \rtimes \Gamma$ of

Examples of factors

- $\mathbb{B}(K)$ is a factor of type I, for any Hilbert space K.
- Any factor M of type I is isomorphic to $\mathbb{B}(K)$, for a Hilbert space K.
  Proof. Let $p \in M$ be a minimal projection. Let $\{p_i\}_{i \in I}$ be a maximal family of mutually orthogonal proj. in M which are equivalent to p.
  Put $q = 1 - \sum_{i \in I} p_i$. Then $p \preceq q$ or $q \preceq p$. Maximality $\Rightarrow q \preceq p$.
  Since p is minimal $\Rightarrow q = 0$ or $q = p$ (impossible by maximality).
  Thus, $q = 0$ and hence $\sum_{i \in I} p_i = 1$. **(a)**
  Since p is minimal $\Rightarrow pMp = \mathbb{C}p$. **(b)**
  Combining **(a)** and **(b)** $\Rightarrow M \cong \mathbb{B}(\ell^2 I)$.

  Let Γ an infinite countable group, (X, μ) non-atomic measure space.
- Γ infinite conjugacy class $\rightsquigarrow$ II_1 factor $L(\Gamma)$.
- $\Gamma \curvearrowright (X, \mu)$ non-singular free ergodic action $\rightsquigarrow$ factor $L^\infty(X) \rtimes \Gamma$ of
  1. type II_1 iff $\exists$ Γ-invariant probability measure on X equivalent to μ.

Examples of factors

- $\mathbb{B}(K)$ is a factor of type I, for any Hilbert space K.
- Any factor M of type I is isomorphic to $\mathbb{B}(K)$, for a Hilbert space K.
  Proof. Let $p \in M$ be a minimal projection. Let $\{p_i\}_{i\in I}$ be a maximal family of mutually orthogonal proj. in M which are equivalent to p.
  Put $q = 1 - \sum_{i\in I} p_i$. Then $p \preceq q$ or $q \preceq p$. Maximality $\Rightarrow q \preceq p$.
  Since p is minimal $\Rightarrow q = 0$ or $q = p$ (impossible by maximality).
  Thus, $q = 0$ and hence $\sum_{i\in I} p_i = 1$. **(a)**
  Since p is minimal $\Rightarrow pMp = \mathbb{C}p$. **(b)**
  Combining **(a)** and **(b)** $\Rightarrow M \cong \mathbb{B}(\ell^2 I)$.

  Let Γ an infinite countable group, (X, μ) non-atomic measure space.
- Γ infinite conjugacy class $\rightsquigarrow$ II_1 factor $L(\Gamma)$.
- $\Gamma \curvearrowright (X, \mu)$ non-singular free ergodic action $\rightsquigarrow$ factor $L^\infty(X) \rtimes \Gamma$ of
  1. type II_1 iff $\exists$ Γ-invariant probability measure on X equivalent to μ.
  2. type II_∞ iff $\exists$ Γ-invariant infinite measure on X equivalent to ν.

Examples of factors

- $\mathbb{B}(K)$ is a factor of type I, for any Hilbert space K.
- Any factor M of type I is isomorphic to $\mathbb{B}(K)$, for a Hilbert space K.
  Proof. Let $p \in M$ be a minimal projection. Let $\{p_i\}_{i\in I}$ be a maximal family of mutually orthogonal proj. in M which are equivalent to p.
  Put $q = 1 - \sum_{i\in I} p_i$. Then $p \preceq q$ or $q \preceq p$. Maximality $\Rightarrow q \preceq p$.
  Since p is minimal $\Rightarrow q = 0$ or $q = p$ (impossible by maximality).
  Thus, $q = 0$ and hence $\sum_{i\in I} p_i = 1$. **(a)**
  Since p is minimal $\Rightarrow pMp = \mathbb{C}p$. **(b)**
  Combining **(a)** and **(b)** $\Rightarrow M \cong \mathbb{B}(\ell^2 I)$.

  Let Γ an infinite countable group, (X, μ) non-atomic measure space.

- Γ infinite conjugacy class $\rightsquigarrow$ II_1 factor $L(\Gamma)$.
- $\Gamma \curvearrowright (X, \mu)$ non-singular free ergodic action $\rightsquigarrow$ factor $L^\infty(X) \rtimes \Gamma$ of
  1. type II_1 iff $\exists$ Γ-invariant probability measure on X equivalent to μ.
  2. type II_∞ iff $\exists$ Γ-invariant infinite measure on X equivalent to ν.
  3. type III iff no measure on X equivalent to μ is Γ-invariant.

II_1 factors are tracial

Theorem

A factor M is of type $\mathrm{II}_1 \iff$ it is an infinite dimensional tracial factor.

II_1 factors are tracial

Theorem

A factor M is of type II_1 $\iff$ it is an infinite dimensional tracial factor.

Moreover, if M is a II_1 factor, then $\exists!$ tracial faithful normal state $\tau : M \to \mathbb{C}$ (called the **trace** of M).

II_1 factors are tracial

Theorem

A factor M is of type II_1 $\iff$ it is an infinite dimensional tracial factor.

Moreover, if M is a II_1 factor, then $\exists!$ tracial faithful normal state $\tau : M \to \mathbb{C}$ (called the **trace** of M).

Proof of $\Leftarrow$. Assume that $\exists$ a trace $\tau : M \to \mathbb{C}$.

II_1 factors are tracial

Theorem

A factor M is of type II_1 $\iff$ it is an infinite dimensional tracial factor.

Moreover, if M is a II_1 factor, then $\exists!$ tracial faithful normal state $\tau : M \to \mathbb{C}$ (called the **trace** of M).

Proof of $\Leftarrow$. Assume that $\exists$ a trace $\tau : M \to \mathbb{C}$.
Goal. M is type II_1: **(a)** $1 \in M$ is a finite projection and **(b)** M not type I.

II_1 factors are tracial

Theorem

A factor M is of type II_1 $\iff$ it is an infinite dimensional tracial factor.

Moreover, if M is a II_1 factor, then $\exists!$ tracial faithful normal state $\tau : M \to \mathbb{C}$ (called the **trace** of M).

Proof of $\Leftarrow$. Assume that $\exists$ a trace $\tau : M \to \mathbb{C}$.

Goal. M is type II_1: **(a)** $1 \in M$ is a finite projection and **(b)** M not type I.

(a) Let $p \in M$ such that $p \sim 1 \Rightarrow \exists\ v \in M$ with $v^*v = 1, vv^* = p$.

II_1 factors are tracial

Theorem

A factor M is of type II_1 $\iff$ it is an infinite dimensional tracial factor.

Moreover, if M is a II_1 factor, then $\exists!$ tracial faithful normal state $\tau : M \to \mathbb{C}$ (called the **trace** of M).

Proof of $\Leftarrow$. Assume that $\exists$ a trace $\tau : M \to \mathbb{C}$.
Goal. M is type II_1: **(a)** $1 \in M$ is a finite projection and **(b)** M not type I.
(a) Let $p \in M$ such that $p \sim 1 \Rightarrow \exists\ v \in M$ with $v^*v = 1, vv^* = p$.
Thus, $\tau(1-p) = \tau(v^*v - vv^*) = \tau(v^*v) - \tau(vv^*) = 0$.

II_1 factors are tracial

Theorem

A factor M is of type $\mathrm{II}_1 \iff$ it is an infinite dimensional tracial factor.

Moreover, if M is a II_1 factor, then $\exists!$ tracial faithful normal state $\tau : M \to \mathbb{C}$ (called the **trace** of M).

Proof of $\Leftarrow$. Assume that $\exists$ a trace $\tau : M \to \mathbb{C}$.
Goal. M is type II_1: **(a)** $1 \in M$ is a finite projection and **(b)** M not type I.
(a) Let $p \in M$ such that $p \sim 1 \Rightarrow \exists\ v \in M$ with $v^*v = 1, vv^* = p$.
Thus, $\tau(1-p) = \tau(v^*v - vv^*) = \tau(v^*v) - \tau(vv^*) = 0$.
Since $1 - p \geq 0$ and τ is faithful $\Rightarrow p = 1 \Rightarrow 1$ is finite.

II_1 factors are tracial

Theorem

A factor M is of type $\mathrm{II}_1 \iff$ it is an infinite dimensional tracial factor.

Moreover, if M is a II_1 factor, then $\exists!$ tracial faithful normal state $\tau : M \to \mathbb{C}$ (called the **trace** of M).

Proof of $\Leftarrow$. Assume that $\exists$ a trace $\tau : M \to \mathbb{C}$.
Goal. M is type II_1: **(a)** $1 \in M$ is a finite projection and **(b)** M not type I.
(a) Let $p \in M$ such that $p \sim 1 \Rightarrow \exists\ v \in M$ with $v^*v = 1, vv^* = p$.
Thus, $\tau(1-p) = \tau(v^*v - vv^*) = \tau(v^*v) - \tau(vv^*) = 0$.
Since $1 - p \geq 0$ and τ is faithful $\Rightarrow p = 1 \Rightarrow 1$ is finite. **(b)** If M is type I, since $\dim M = \infty \Rightarrow M \cong \mathbb{B}(K)$, where $\dim K = \infty$. But $1 \in \mathbb{B}(K)$, with $\dim K = \infty$, is not a finite projection, contradiction.

II_1 factors are tracial

Theorem

A factor M is of type II_1 $\iff$ it is an infinite dimensional tracial factor.

Moreover, if M is a II_1 factor, then $\exists!$ tracial faithful normal state $\tau : M \to \mathbb{C}$ (called the **trace** of M).

Proof of $\Leftarrow$. Assume that $\exists$ a trace $\tau : M \to \mathbb{C}$.
Goal. M is type II_1: **(a)** $1 \in M$ is a finite projection and **(b)** M not type I.
(a) Let $p \in M$ such that $p \sim 1 \Rightarrow \exists\ v \in M$ with $v^*v = 1, vv^* = p$.
Thus, $\tau(1-p) = \tau(v^*v - vv^*) = \tau(v^*v) - \tau(vv^*) = 0$.
Since $1 - p \geq 0$ and τ is faithful $\Rightarrow p = 1 \Rightarrow 1$ is finite. **(b)** If M is type I, since $\dim M = \infty \Rightarrow M \cong \mathbb{B}(K)$, where $\dim K = \infty$. But $1 \in \mathbb{B}(K)$, with $\dim K = \infty$, is not a finite projection, contradiction.

Notation. If M is a II_1 factor and τ its trace, we put $\|x\|_2 := \tau(x^*x)^{1/2}$.
$L^2(M)$ is the closure of M w.r.t. $\|.\|_2$.

II_1 factors are tracial

Theorem

A factor M is of type $\mathrm{II}_1 \iff$ it is an infinite dimensional tracial factor.

Moreover, if M is a II_1 factor, then $\exists !$ tracial faithful normal state $\tau : M \to \mathbb{C}$ (called the **trace** of M).

Proof of $\Leftarrow$. Assume that $\exists$ a trace $\tau : M \to \mathbb{C}$.
Goal. M is type II_1: **(a)** $1 \in M$ is a finite projection and **(b)** M not type I.
(a) Let $p \in M$ such that $p \sim 1 \Rightarrow \exists\ v \in M$ with $v^*v = 1, vv^* = p$.
Thus, $\tau(1-p) = \tau(v^*v - vv^*) = \tau(v^*v) - \tau(vv^*) = 0$.
Since $1 - p \geq 0$ and τ is faithful $\Rightarrow p = 1 \Rightarrow 1$ is finite. **(b)** If M is type I, since $\dim M = \infty \Rightarrow M \cong \mathbb{B}(K)$, where $\dim K = \infty$. But $1 \in \mathbb{B}(K)$, with $\dim K = \infty$, is not a finite projection, contradiction.

Notation. If M is a II_1 factor and τ its trace, we put $\|x\|_2 := \tau(x^*x)^{1/2}$.
$L^2(M)$ is the closure of M w.r.t. $\|.\|_2$.
We have an injective SOT-cont. $*$-homomorphism $\pi : M \to \mathbb{B}(L^2(M))$ given by $\pi(x)(y) = xy$, for $x, y \in M$. (the standard representation)

Let Γ be a countable group.

Group von Neumann algebras, I

Let Γ be a countable group.
Put $\ell^2\Gamma = \{f : \Gamma \to \mathbb{C} |\ \|f\|_2 := (\sum_{g\in\Gamma} |f(g)|^2)^{1/2} < \infty\}$.

Group von Neumann algebras, I

Let Γ be a countable group.
Put $\ell^2\Gamma = \{f : \Gamma \to \mathbb{C} |\ \|f\|_2 := (\sum_{g\in\Gamma} |f(g)|^2)^{1/2} < \infty\}$.
An orthonormal basis for $\ell^2\Gamma$ is $\{\delta_g\}_{g\in\Gamma}$.

Group von Neumann algebras, I

Let Γ be a countable group.
Put $\ell^2\Gamma = \{f : \Gamma \to \mathbb{C} | \; \|f\|_2 := (\sum_{g\in\Gamma} |f(g)|^2)^{1/2} < \infty\}$.
An orthonormal basis for $\ell^2\Gamma$ is $\{\delta_g\}_{g\in\Gamma}$.

The **left regular representation** $\lambda : \Gamma \to \mathcal{U}(\ell^2\Gamma)$ is given by $\lambda(g)(\delta_h) = \delta_{gh}$, for all $g, h \in \Gamma$.

Group von Neumann algebras, I

Let Γ be a countable group.
Put $\ell^2\Gamma = \{f : \Gamma \to \mathbb{C} | \ \|f\|_2 := (\sum_{g\in\Gamma} |f(g)|^2)^{1/2} < \infty\}$.
An orthonormal basis for $\ell^2\Gamma$ is $\{\delta_g\}_{g\in\Gamma}$.

The **left regular representation** $\lambda : \Gamma \to \mathcal{U}(\ell^2\Gamma)$ is given by $\lambda(g)(\delta_h) = \delta_{gh}$, for all $g, h \in \Gamma$. **Note.** λ is a unitary representation: $\lambda(g)$ is a unitary operator and $\lambda(gh) = \lambda(g)\lambda(h)$, for all $g, h \in \Gamma$.

Group von Neumann algebras, I

Let Γ be a countable group.
Put $\ell^2\Gamma = \{f : \Gamma \to \mathbb{C} |\ \|f\|_2 := (\sum_{g\in\Gamma} |f(g)|^2)^{1/2} < \infty\}$.
An orthonormal basis for $\ell^2\Gamma$ is $\{\delta_g\}_{g\in\Gamma}$.

The **left regular representation** $\lambda : \Gamma \to \mathcal{U}(\ell^2\Gamma)$ is given by $\lambda(g)(\delta_h) = \delta_{gh}$, for all $g, h \in \Gamma$. **Note.** λ is a unitary representation: $\lambda(g)$ is a unitary operator and $\lambda(gh) = \lambda(g)\lambda(h)$, for all $g, h \in \Gamma$.

Define $\mathcal{A} := \{\sum_{g\in F} a_g\lambda(g) |\ F \subset \Gamma \text{ finite}, a_g \in \mathbb{C}, \forall g \in F\}$.

Group von Neumann algebras, I

Let Γ be a countable group.
Put $\ell^2\Gamma = \{f : \Gamma \to \mathbb{C} | \ \|f\|_2 := (\sum_{g\in\Gamma} |f(g)|^2)^{1/2} < \infty\}$.
An orthonormal basis for $\ell^2\Gamma$ is $\{\delta_g\}_{g\in\Gamma}$.

The **left regular representation** $\lambda : \Gamma \to \mathcal{U}(\ell^2\Gamma)$ is given by $\lambda(g)(\delta_h) = \delta_{gh}$, for all $g, h \in \Gamma$. **Note.** λ is a unitary representation: $\lambda(g)$ is a unitary operator and $\lambda(gh) = \lambda(g)\lambda(h)$, for all $g, h \in \Gamma$.

Define $\mathcal{A} := \{\sum_{g\in F} a_g\lambda(g) | \ F \subset \Gamma \text{ finite}, a_g \in \mathbb{C}, \forall g \in F\}$.
Then $\mathcal{A} \subset \mathbb{B}(\ell^2\Gamma)$ is a $*$-subalgebra.
Remark. $\mathcal{A}$ is isomorphic to the complex group algebra $\mathbb{C}[\Gamma]$.

Group von Neumann algebras, I

Let Γ be a countable group.
Put $\ell^2\Gamma = \{f : \Gamma \to \mathbb{C} | \; \|f\|_2 := (\sum_{g\in\Gamma} |f(g)|^2)^{1/2} < \infty\}$.
An orthonormal basis for $\ell^2\Gamma$ is $\{\delta_g\}_{g\in\Gamma}$.

The **left regular representation** $\lambda : \Gamma \to \mathcal{U}(\ell^2\Gamma)$ is given by $\lambda(g)(\delta_h) = \delta_{gh}$, for all $g, h \in \Gamma$. **Note.** λ is a unitary representation: $\lambda(g)$ is a unitary operator and $\lambda(gh) = \lambda(g)\lambda(h)$, for all $g, h \in \Gamma$.

Define $\mathcal{A} := \{\sum_{g\in F} a_g\lambda(g) | \; F \subset \Gamma \text{ finite}, a_g \in \mathbb{C}, \forall g \in F\}$.
Then $\mathcal{A} \subset \mathbb{B}(\ell^2\Gamma)$ is a $*$-subalgebra.
Remark. $\mathcal{A}$ is isomorphic to the complex group algebra $\mathbb{C}[\Gamma]$.

Definition

$L(\Gamma) := \overline{\mathcal{A}}^{WOT} \subset \mathbb{B}(\ell^2\Gamma)$ is called the **group von Neumann algebra of** Γ.

Group von Neumann algebras, II

Proposition 1

$\tau : L(\Gamma) \to \mathbb{C}$ given by $\tau(x) = \langle x(\delta_e), \delta_e \rangle$ is a trace.

Proof. τ is a state, since $\|\delta_e\| = 1$.

Group von Neumann algebras, II

Proposition 1

$\tau : L(\Gamma) \to \mathbb{C}$ given by $\tau(x) = \langle x(\delta_e), \delta_e \rangle$ is a trace.

Proof. τ is a state, since $\|\delta_e\| = 1$.

τ is faithful: let $x \in L(\Gamma)$, $x \geq 0$ s.t. $\tau(x) = 0$. Let $y \in L(\Gamma)$ s.t. $x = y^* y$.

Group von Neumann algebras, II

Proposition 1

$\tau : L(\Gamma) \to \mathbb{C}$ given by $\tau(x) = \langle x(\delta_e), \delta_e \rangle$ is a trace.

Proof. τ is a state, since $\|\delta_e\| = 1$.

τ is faithful: let $x \in L(\Gamma)$, $x \geq 0$ s.t. $\tau(x) = 0$. Let $y \in L(\Gamma)$ s.t. $x = y^* y$. Then $0 = \tau(x) = \langle x\delta_e, \delta_e \rangle = \langle y^* y \delta_e, \delta_e \rangle = \|y\delta_e\|^2 \Rightarrow y\delta_e = 0 \Rightarrow x\delta_e = 0$.

Group von Neumann algebras, II

Proposition 1

$\tau : L(\Gamma) \to \mathbb{C}$ given by $\tau(x) = \langle x(\delta_e), \delta_e \rangle$ is a trace.

Proof. τ is a state, since $\|\delta_e\| = 1$.

τ is faithful: let $x \in L(\Gamma)$, $x \geq 0$ s.t. $\tau(x) = 0$. Let $y \in L(\Gamma)$ s.t. $x = y^* y$.
Then $0 = \tau(x) = \langle x\delta_e, \delta_e \rangle = \langle y^* y \delta_e, \delta_e \rangle = \|y\delta_e\|^2 \Rightarrow y\delta_e = 0 \Rightarrow x\delta_e = 0$.

Define the **right regular rep.** $\rho : \Gamma \to \mathcal{U}(\ell^2\Gamma)$ by $\rho(k)(\delta_h) = \delta_{hk^{-1}}$.
Then $\rho(k)$ commutes with $\lambda(g)$, for all $g \in \Gamma \Rightarrow \rho(k) \in L(\Gamma)'$.

Group von Neumann algebras, II

Proposition 1

$\tau : L(\Gamma) \to \mathbb{C}$ given by $\tau(x) = \langle x(\delta_e), \delta_e \rangle$ is a trace.

Proof. τ is a state, since $\|\delta_e\| = 1$.

τ is faithful: let $x \in L(\Gamma)$, $x \geq 0$ s.t. $\tau(x) = 0$. Let $y \in L(\Gamma)$ s.t. $x = y^* y$.
Then $0 = \tau(x) = \langle x\delta_e, \delta_e \rangle = \langle y^* y \delta_e, \delta_e \rangle = \|y\delta_e\|^2 \Rightarrow y\delta_e = 0 \Rightarrow x\delta_e = 0$.
Define the **right regular rep.** $\rho : \Gamma \to \mathcal{U}(\ell^2\Gamma)$ by $\rho(k)(\delta_h) = \delta_{hk^{-1}}$.
Then $\rho(k)$ commutes with $\lambda(g)$, for all $g \in \Gamma \Rightarrow \rho(k) \in L(\Gamma)'$.
Hence $x(\delta_k) = x(\rho(k^{-1})\delta_e) = \rho(k^{-1})(x\delta_e) = 0$, for all $k \in \Gamma \Rightarrow x = 0$.

Proposition 1

$\tau : L(\Gamma) \to \mathbb{C}$ given by $\tau(x) = \langle x(\delta_e), \delta_e \rangle$ is a trace.

Proof. τ is a state, since $\|\delta_e\| = 1$.

τ is faithful: let $x \in L(\Gamma)$, $x \geq 0$ s.t. $\tau(x) = 0$. Let $y \in L(\Gamma)$ s.t. $x = y^* y$.
Then $0 = \tau(x) = \langle x\delta_e, \delta_e \rangle = \langle y^* y \delta_e, \delta_e \rangle = \|y\delta_e\|^2 \Rightarrow y\delta_e = 0 \Rightarrow x\delta_e = 0$.

Define the **right regular rep.** $\rho : \Gamma \to \mathcal{U}(\ell^2\Gamma)$ by $\rho(k)(\delta_h) = \delta_{hk^{-1}}$.
Then $\rho(k)$ commutes with $\lambda(g)$, for all $g \in \Gamma \Rightarrow \rho(k) \in L(\Gamma)'$.
Hence $x(\delta_k) = x(\rho(k^{-1})\delta_e) = \rho(k^{-1})(x\delta_e) = 0$, for all $k \in \Gamma \Rightarrow x = 0$.

τ is tracial: if $g \in \Gamma$, then $\tau(\lambda(g)) = \langle \lambda(g)\delta_e, \delta_e \rangle = \langle \delta_g, \delta_e \rangle = \delta_{g,e}$.

Group von Neumann algebras, II

Proposition 1

$\tau : L(\Gamma) \to \mathbb{C}$ given by $\tau(x) = \langle x(\delta_e), \delta_e \rangle$ is a trace.

Proof. τ is a state, since $\|\delta_e\| = 1$.

τ is faithful: let $x \in L(\Gamma)$, $x \geq 0$ s.t. $\tau(x) = 0$. Let $y \in L(\Gamma)$ s.t. $x = y^*y$.
Then $0 = \tau(x) = \langle x\delta_e, \delta_e \rangle = \langle y^*y\delta_e, \delta_e \rangle = \|y\delta_e\|^2 \Rightarrow y\delta_e = 0 \Rightarrow x\delta_e = 0$.
Define the **right regular rep.** $\rho : \Gamma \to \mathcal{U}(\ell^2\Gamma)$ by $\rho(k)(\delta_h) = \delta_{hk^{-1}}$.
Then $\rho(k)$ commutes with $\lambda(g)$, for all $g \in \Gamma \Rightarrow \rho(k) \in L(\Gamma)'$.
Hence $x(\delta_k) = x(\rho(k^{-1})\delta_e) = \rho(k^{-1})(x\delta_e) = 0$, for all $k \in \Gamma \Rightarrow x = 0$.

τ is tracial: if $g \in \Gamma$, then $\tau(\lambda(g)) = \langle \lambda(g)\delta_e, \delta_e \rangle = \langle \delta_g, \delta_e \rangle = \delta_{g,e}$.
Thus, if $g, h \in \Gamma$, we have $\tau(\lambda(g)\lambda(h)) = \delta_{gh,e} = \delta_{hg,e} = \tau(\lambda(h)\lambda(g))$.
Since τ is WOT-continuous $\Rightarrow \tau$ is tracial.

Group von Neumann algebras, II

Proposition 1

$\tau : L(\Gamma) \to \mathbb{C}$ given by $\tau(x) = \langle x(\delta_e), \delta_e \rangle$ is a trace.

Proof. τ is a state, since $\|\delta_e\| = 1$.

τ is faithful: let $x \in L(\Gamma)$, $x \geq 0$ s.t. $\tau(x) = 0$. Let $y \in L(\Gamma)$ s.t. $x = y^* y$.
Then $0 = \tau(x) = \langle x\delta_e, \delta_e \rangle = \langle y^* y \delta_e, \delta_e \rangle = \|y\delta_e\|^2 \Rightarrow y\delta_e = 0 \Rightarrow x\delta_e = 0$.
Define the **right regular rep.** $\rho : \Gamma \to \mathcal{U}(\ell^2\Gamma)$ by $\rho(k)(\delta_h) = \delta_{hk^{-1}}$.
Then $\rho(k)$ commutes with $\lambda(g)$, for all $g \in \Gamma \Rightarrow \rho(k) \in L(\Gamma)'$.
Hence $x(\delta_k) = x(\rho(k^{-1})\delta_e) = \rho(k^{-1})(x\delta_e) = 0$, for all $k \in \Gamma \Rightarrow x = 0$.

τ is tracial: if $g \in \Gamma$, then $\tau(\lambda(g)) = \langle \lambda(g)\delta_e, \delta_e \rangle = \langle \delta_g, \delta_e \rangle = \delta_{g,e}$.
Thus, if $g, h \in \Gamma$, we have $\tau(\lambda(g)\lambda(h)) = \delta_{gh,e} = \delta_{hg,e} = \tau(\lambda(h)\lambda(g))$.
Since τ is WOT-continuous $\Rightarrow \tau$ is tracial.

Since τ is normal, we are done.

Group von Neumann algebras, III

Proposition 2

Let Γ be a countable **abelian** group. Let $\hat{\Gamma}$ be the Pontryagin dual of Γ, i.e. $\hat{\Gamma} = \{h : \Gamma \to \mathbb{C} |\ h \text{ homomorphism such that } |h(g)| = 1, \forall g \in \Gamma\}$.

Group von Neumann algebras, III

Proposition 2

Let Γ be a countable **abelian** group. Let $\hat{\Gamma}$ be the Pontryagin dual of Γ, i.e. $\hat{\Gamma} = \{h : \Gamma \to \mathbb{C} |\ h \text{ homomorphism such that } |h(g)| = 1, \forall g \in \Gamma\}$.
(**Note.** $\hat{\Gamma}$ is a compact metrizable group.)

Proposition 2

Let Γ be a countable **abelian** group. Let $\hat{\Gamma}$ be the Pontryagin dual of Γ, i.e. $\hat{\Gamma} = \{h : \Gamma \to \mathbb{C} |\ h \text{ homomorphism such that } |h(g)| = 1, \forall g \in \Gamma\}$. (**Note.** $\hat{\Gamma}$ is a compact metrizable group.)

Then $L(\Gamma)$ is $*$-isomorphic to $L^\infty(\hat{\Gamma}, \mu)$, where μ is the Haar measure of Γ.

Group von Neumann algebras, III

Proposition 2

Let Γ be a countable **abelian** group. Let $\hat{\Gamma}$ be the Pontryagin dual of Γ, i.e. $\hat{\Gamma} = \{h : \Gamma \to \mathbb{C} |\ h \text{ homomorphism such that } |h(g)| = 1, \forall g \in \Gamma\}$.
(**Note.** $\hat{\Gamma}$ is a compact metrizable group.)

Then $L(\Gamma)$ is $*$-isomorphic to $L^\infty(\hat{\Gamma}, \mu)$, where μ is the Haar measure of Γ.

Sketch of proof. For $g \in \Gamma$, define $\hat{g} \in C(\hat{\Gamma})$ by $\hat{g}(h) := h(g)$.

Group von Neumann algebras, III

Proposition 2

Let Γ be a countable **abelian** group. Let $\hat{\Gamma}$ be the Pontryagin dual of Γ, i.e. $\hat{\Gamma} = \{h : \Gamma \to \mathbb{C} | \ h \text{ homomorphism such that } |h(g)| = 1, \forall g \in \Gamma\}$.
(**Note.** $\hat{\Gamma}$ is a compact metrizable group.)

Then $L(\Gamma)$ is $*$-isomorphic to $L^{\infty}(\hat{\Gamma}, \mu)$, where μ is the Haar measure of Γ.

Sketch of proof. For $g \in \Gamma$, define $\hat{g} \in C(\hat{\Gamma})$ by $\hat{g}(h) := h(g)$.

Claim. If $g \in \Gamma \setminus \{e\}$, then $\exists \ h \in \hat{\Gamma}$ such that $h(g) \neq 1$.

Group von Neumann algebras, III

Proposition 2

Let Γ be a countable **abelian** group. Let $\hat{\Gamma}$ be the Pontryagin dual of Γ, i.e. $\hat{\Gamma} = \{h : \Gamma \to \mathbb{C} | \ h \text{ homomorphism such that } |h(g)| = 1, \forall g \in \Gamma\}$. (**Note.** $\hat{\Gamma}$ is a compact metrizable group.)

Then $L(\Gamma)$ is $*$-isomorphic to $L^{\infty}(\hat{\Gamma}, \mu)$, where μ is the Haar measure of Γ.

Sketch of proof. For $g \in \Gamma$, define $\hat{g} \in C(\hat{\Gamma})$ by $\hat{g}(h) := h(g)$.

Claim. If $g \in \Gamma \setminus \{e\}$, then $\exists \ h \in \hat{\Gamma}$ such that $h(g) \neq 1$.

Claim $+$ Stone-Weierstrass $\Rightarrow$ $\text{span}\{\hat{g} | g \in \Gamma\}$ is dense in $C(\hat{\Gamma})$

Group von Neumann algebras, III

Proposition 2

Let Γ be a countable **abelian** group. Let $\hat{\Gamma}$ be the Pontryagin dual of Γ, i.e. $\hat{\Gamma} = \{h : \Gamma \to \mathbb{C} |\ h$ homomorphism such that $|h(g)| = 1, \forall g \in \Gamma\}$.
(**Note.** $\hat{\Gamma}$ is a compact metrizable group.)

Then $L(\Gamma)$ is $*$-isomorphic to $L^\infty(\hat{\Gamma}, \mu)$, where μ is the Haar measure of Γ.

Sketch of proof. For $g \in \Gamma$, define $\hat{g} \in C(\hat{\Gamma})$ by $\hat{g}(h) := h(g)$.

Claim. If $g \in \Gamma \setminus \{e\}$, then $\exists\ h \in \hat{\Gamma}$ such that $h(g) \neq 1$.

Claim + Stone-Weierstrass $\Rightarrow$ span$\{\hat{g} | g \in \Gamma\}$ is dense in $C(\hat{\Gamma}) \Rightarrow$ span$\{\hat{g} | g \in \Gamma\}$ is dense in $L^2(\hat{\Gamma}, \mu) \Rightarrow \{\hat{g} | g \in \Gamma\}$ is an O.B. for $L^2(\hat{\Gamma}, \mu)$.

Group von Neumann algebras, III

Proposition 2

Let Γ be a countable **abelian** group. Let $\hat{\Gamma}$ be the Pontryagin dual of Γ, i.e. $\hat{\Gamma} = \{h : \Gamma \to \mathbb{C} |\ h \text{ homomorphism such that } |h(g)| = 1, \forall g \in \Gamma\}$. (**Note.** $\hat{\Gamma}$ is a compact metrizable group.)

Then $L(\Gamma)$ is $*$-isomorphic to $L^\infty(\hat{\Gamma}, \mu)$, where μ is the Haar measure of Γ.

Sketch of proof. For $g \in \Gamma$, define $\hat{g} \in C(\hat{\Gamma})$ by $\hat{g}(h) := h(g)$.

Claim. If $g \in \Gamma \setminus \{e\}$, then $\exists\ h \in \hat{\Gamma}$ such that $h(g) \neq 1$.

Claim + Stone-Weierstrass $\Rightarrow$ span$\{\hat{g}|g \in \Gamma\}$ is dense in $C(\hat{\Gamma}) \Rightarrow$ span$\{\hat{g}|g \in \Gamma\}$ is dense in $L^2(\hat{\Gamma}, \mu) \Rightarrow \{\hat{g}|g \in \Gamma\}$ is an O.B. for $L^2(\hat{\Gamma}, \mu)$.

Thus, $U(\delta_g) = \hat{g}$ defines a unitary $U : \ell^2\Gamma \to L^2(\hat{\Gamma}, \mu)$.

Proposition 2

Let Γ be a countable **abelian** group. Let $\hat{\Gamma}$ be the Pontryagin dual of Γ, i.e. $\hat{\Gamma} = \{h : \Gamma \to \mathbb{C} |\ h \text{ homomorphism such that } |h(g)| = 1, \forall g \in \Gamma\}$. (**Note.** $\hat{\Gamma}$ is a compact metrizable group.)

Then $L(\Gamma)$ is $*$-isomorphic to $L^\infty(\hat{\Gamma}, \mu)$, where μ is the Haar measure of Γ.

Sketch of proof. For $g \in \Gamma$, define $\hat{g} \in C(\hat{\Gamma})$ by $\hat{g}(h) := h(g)$.

Claim. If $g \in \Gamma \setminus \{e\}$, then $\exists\ h \in \hat{\Gamma}$ such that $h(g) \neq 1$.

Claim + Stone-Weierstrass $\Rightarrow$ $\text{span}\{\hat{g}|g \in \Gamma\}$ is dense in $C(\hat{\Gamma})$$\Rightarrow$ $\text{span}\{\hat{g}|g \in \Gamma\}$ is dense in $L^2(\hat{\Gamma}, \mu)$ $\Rightarrow$ $\{\hat{g}|g \in \Gamma\}$ is an O.B. for $L^2(\hat{\Gamma}, \mu)$.

Thus, $U(\delta_g) = \hat{g}$ defines a unitary $U : \ell^2\Gamma \to L^2(\hat{\Gamma}, \mu)$.
Since $U\lambda(g)U^* = \hat{g}$, for $g \in \Gamma$, we have $C(\hat{\Gamma}) \subset U\ L(\Gamma)\ U^* \subset L^\infty(\hat{\Gamma}, \mu)$.

Group von Neumann algebras, III

Proposition 2

Let Γ be a countable **abelian** group. Let $\hat{\Gamma}$ be the Pontryagin dual of Γ, i.e. $\hat{\Gamma} = \{h : \Gamma \to \mathbb{C} |\ h \text{ homomorphism such that } |h(g)| = 1, \forall g \in \Gamma\}$.
(**Note.** $\hat{\Gamma}$ is a compact metrizable group.)

Then $L(\Gamma)$ is $*$-isomorphic to $L^\infty(\hat{\Gamma}, \mu)$, where μ is the Haar measure of Γ.

Sketch of proof. For $g \in \Gamma$, define $\hat{g} \in C(\hat{\Gamma})$ by $\hat{g}(h) := h(g)$.

Claim. If $g \in \Gamma \setminus \{e\}$, then $\exists\ h \in \hat{\Gamma}$ such that $h(g) \neq 1$.

Claim + Stone-Weierstrass $\Rightarrow$ span$\{\hat{g} | g \in \Gamma\}$ is dense in $C(\hat{\Gamma}) \Rightarrow$ span$\{\hat{g} | g \in \Gamma\}$ is dense in $L^2(\hat{\Gamma}, \mu) \Rightarrow \{\hat{g} | g \in \Gamma\}$ is an O.B. for $L^2(\hat{\Gamma}, \mu)$.

Thus, $U(\delta_g) = \hat{g}$ defines a unitary $U : \ell^2\Gamma \to L^2(\hat{\Gamma}, \mu)$.
Since $U\lambda(g)U^* = \hat{g}$, for $g \in \Gamma$, we have $C(\hat{\Gamma}) \subset U\ L(\Gamma)\ U^* \subset L^\infty(\hat{\Gamma}, \mu)$.

Since $C(\hat{\Gamma})$ is WOT-dense in $L^\infty(\hat{\Gamma}, \mu)$, we get $U\ L(\Gamma)\ U^* = L^\infty(\hat{\Gamma}, \mu)$

Definition. Let $x \in L(\Gamma)$ and write $x\delta_e = \sum_{g\in\Gamma} x_g\delta_g \in \ell^2\Gamma$.
Then $\{x_g\}_{g\in\Gamma}$ are called the **Fourier coefficients** of x.

Definition. Let $x \in L(\Gamma)$ and write $x\delta_e = \sum_{g\in\Gamma} x_g \delta_g \in \ell^2\Gamma$.
Then $\{x_g\}_{g\in\Gamma}$ are called the **Fourier coefficients** of x.

Remark. Since $\|x\|_2 = \tau(x^*x)^{1/2} = \|x\delta_e\|_2$ and $x_g\delta_g = (x_g\lambda(g))\delta_e$, we have that $x = \sum_{g\in\Gamma} x_g\lambda(g)$, w.r.t. convergence in $\|.\|_2$.

Definition. Let $x \in L(\Gamma)$ and write $x\delta_e = \sum_{g\in\Gamma} x_g\delta_g \in \ell^2\Gamma$.
Then $\{x_g\}_{g\in\Gamma}$ are called the **Fourier coefficients** of x.

Remark. Since $\|x\|_2 = \tau(x^*x)^{1/2} = \|x\delta_e\|_2$ and $x_g\delta_g = (x_g\lambda(g))\delta_e$, we have that $x = \sum_{g\in\Gamma} x_g\lambda(g)$, w.r.t. convergence in $\|.\|_2$.

Exercise 7. Let $x, y \in L(\Gamma)$ and $x = \sum_{g\in\Gamma} x_g\lambda(g)$, $y = \sum_{h\in\Gamma} y_h\lambda(h)$ be their Fourier expansions. Show that $xy = \sum_{g\in\Gamma}(\Sigma_{h\in\Gamma} x_h y_{h^{-1}g})\lambda(g)$.

Group von Neumann algebras, IV

Definition. Let $x \in L(\Gamma)$ and write $x\delta_e = \sum_{g\in\Gamma} x_g\delta_g \in \ell^2\Gamma$.
Then $\{x_g\}_{g\in\Gamma}$ are called the **Fourier coefficients** of x.

Remark. Since $\|x\|_2 = \tau(x^*x)^{1/2} = \|x\delta_e\|_2$ and $x_g\delta_g = (x_g\lambda(g))\delta_e$, we have that $x = \sum_{g\in\Gamma} x_g\lambda(g)$, w.r.t. convergence in $\|.\|_2$.

Exercise 7. Let $x, y \in L(\Gamma)$ and $x = \sum_{g\in\Gamma} x_g\lambda(g)$, $y = \sum_{h\in\Gamma} y_h\lambda(h)$ be their Fourier expansions. Show that $xy = \sum_{g\in\Gamma}(\Sigma_{h\in\Gamma} x_h y_{h^{-1}g})\lambda(g)$.

Proposition 3

$L(\Gamma)$ is a II_1 factor $\Leftrightarrow$ Γ is infinite conjugacy class (icc):
$\{hgh^{-1} | h \in \Gamma\}$ is infinite, for all $g \in \Gamma \setminus \{e\}$.

Proof of $\Rightarrow$. If $C = \{hgh^{-1} | h \in \Gamma\}$ is finite, for some $g \in \Gamma \setminus \{e\}$, then $x = \sum_{k\in C} \lambda(k) \in \mathcal{Z}(L(\Gamma))$.

Group von Neumann algebras, IV

Definition. Let $x \in L(\Gamma)$ and write $x\delta_e = \sum_{g\in\Gamma} x_g\delta_g \in \ell^2\Gamma$. Then $\{x_g\}_{g\in\Gamma}$ are called the **Fourier coefficients** of x.

Remark. Since $\|x\|_2 = \tau(x^*x)^{1/2} = \|x\delta_e\|_2$ and $x_g\delta_g = (x_g\lambda(g))\delta_e$, we have that $x = \sum_{g\in\Gamma} x_g\lambda(g)$, w.r.t. convergence in $\|.\|_2$.

Exercise 7. Let $x, y \in L(\Gamma)$ and $x = \sum_{g\in\Gamma} x_g\lambda(g)$, $y = \sum_{h\in\Gamma} y_h\lambda(h)$ be their Fourier expansions. Show that $xy = \sum_{g\in\Gamma}(\Sigma_{h\in\Gamma} x_h y_{h^{-1}g})\lambda(g)$.

Proposition 3

$L(\Gamma)$ is a II_1 factor $\Leftrightarrow$ Γ is infinite conjugacy class (icc): $\{hgh^{-1}|h \in \Gamma\}$ is infinite, for all $g \in \Gamma \setminus \{e\}$.

Proof of $\Rightarrow$. If $C = \{hgh^{-1}|h \in \Gamma\}$ is finite, for some $g \in \Gamma \setminus \{e\}$, then $x = \sum_{k\in C} \lambda(k) \in \mathcal{Z}(L(\Gamma))$.

Since $L(\Gamma)$ is a factor $\Rightarrow x \in \mathbb{C}1 \Rightarrow x = \tau(x)1 = 0$, a contradiction.

Proof of $\Leftarrow$. Assume Γ is icc and let $x \in \mathcal{Z}(L(\Gamma))$.

Proof of $\Leftarrow$. Assume Γ is icc and let $x \in \mathcal{Z}(L(\Gamma))$.
Let $x = \sum_{g \in \Gamma} x_g \lambda(g)$ be the Fourier expansion of x.

Proof of $\Leftarrow$. Assume Γ is icc and let $x \in \mathcal{Z}(L(\Gamma))$.
Let $x = \sum_{g\in\Gamma} x_g \lambda(g)$ be the Fourier expansion of x.
If $h \in \Gamma$, then $x = \lambda(h) x \lambda(h^{-1}) = \sum_{g\in\Gamma} x_{h^{-1}gh} \lambda(g)$

Proof of $\Leftarrow$. Assume Γ is icc and let $x \in \mathcal{Z}(L(\Gamma))$.
Let $x = \sum_{g \in \Gamma} x_g \lambda(g)$ be the Fourier expansion of x.

If $h \in \Gamma$, then $x = \lambda(h) x \lambda(h^{-1}) = \sum_{g \in \Gamma} x_{h^{-1}gh} \lambda(g)$
Uniqueness of the Fourier expansion $\Rightarrow x_{h^{-1}gh} = x_g$, for all $g, h \in \Gamma$.

Proof of $\Leftarrow$. Assume Γ is icc and let $x \in \mathcal{Z}(L(\Gamma))$.
Let $x = \sum_{g\in\Gamma} x_g \lambda(g)$ be the Fourier expansion of x.

If $h \in \Gamma$, then $x = \lambda(h)x\lambda(h^{-1}) = \sum_{g\in\Gamma} x_{h^{-1}gh}\lambda(g)$
Uniqueness of the Fourier expansion $\Rightarrow x_{h^{-1}gh} = x_g$, for all $g, h \in \Gamma$.

Since $\sum_{g\in\Gamma} |x_g|^2 = \| \sum_{g\in\Gamma} x_g\delta_g\|^2 = \|x\delta_e\|^2 < \infty$ and Γ is icc
we get that $x_g = 0$, for all $g \in \Gamma \setminus \{e\}$. Hence $x = x_e 1 \in \mathbb{C}1$.

Proof of $\Leftarrow$. Assume Γ is icc and let $x \in \mathcal{Z}(L(\Gamma))$.
Let $x = \sum_{g \in \Gamma} x_g \lambda(g)$ be the Fourier expansion of x.

If $h \in \Gamma$, then $x = \lambda(h) x \lambda(h^{-1}) = \sum_{g \in \Gamma} x_{h^{-1}gh} \lambda(g)$
Uniqueness of the Fourier expansion $\Rightarrow x_{h^{-1}gh} = x_g$, for all $g, h \in \Gamma$.

Since $\sum_{g \in \Gamma} |x_g|^2 = \| \sum_{g \in \Gamma} x_g \delta_g \|^2 = \|x \delta_e\|^2 < \infty$ and Γ is icc
we get that $x_g = 0$, for all $g \in \Gamma \setminus \{e\}$. Hence $x = x_e 1 \in \mathbb{C}1$.

Exercise 8. Show that the following groups are icc:

- $S_\infty = \{\pi \text{ permutation of } \mathbb{N} | \ \pi(n) \neq n, \text{ for finitely many } n\}$.

Proof of $\Leftarrow$. Assume Γ is icc and let $x \in \mathcal{Z}(L(\Gamma))$.
Let $x = \sum_{g\in\Gamma} x_g \lambda(g)$ be the Fourier expansion of x.

If $h \in \Gamma$, then $x = \lambda(h)x\lambda(h^{-1}) = \sum_{g\in\Gamma} x_{h^{-1}gh}\lambda(g)$
Uniqueness of the Fourier expansion $\Rightarrow x_{h^{-1}gh} = x_g$, for all $g, h \in \Gamma$.

Since $\sum_{g\in\Gamma} |x_g|^2 = \|\sum_{g\in\Gamma} x_g \delta_g\|^2 = \|x\delta_e\|^2 < \infty$ and Γ is icc
we get that $x_g = 0$, for all $g \in \Gamma \setminus \{e\}$. Hence $x = x_e 1 \in \mathbb{C}1$.

Exercise 8. Show that the following groups are icc:

- $S_\infty = \{\pi$ permutation of $\mathbb{N} | \ \pi(n) \neq n,$ for finitely many $n\}$.
- the free group $\mathbb{F}_n = \langle a_1, ..., a_n \rangle$ on $n \geq 2$ generators.

Proof of $\Leftarrow$. Assume Γ is icc and let $x \in \mathcal{Z}(L(\Gamma))$.
Let $x = \sum_{g\in\Gamma} x_g\lambda(g)$ be the Fourier expansion of x.

If $h \in \Gamma$, then $x = \lambda(h)x\lambda(h^{-1}) = \sum_{g\in\Gamma} x_{h^{-1}gh}\lambda(g)$
Uniqueness of the Fourier expansion $\Rightarrow x_{h^{-1}gh} = x_g$, for all $g, h \in \Gamma$.

Since $\sum_{g\in\Gamma} |x_g|^2 = \| \sum_{g\in\Gamma} x_g\delta_g\|^2 = \|x\delta_e\|^2 < \infty$ and Γ is icc
we get that $x_g = 0$, for all $g \in \Gamma \setminus \{e\}$. Hence $x = x_e 1 \in \mathbb{C}1$.

Exercise 8. Show that the following groups are icc:

- $S_\infty = \{\pi$ permutation of $\mathbb{N} | \ \pi(n) \neq n,$ for finitely many $n\}$.
- the free group $\mathbb{F}_n = \langle a_1, ..., a_n\rangle$ on $n \geq 2$ generators.
- $\mathrm{SL}_3(\mathbb{Z})$; more generally, $\mathrm{PSL}_n(\mathbb{Z})$, for $n \geq 2$.

Group measure space von Neumann algebras, I

Let $\Gamma \curvearrowright (X, \mu)$ be a measure preserving action of a countable group Γ on a probability space (X, μ). For $f \in L^2(X), g \in \Gamma$, let $\sigma_g(f)(x) = f(g^{-1}x)$.

Group measure space von Neumann algebras, I

Let $\Gamma \curvearrowright (X, \mu)$ be a measure preserving action of a countable group Γ on a probability space (X, μ). For $f \in L^2(X), g \in \Gamma$, let $\sigma_g(f)(x) = f(g^{-1}x)$. Define the Hilbert space $H = L^2(X, \mu) \otimes \ell^2\Gamma$

$$\langle \xi_1 \otimes \delta_{g_1}, \xi_2 \otimes \delta_{g_2} \rangle = \langle \xi_1, \xi_2 \rangle \langle \delta_{g_1}, \delta_{g_2} \rangle = \delta_{g_1,g_2} \int_X \xi_1 \bar{\xi_2} \, \mathrm{d}\mu$$

Group measure space von Neumann algebras, I

Let $\Gamma \curvearrowright (X, \mu)$ be a measure preserving action of a countable group Γ on a probability space (X, μ). For $f \in L^2(X), g \in \Gamma$, let $\sigma_g(f)(x) = f(g^{-1}x)$. Define the Hilbert space $H = L^2(X, \mu) \otimes \ell^2\Gamma$

$$\langle \xi_1 \otimes \delta_{g_1}, \xi_2 \otimes \delta_{g_2} \rangle = \langle \xi_1, \xi_2 \rangle \langle \delta_{g_1}, \delta_{g_2} \rangle = \delta_{g_1,g_2} \int_X \xi_1 \bar{\xi_2} \, \mathrm{d}\mu$$

a $*$-homomorphism $\pi : L^\infty(X) \to \mathbb{B}(H)$ and a unitary rep. $u : \Gamma \to \mathcal{U}(H)$

$$\pi(f)(\xi \otimes \delta_h) = f\xi \otimes \delta_h \quad \text{and} \quad u_g(\xi \otimes \delta_h) = \sigma_g(\xi) \otimes \delta_{gh}.$$

Group measure space von Neumann algebras, I

Let $\Gamma \curvearrowright (X, \mu)$ be a measure preserving action of a countable group Γ on a probability space (X, μ). For $f \in L^2(X), g \in \Gamma$, let $\sigma_g(f)(x) = f(g^{-1}x)$. Define the Hilbert space $H = L^2(X, \mu) \otimes \ell^2\Gamma$

$$\langle \xi_1 \otimes \delta_{g_1}, \xi_2 \otimes \delta_{g_2} \rangle = \langle \xi_1, \xi_2 \rangle \langle \delta_{g_1}, \delta_{g_2} \rangle = \delta_{g_1,g_2} \int_X \xi_1 \bar{\xi_2} \, \mathrm{d}\mu$$

a $*$-homomorphism $\pi : L^\infty(X) \to \mathbb{B}(H)$ and a unitary rep. $u : \Gamma \to \mathcal{U}(H)$

$$\pi(f)(\xi \otimes \delta_h) = f\xi \otimes \delta_h \quad \text{and} \quad u_g(\xi \otimes \delta_h) = \sigma_g(\xi) \otimes \delta_{gh}.$$

Fact. View $L^\infty(X) \subset \mathbb{B}(H)$. Then $u_g f u_g^* = \sigma_g(f)$, for $f \in L^\infty(X), g \in \Gamma$.

Group measure space von Neumann algebras, I

Let $\Gamma \curvearrowright (X, \mu)$ be a measure preserving action of a countable group Γ on a probability space (X, μ). For $f \in L^2(X), g \in \Gamma$, let $\sigma_g(f)(x) = f(g^{-1}x)$. Define the Hilbert space $H = L^2(X, \mu) \otimes \ell^2\Gamma$

$$\langle \xi_1 \otimes \delta_{g_1}, \xi_2 \otimes \delta_{g_2} \rangle = \langle \xi_1, \xi_2 \rangle \langle \delta_{g_1}, \delta_{g_2} \rangle = \delta_{g_1, g_2} \int_X \xi_1 \bar{\xi_2} \, \mathrm{d}\mu$$

a $*$-homomorphism $\pi : L^\infty(X) \to \mathbb{B}(H)$ and a unitary rep. $u : \Gamma \to \mathcal{U}(H)$

$$\pi(f)(\xi \otimes \delta_h) = f\xi \otimes \delta_h \quad \text{and} \quad u_g(\xi \otimes \delta_h) = \sigma_g(\xi) \otimes \delta_{gh}.$$

Fact. View $L^\infty(X) \subset \mathbb{B}(H)$. Then $u_g f u_g^* = \sigma_g(f)$, for $f \in L^\infty(X), g \in \Gamma$.

Let $\mathcal{A} := \{\sum_{g \in F} a_g u_g | \; F \subset \Gamma \text{ finite}, \; a_g \in L^\infty(X), \forall g \in \Gamma\}$.

Group measure space von Neumann algebras, I

Let $\Gamma \curvearrowright (X,\mu)$ be a measure preserving action of a countable group Γ on a probability space (X,μ). For $f \in L^2(X), g \in \Gamma$, let $\sigma_g(f)(x) = f(g^{-1}x)$. Define the Hilbert space $H = L^2(X,\mu) \otimes \ell^2\Gamma$

$$\langle \xi_1 \otimes \delta_{g_1}, \xi_2 \otimes \delta_{g_2} \rangle = \langle \xi_1, \xi_2 \rangle \langle \delta_{g_1}, \delta_{g_2} \rangle = \delta_{g_1,g_2} \int_X \xi_1 \bar{\xi_2} \, \mathrm{d}\mu$$

a $*$-homomorphism $\pi : L^\infty(X) \to \mathbb{B}(H)$ and a unitary rep. $u : \Gamma \to \mathcal{U}(H)$

$$\pi(f)(\xi \otimes \delta_h) = f\xi \otimes \delta_h \quad \text{and} \quad u_g(\xi \otimes \delta_h) = \sigma_g(\xi) \otimes \delta_{gh}.$$

Fact. View $L^\infty(X) \subset \mathbb{B}(H)$. Then $u_g f u_g^* = \sigma_g(f)$, for $f \in L^\infty(X), g \in \Gamma$.

Let $\mathcal{A} := \{\sum_{g \in F} a_g u_g \mid F \subset \Gamma \text{ finite}, \ a_g \in L^\infty(X), \forall g \in \Gamma\}$.
Then $\mathcal{A} \subset \mathbb{B}(H)$ is a $*$-subalgebra.

Group measure space von Neumann algebras, I

Let $\Gamma \curvearrowright (X,\mu)$ be a measure preserving action of a countable group Γ on a probability space (X,μ). For $f \in L^2(X), g \in \Gamma$, let $\sigma_g(f)(x) = f(g^{-1}x)$. Define the Hilbert space $H = L^2(X,\mu) \otimes \ell^2\Gamma$

$$\langle \xi_1 \otimes \delta_{g_1}, \xi_2 \otimes \delta_{g_2} \rangle = \langle \xi_1, \xi_2 \rangle \langle \delta_{g_1}, \delta_{g_2} \rangle = \delta_{g_1,g_2} \int_X \xi_1 \bar{\xi_2} \, \mathrm{d}\mu$$

a $*$-homomorphism $\pi : L^\infty(X) \to \mathbb{B}(H)$ and a unitary rep. $u : \Gamma \to \mathcal{U}(H)$

$$\pi(f)(\xi \otimes \delta_h) = f\xi \otimes \delta_h \quad \text{and} \quad u_g(\xi \otimes \delta_h) = \sigma_g(\xi) \otimes \delta_{gh}.$$

Fact. View $L^\infty(X) \subset \mathbb{B}(H)$. Then $u_g f u_g^* = \sigma_g(f)$, for $f \in L^\infty(X), g \in \Gamma$.

Let $\mathcal{A} := \{\sum_{g\in F} a_g u_g |\ F \subset \Gamma \text{ finite}, \ a_g \in L^\infty(X), \forall g \in \Gamma\}$.
Then $\mathcal{A} \subset \mathbb{B}(H)$ is a $*$-subalgebra.

Definition

$L^\infty(X) \rtimes \Gamma := \overline{\mathcal{A}}^{WOT} \subset \mathbb{B}(H)$ is called the **group measure space von Neumann algebra** (or **crossed product vNa**) of $\Gamma \curvearrowright (X,\mu)$.

Group measure space von Neumann algebras, II

Proposition 4

$\tau : L^\infty(X) \rtimes \Gamma \to \mathbb{C}$ given by $\tau(x) = \langle x(1 \otimes \delta_e), 1 \otimes \delta_e \rangle$ is a trace.

Group measure space von Neumann algebras, II

Proposition 4

$\tau : L^\infty(X) \rtimes \Gamma \to \mathbb{C}$ given by $\tau(x) = \langle x(1 \otimes \delta_e), 1 \otimes \delta_e \rangle$ is a trace.

Definition. Let $a \in L^\infty(X) \rtimes \Gamma$. Since $a(1 \otimes \delta_e) \in L^2(X) \otimes \ell^2\Gamma$, we can write $a(1 \otimes \delta_e) = \sum_{g \in \Gamma} a_g \otimes \delta_g$, where $a_g \in L^2(X)$.

Proposition 4

$\tau : L^\infty(X) \rtimes \Gamma \to \mathbb{C}$ given by $\tau(x) = \langle x(1 \otimes \delta_e), 1 \otimes \delta_e \rangle$ is a trace.

Definition. Let $a \in L^\infty(X) \rtimes \Gamma$. Since $a(1 \otimes \delta_e) \in L^2(X) \otimes \ell^2\Gamma$, we can write $a(1 \otimes \delta_e) = \sum_{g \in \Gamma} a_g \otimes \delta_g$, where $a_g \in L^2(X)$.

Remark. We have $a = \sum_{g \in \Gamma} a_g u_g$, w.r.t. $\|.\|_2$-convergence.

Group measure space von Neumann algebras, II

Proposition 4

$\tau : L^\infty(X) \rtimes \Gamma \to \mathbb{C}$ given by $\tau(x) = \langle x(1 \otimes \delta_e), 1 \otimes \delta_e \rangle$ is a trace.

Definition. Let $a \in L^\infty(X) \rtimes \Gamma$. Since $a(1 \otimes \delta_e) \in L^2(X) \otimes \ell^2\Gamma$, we can write $a(1 \otimes \delta_e) = \sum_{g \in \Gamma} a_g \otimes \delta_g$, where $a_g \in L^2(X)$.

Remark. We have $a = \sum_{g \in \Gamma} a_g u_g$, w.r.t. $\|.\|_2$-convergence.

Exercise 9. If $a = \sum_{g \in \Gamma} a_g u_g, b = \sum_{g \in \Gamma} b_g u_g \in L^\infty(X) \rtimes \Gamma$, then $ab = \sum_{g \in \Gamma} (\sum_{h \in \Gamma} a_h \sigma_h(b_{h^{-1}g}))$.

Group measure space von Neumann algebras, II

Proposition 4

$\tau : L^\infty(X) \rtimes \Gamma \to \mathbb{C}$ given by $\tau(x) = \langle x(1 \otimes \delta_e), 1 \otimes \delta_e \rangle$ is a trace.

Definition. Let $a \in L^\infty(X) \rtimes \Gamma$. Since $a(1 \otimes \delta_e) \in L^2(X) \otimes \ell^2\Gamma$, we can write $a(1 \otimes \delta_e) = \sum_{g \in \Gamma} a_g \otimes \delta_g$, where $a_g \in L^2(X)$.

Remark. We have $a = \sum_{g \in \Gamma} a_g u_g$, w.r.t. $\|.\|_2$-convergence.

Exercise 9. If $a = \sum_{g \in \Gamma} a_g u_g, b = \sum_{g \in \Gamma} b_g u_g \in L^\infty(X) \rtimes \Gamma$, then $ab = \sum_{g \in \Gamma} (\sum_{h \in \Gamma} a_h \sigma_h(b_{h^{-1}g}))$.

Definition

A p.m.p. (probability measure preserving) action $\Gamma \curvearrowright (X, \mu)$ is

- **free** if $\{x \in X | \, gx = x\}$ is a μ-null set, for all $g \in \Gamma \setminus \{e\}$.
- **ergodic** if any Γ-invariant mess. set $A \subset X$ has $\mu(A) \in \{0, 1\}$.

Proposition 5

Let $\Gamma \curvearrowright (X, \mu)$ be a free ergodic p.m.p. action.
Then $L^\infty(X) \rtimes \Gamma$ is a II_1 factor.

Proof. Let $a \in \mathcal{Z}(L^\infty(X) \rtimes \Gamma)$. Write $a = \sum_{g \in \Gamma} a_g u_g$.

Proposition 5

Let $\Gamma \curvearrowright (X, \mu)$ be a free ergodic p.m.p. action.
Then $L^\infty(X) \rtimes \Gamma$ is a II_1 factor.

Proof. Let $a \in \mathcal{Z}(L^\infty(X) \rtimes \Gamma)$. Write $a = \sum_{g \in \Gamma} a_g u_g$.

Claim 1. $a \in L^\infty(X)$.

Group measure space von Neumann algebras, III

Proposition 5

Let $\Gamma \curvearrowright (X, \mu)$ be a free ergodic p.m.p. action.
Then $L^\infty(X) \rtimes \Gamma$ is a II_1 factor.

Proof. Let $a \in \mathcal{Z}(L^\infty(X) \rtimes \Gamma)$. Write $a = \sum_{g \in \Gamma} a_g u_g$.

Claim 1. $a \in L^\infty(X)$.
If $b \in L^\infty(X)$, then $\sum_{g \in \Gamma} b a_g u_g = ba = ab = \sum_{g \in \Gamma} a_g \sigma_g(b) u_g$.

Group measure space von Neumann algebras, III

Proposition 5

Let $\Gamma \curvearrowright (X, \mu)$ be a free ergodic p.m.p. action.
Then $L^\infty(X) \rtimes \Gamma$ is a II_1 factor.

Proof. Let $a \in \mathcal{Z}(L^\infty(X) \rtimes \Gamma)$. Write $a = \sum_{g \in \Gamma} a_g u_g$.

Claim 1. $a \in L^\infty(X)$.
If $b \in L^\infty(X)$, then $\sum_{g \in \Gamma} b a_g u_g = ba = ab = \sum_{g \in \Gamma} a_g \sigma_g(b) u_g$.
Identifying coefficients $\Rightarrow a_g b = a_g \sigma_g(b)$.

Group measure space von Neumann algebras, III

Proposition 5

Let $\Gamma \curvearrowright (X, \mu)$ be a free ergodic p.m.p. action.
Then $L^\infty(X) \rtimes \Gamma$ is a II_1 factor.

Proof. Let $a \in \mathcal{Z}(L^\infty(X) \rtimes \Gamma)$. Write $a = \sum_{g \in \Gamma} a_g u_g$.

Claim 1. $a \in L^\infty(X)$.
If $b \in L^\infty(X)$, then $\sum_{g \in \Gamma} b a_g u_g = ba = ab = \sum_{g \in \Gamma} a_g \sigma_g(b) u_g$.
Identifying coefficients $\Rightarrow a_g b = a_g \sigma_g(b)$. Let $Y_g = \{x \in X |\ a_g(x) \neq 0\}$.
For all $b \in L^\infty(X)$ we have $b(x) = b(g^{-1}x)$, for almost every $x \in Y_g$.

Group measure space von Neumann algebras, III

Proposition 5

Let $\Gamma \curvearrowright (X, \mu)$ be a free ergodic p.m.p. action.
Then $L^\infty(X) \rtimes \Gamma$ is a II_1 factor.

Proof. Let $a \in \mathcal{Z}(L^\infty(X) \rtimes \Gamma)$. Write $a = \sum_{g \in \Gamma} a_g u_g$.

Claim 1. $a \in L^\infty(X)$.
If $b \in L^\infty(X)$, then $\sum_{g \in \Gamma} b a_g u_g = ba = ab = \sum_{g \in \Gamma} a_g \sigma_g(b) u_g$.
Identifying coefficients $\Rightarrow a_g b = a_g \sigma_g(b)$. Let $Y_g = \{x \in X |\ a_g(x) \neq 0\}$.
For all $b \in L^\infty(X)$ we have $b(x) = b(g^{-1}x)$, for almost every $x \in Y_g$.
The action is free $\Rightarrow \mu(Y_g) = 0$ hence $a_g = 0$, for all $g \in \Gamma \setminus \{e\}$.
Thus, $a = a_e 1 \in L^\infty(X)$.

Proposition 5

Let $\Gamma \curvearrowright (X, \mu)$ be a free ergodic p.m.p. action.
Then $L^\infty(X) \rtimes \Gamma$ is a II_1 factor.

Proof. Let $a \in \mathcal{Z}(L^\infty(X) \rtimes \Gamma)$. Write $a = \sum_{g \in \Gamma} a_g u_g$.

Claim 1. $a \in L^\infty(X)$.
If $b \in L^\infty(X)$, then $\sum_{g\in\Gamma} b a_g u_g = ba = ab = \sum_{g \in \Gamma} a_g \sigma_g(b) u_g$.
Identifying coefficients $\Rightarrow a_g b = a_g \sigma_g(b)$. Let $Y_g = \{x \in X |\ a_g(x) \neq 0\}$.
For all $b \in L^\infty(X)$ we have $b(x) = b(g^{-1}x)$, for almost every $x \in Y_g$.
The action is free $\Rightarrow \mu(Y_g) = 0$ hence $a_g = 0$, for all $g \in \Gamma \setminus \{e\}$.
Thus, $a = a_e 1 \in L^\infty(X)$.

Claim 2. a is a constant function, i.e. $a \in \mathbb{C}1$.

Proposition 5

Let $\Gamma \curvearrowright (X, \mu)$ be a free ergodic p.m.p. action.
Then $L^\infty(X) \rtimes \Gamma$ is a II_1 factor.

Proof. Let $a \in \mathcal{Z}(L^\infty(X) \rtimes \Gamma)$. Write $a = \sum_{g \in \Gamma} a_g u_g$.

Claim 1. $a \in L^\infty(X)$.
If $b \in L^\infty(X)$, then $\sum_{g \in \Gamma} b a_g u_g = ba = ab = \sum_{g \in \Gamma} a_g \sigma_g(b) u_g$.
Identifying coefficients $\Rightarrow a_g b = a_g \sigma_g(b)$. Let $Y_g = \{x \in X | \, a_g(x) \neq 0\}$.
For all $b \in L^\infty(X)$ we have $b(x) = b(g^{-1}x)$, for almost every $x \in Y_g$.
The action is free $\Rightarrow \mu(Y_g) = 0$ hence $a_g = 0$, for all $g \in \Gamma \setminus \{e\}$.
Thus, $a = a_e 1 \in L^\infty(X)$.

Claim 2. a is a constant function, i.e. $a \in \mathbb{C}1$.
For all $g \in \Gamma$, $\sigma_g(a) = u_g a u_g^* = a$, hence $a(g^{-1}x) = a(x)$, a.e. $x \in X$.

Group measure space von Neumann algebras, III

Proposition 5

Let $\Gamma \curvearrowright (X,\mu)$ be a free ergodic p.m.p. action.
Then $L^\infty(X) \rtimes \Gamma$ is a II_1 factor.

Proof. Let $a \in \mathcal{Z}(L^\infty(X) \rtimes \Gamma)$. Write $a = \sum_{g\in\Gamma} a_g u_g$.

Claim 1. $a \in L^\infty(X)$.
If $b \in L^\infty(X)$, then $\sum_{g\in\Gamma} b a_g u_g = ba = ab = \sum_{g\in\Gamma} a_g \sigma_g(b) u_g$.
Identifying coefficients $\Rightarrow a_g b = a_g \sigma_g(b)$. Let $Y_g = \{x \in X |\ a_g(x) \neq 0\}$.
For all $b \in L^\infty(X)$ we have $b(x) = b(g^{-1}x)$, for almost every $x \in Y_g$.
The action is free $\Rightarrow \mu(Y_g) = 0$ hence $a_g = 0$, for all $g \in \Gamma \setminus \{e\}$.
Thus, $a = a_e 1 \in L^\infty(X)$.

Claim 2. a is a constant function, i.e. $a \in \mathbb{C}1$.
For all $g \in \Gamma$, $\sigma_g(a) = u_g a u_g^* = a$, hence $a(g^{-1}x) = a(x)$, a.e. $x \in X$.
If a has two essential values $v_1 \neq v_2$, put $\delta = |v_1 - v_2|/2$.
Then $X_i = \{x |\ |f(x) - v_i| < \delta\}$, $i = 1,2$, is non-negligible and Γ-invariant.

Proposition 5

Let $\Gamma \curvearrowright (X, \mu)$ be a free ergodic p.m.p. action.
Then $L^\infty(X) \rtimes \Gamma$ is a II_1 factor.

Proof. Let $a \in \mathcal{Z}(L^\infty(X) \rtimes \Gamma)$. Write $a = \sum_{g \in \Gamma} a_g u_g$.

Claim 1. $a \in L^\infty(X)$.
If $b \in L^\infty(X)$, then $\sum_{g\in\Gamma} b a_g u_g = ba = ab = \sum_{g\in\Gamma} a_g \sigma_g(b) u_g$.
Identifying coefficients $\Rightarrow a_g b = a_g \sigma_g(b)$. Let $Y_g = \{x \in X |\ a_g(x) \neq 0\}$.
For all $b \in L^\infty(X)$ we have $b(x) = b(g^{-1}x)$, for almost every $x \in Y_g$.
The action is free $\Rightarrow \mu(Y_g) = 0$ hence $a_g = 0$, for all $g \in \Gamma \setminus \{e\}$.
Thus, $a = a_e 1 \in L^\infty(X)$.

Claim 2. a is a constant function, i.e. $a \in \mathbb{C}1$.
For all $g \in \Gamma$, $\sigma_g(a) = u_g a u_g^* = a$, hence $a(g^{-1}x) = a(x)$, a.e. $x \in X$.
If a has two essential values $v_1 \neq v_2$, put $\delta = |v_1 - v_2|/2$.
Then $X_i = \{x|\ |f(x) - v_i| < \delta\}$, $i = 1, 2$, is non-negligible and Γ-invariant.
Since $X_1 \cap X_2 = \emptyset$, this contradicts that the action is ergodic.

Examples of free ergodic p.m.p. actions

The following actions are free ergodic p.m.p.

- Let Γ be a countable ∞ group and put $(X, \mu) = ([0,1], \mathbf{Leb})^\Gamma$.

Examples of free ergodic p.m.p. actions

The following actions are free ergodic p.m.p.

- Let Γ be a countable ∞ group and put $(X, \mu) = ([0,1], \mathbf{Leb})^\Gamma$.

  The **Bernoulli action** $\Gamma \curvearrowright (X, \mu)$ is given by:
  $g \cdot x = (x_{g^{-1}h})_{h \in \Gamma}$ for all $g \in \Gamma$ and $x = (x_g)_{g \in \Gamma} \in X$.

Examples of free ergodic p.m.p. actions

The following actions are free ergodic p.m.p.

- Let Γ be a countable ∞ group and put $(X, \mu) = ([0,1], \mathbf{Leb})^{\Gamma}$.

  The **Bernoulli action** $\Gamma \curvearrowright (X, \mu)$ is given by:
  $g \cdot x = (x_{g^{-1}h})_{h \in \Gamma}$ for all $g \in \Gamma$ and $x = (x_g)_{g \in \Gamma} \in X$.
- Let $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ and $\mathbb{T} = \{z \in \mathbb{C} | \, |z| = 1\}$.

Examples of free ergodic p.m.p. actions

The following actions are free ergodic p.m.p.

- Let Γ be a countable ∞ group and put $(X, \mu) = ([0,1], \mathbf{Leb})^{\Gamma}$.

  The **Bernoulli action** $\Gamma \curvearrowright (X, \mu)$ is given by:
  $g \cdot x = (x_{g^{-1}h})_{h\in\Gamma}$ for all $g \in \Gamma$ and $x = (x_g)_{g\in\Gamma} \in X$.

- Let $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ and $\mathbb{T} = \{z \in \mathbb{C} | \, |z| = 1\}$.

  The **irrational rotation action** $\mathbb{Z} \curvearrowright (\mathbb{T}, \mathbf{Leb})$ is given by:
  $n \cdot z = \exp(2\pi i n\alpha)z$, for all $n \in \mathbb{Z}$ and $z \in \mathbb{T}$.

Examples of free ergodic p.m.p. actions

The following actions are free ergodic p.m.p.

- Let Γ be a countable ∞ group and put $(X, \mu) = ([0,1], \mathbf{Leb})^{\Gamma}$.

  The **Bernoulli action** $\Gamma \curvearrowright (X, \mu)$ is given by:
  $g \cdot x = (x_{g^{-1}h})_{h \in \Gamma}$ for all $g \in \Gamma$ and $x = (x_g)_{g \in \Gamma} \in X$.

- Let $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ and $\mathbb{T} = \{z \in \mathbb{C} | \, |z| = 1\}$.

  The **irrational rotation action** $\mathbb{Z} \curvearrowright (\mathbb{T}, \mathbf{Leb})$ is given by:
  $n \cdot z = \exp(2\pi i n \alpha) z$, for all $n \in \mathbb{Z}$ and $z \in \mathbb{T}$.

- The usual **matrix multiplication** action
  $\mathrm{SL}_n(\mathbb{Z}) \curvearrowright (\mathbb{T}^n \equiv \mathbb{R}^n/\mathbb{Z}^n, \mathbf{Leb})$, for $n \geq 2$.