

ALGEBRAIC INDEPENDENCE RELATIONS IN RANDOMIZATIONS

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ABSTRACT. We study the properties of algebraic independence and pointwise algebraic independence in a class of continuous theories, the randomizations T^R of complete first order theories T . If algebraic and definable closure coincide in T , then algebraic independence in T^R satisfies extension and has local character with the smallest possible bound, but has neither finite character nor base monotonicity. For arbitrary T , pointwise algebraic independence in T^R satisfies extension for countable sets, has finite character, has local character with the smallest possible bound, and satisfies base monotonicity if and only if algebraic independence in T does.

1. INTRODUCTION

The randomization of a complete first order theory T is the complete continuous theory T^R with two sorts, a sort for random elements of models of T , and a sort for events in an underlying probability space. The aim of this paper is to investigate algebraic independence relations in randomizations of first order theories. We will use results from our earlier papers [AGK1], which characterizes definability in randomizations, and [AGK2], where it is shown that the randomization of every o-minimal theory is real rosy, that is, has a strict independence relation.

We focus on the independence axioms introduced by Adler [Ad2] (see Definition 2.1 below). In first order model theory, algebraic independence is anti-reflexive and satisfies all of Adler's axioms except perhaps base monotonicity, and also satisfies *small local character*, a property that implies local character with the smallest possible bound $\kappa(D) = (|D| + \aleph_0)^+$. It was shown in [BBHU] and [EG] that for any complete continuous theory, the algebraic independence relation satisfies all of the Adler's axioms except perhaps base monotonicity, extension, and finite character, and also satisfies countable character (a weakening of finite character), has local character with bound $\kappa(D) = ((|D| + 2)^{\aleph_0})^+$, and is anti-reflexive. We show here that if the underlying first order theory T has $\text{acl} = \text{dcl}$ (that is, algebraic closure coincides with definable closure), then algebraic closure in T^R also satisfies extension and small local character. However, for every T , algebraic independence in T^R never has finite character and never satisfies base monotonicity.

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Another relation on models of T^R is *pointwise algebraic independence*, which was introduced in [AGK2] and roughly means algebraic independence almost everywhere. We show that for arbitrary T (rather than just when T has $\text{acl} = \text{dcl}$), pointwise algebraic independence in T^R satisfies all of Adler's axioms except perhaps base monotonicity and extension. In particular, it does have finite character. Moreover, pointwise algebraic independence satisfies extension for countable sets, has small local character, and satisfies base monotonicity if and only if algebraic extension in T satisfies base monotonicity. However, pointwise algebraic independence is never anti-reflexive.

This paper is organized as follows. In Section 2 we review Adler's axioms for independence relations and some general results from the literature about algebraic independence in first order and continuous model theory. Section 3 contains some notions and results about the randomization theory T^R that we will need from the papers [AGK1] and [AGK2]. Section 4 contains the proofs of the negative results that in T^R , algebraic independence never has finite character and never satisfies base monotonicity. To better understand why this happens, we take a closer look at the example of dense linear order. Section 5 contains the proof of the result that if T has $\text{acl} = \text{dcl}$ then algebraic independence in T^R satisfies the extension axiom. In Section 6 we prove that if T has $\text{acl} = \text{dcl}$ then algebraic independence in T^R has small local character. On the way to this proof, we introduce the pointwise algebraic independence relation in T^R , and show that it has small local character whether or not T has $\text{acl} = \text{dcl}$. Finally, in Section 7 we prove the other results stated in the preceding paragraph about pointwise algebraic independence in T^R . We also show that in T^R , pointwise algebraic independence never implies algebraic independence, and algebraic independence implies pointwise algebraic independence only in the trivial case that the models of T are finite.

For background in continuous model theory in its current form we refer to the papers [BBHU] and [BU]. We assume the reader is familiar with the basics of continuous model theory, including the notions of a theory, model, pre-model, reduction, and completion. For background on randomizations of models we refer to the papers [Ke] and [BK]. We follow the terminology of [AGK2]. A continuous pre-model is called *pre-complete* if its reduction is its completion. The set of all finite tuples in a set A is denoted by $A^{<\mathbb{N}}$. We assume throughout this paper that T is a complete first order theory with countable signature L and models of cardinality > 1 , and that v is an uncountable inaccessible cardinal that is held fixed. We let \mathcal{M} be the *big model* of T , that is, the (unique up to isomorphism) saturated model $\mathcal{M} \models T$ that is finite or of cardinality $|\mathcal{N}| = v$. We call a set *small* if it has cardinality $< v$, and *large* otherwise.

2. INDEPENDENCE

2.1. Abstract Independence Relations. Since the various properties of independence are given some slightly different names in various parts of the literature, we take this opportunity to declare that we are following the terminology established in [Ad2], which is repeated here for the reader's convenience. In this paper, we will sometimes write AB for $A \cup B$, and write $[A, B]$ for $\{D: A \subseteq D \wedge D \subseteq B\}$

Definition 2.1 (Adler). Let \mathcal{N} be the big model of a continuous or first order theory. By a *ternary relation over \mathcal{N}* we mean a ternary relation \perp on the small subsets of \mathcal{N} . We say that \perp is an *independence relation* if it satisfies the following *axioms for independence relations* for all small sets:

- (1) (Invariance) If $A \perp_C B$ and $(A', B', C') \equiv (A, B, C)$, then $A' \perp_{C'} B'$.
- (2) (Monotonicity) If $A \perp_C B$, $A' \subseteq A$, and $B' \subseteq B$, then $A' \perp_C B'$.
- (3) (Base monotonicity) Suppose $C \in [D, B]$. If $A \perp_D B$, then $A \perp_C B$.
- (4) (Transitivity) Suppose $C \in [D, B]$. If $B \perp_C A$ and $C \perp_D A$, then $B \perp_D A$.
- (5) (Normality) $A \perp_C B$ implies $AC \perp_C B$.
- (6) (Extension) If $A \perp_C B$ and $\widehat{B} \supseteq B$, then there is $A' \equiv_{BC} A$ such that $A' \perp_C \widehat{B}$.
- (7) (Finite character) If $A_0 \perp_C B$ for all finite $A_0 \subseteq A$, then $A \perp_C B$.
- (8) (Local character) For every A , there is a cardinal $\kappa(A) < \nu$ such that, for any set B , there is a subset C of B with $|C| < \kappa(A)$ such that $A \perp_C B$.

If finite character is replaced by countable character (which is defined in the obvious way), then we say that \perp is a *countable independence relation*. We will refer to the first five axioms (1)–(5) as the *basic axioms*.

Definition 2.2. An independence relation \perp is *strict* if it satisfies

- (9) (Anti-reflexivity) $a \perp_B a$ implies $a \in \text{acl}(B)$.

There are two other useful properties to consider when studying ternary relations over \mathcal{N} :

Definition 2.3.

- (10) (Full existence) For every A, B, C , there is $A' \equiv_C A$ such that $A' \perp_C B$.
- (11) (Symmetry) For every A, B, C , $A \perp_C B$ implies $B \perp_C A$.

Fact 2.4. (Remarks 2.2.4 in [AGK2]).

- (i) *Whenever \perp satisfies invariance, monotonicity, transitivity, normality, full existence, and symmetry, then \perp also satisfies extension.*
- (ii) *Any countable independence relation is symmetric.*

Definition 2.5.

- (i) We say that \downarrow has *countably local character* if for every countable set A and every small set B , there is a countable subset C of B such that $A \downarrow_C B$.
- (ii) We say that \downarrow has *small local character* if for all small sets A, B, C_0 such that $C_0 \subseteq B$ and $|C_0| \leq |A| + \aleph_0$, there is a set $C \in [C_0, B]$ such that $|C| \leq |A| + \aleph_0$ and $A \downarrow_C B$.

Fact 2.6. (Remark 2.2.7 in [AGK2]).

- (i) If \downarrow has small local character, then \downarrow has local character with bound $\kappa(D) = (|D| + \aleph_0)^+$ (the smallest possible bound). In the presence of monotonicity, the converse is also true.
- (ii) If \downarrow has local character with bound $\kappa(D) = (|D| + \aleph_0)^+$, then \downarrow has countably local character.
- (iii) If \downarrow has invariance, countable character, base monotonicity, and countably local character, then \downarrow has local character with bound $\kappa(D) = ((|D| + 2)^{\aleph_0})^+$.

We say that \downarrow^J is weaker than \downarrow^I , and write $\downarrow^I \Rightarrow \downarrow^J$, if $A \downarrow_C^I B \Rightarrow A \downarrow_C^J B$.

Remark 2.7. Suppose $\downarrow^I \Rightarrow \downarrow^J$. If \downarrow^I has full existence, local character, countably local character, or small local character. Then \downarrow^J has the same property.

2.2. Algebraic Independence.

Definition 2.8. In first order logic, a formula $\varphi(u, \vec{v})$ is *functional* in T if

$$T \models (\forall \vec{v})(\exists^{\leq 1} u)\varphi(u, \vec{v}).$$

$\varphi(u, \vec{v})$ is *algebraical* in T if there exists $n \in \mathbb{N}$ such that

$$T \models (\forall \vec{v})(\exists^{\leq n} u)\varphi(u, \vec{v}).$$

The *definable closure* of A in \mathcal{M} is the set

$$\text{dcl}^{\mathcal{M}}(A) = \{b \in M \mid \mathcal{M} \models \varphi(b, \vec{a}) \text{ for some functional } \varphi \text{ and } \vec{a} \in A^{<\mathbb{N}}\}.$$

The *algebraic closure* of A in \mathcal{M} is the set

$$\text{acl}^{\mathcal{M}}(A) = \{b \in M \mid \mathcal{M} \models \varphi(b, \vec{a}) \text{ for some algebraical } \varphi \text{ and } \vec{a} \in A^{<\mathbb{N}}\}.$$

We refer to [BBHU] for the definitions of the algebraic closure $\text{acl}^{\mathcal{N}}(A)$ and definable closure $\text{dcl}^{\mathcal{N}}(A)$ in a continuous structure \mathcal{N} . If \mathcal{N} is clear from the context, we will sometimes drop the superscript and write dcl, acl instead of $\text{dcl}^{\mathcal{N}}, \text{acl}^{\mathcal{N}}$. We will often use the following facts without explicit mention.

Fact 2.9. (Follows from [BBHU], Exercise 10.8) For every set A , $\text{acl}(A)$ has cardinality at most $(|A| + 2)^{\aleph_0}$. Thus the algebraic closure of a small set is small.

Fact 2.10. (Definable Closure, Exercises 10.10 and 10.11, and Corollary 10.5 in [BBHU])

- (1) If $A \subseteq \mathcal{N}$ then $\text{dcl}(A) = \text{dcl}(\text{dcl}(A))$ and $\text{acl}(A) = \text{acl}(\text{acl}(A))$.
- (2) If A is a dense subset of the topological closure of B and $B \subseteq \mathcal{N}$, then $\text{dcl}(A) = \text{dcl}(B)$ and $\text{acl}(A) = \text{acl}(B)$.

It follows that for any $A \subseteq \mathcal{N}$, $\text{dcl}(A)$ and $\text{acl}(A)$ are topologically closed. In any complete theory (first order or continuous), we define the notion of *algebraic independence*, denoted \downarrow^a , by setting $A \downarrow_C^a B$ to mean $\text{acl}(AC) \cap \text{acl}(BC) = \text{acl}(C)$. In first order logic, \downarrow^a satisfies all axioms for a strict independence relation except for perhaps base monotonicity.

Proposition 2.11. *In any complete continuous theory, \downarrow^a satisfies symmetry and all axioms for a strict countable independence relation except perhaps for base monotonicity and extension.*

Proof. The proof is exactly as in [Ad2], Proposition 1.5, except for some minor modifications. For example, countable character of acl in continuous logic yields countable character of \downarrow^a . Also, in the verification of local character, one needs to take $\kappa(A) := (|A| + 2)^{\aleph_0} +$ instead of $(|A| + \aleph_0)^+$. \square

Question 2.12. Does \downarrow^a always have full existence (or extension) in continuous logic?

The proof that \downarrow^a has full existence in first order logic uses the negation connective, which is not available in continuous logic.

3. RANDOMIZATIONS

3.1. The Theory T^R . The *randomization signature* L^R is the two-sorted continuous signature with sorts \mathbb{K} (for random elements) and \mathbb{B} (for events), an n -ary function symbol $\llbracket \varphi(\cdot) \rrbracket$ of sort $\mathbb{K}^n \rightarrow \mathbb{B}$ for each first order formula φ of L with n free variables, a $[0, 1]$ -valued unary predicate symbol μ of sort \mathbb{B} for probability, and the Boolean operations $\top, \perp, \sqcap, \sqcup, \neg$ of sort \mathbb{B} . The signature L^R also has distance predicates $d_{\mathbb{B}}$ of sort \mathbb{B} and $d_{\mathbb{K}}$ of sort \mathbb{K} . In L^R , we use $\mathbf{B}, \mathbf{C}, \dots$ for variables or parameters of sort \mathbb{B} . $\mathbf{B} \doteq \mathbf{C}$ means $d_{\mathbb{B}}(\mathbf{B}, \mathbf{C}) = 0$, and $\mathbf{B} \sqsubseteq \mathbf{C}$ means $\mathbf{B} \doteq \mathbf{B} \sqcap \mathbf{C}$. A structure with signature L^R will be a pair $\mathcal{N} = (\mathcal{K}, \mathcal{E})$ where \mathcal{K} is the part of sort \mathbb{K} and \mathcal{E} is the part of sort \mathbb{B} .

The following fact, which is a consequence of Proposition 2.1.10 of [AGK1], gives a model-theoretic characterization of T^R .

Fact 3.1. *There is a unique complete theory T^R with signature L^R whose big model $\mathcal{N} = (\mathcal{K}, \mathcal{E})$ is the reduction of a pre-complete-structure $\mathcal{P} = (\mathcal{J}, \mathcal{F})$ equipped with a complete atomless probability space $(\Omega, \mathcal{F}, \mu)$ such that:*

- (1) \mathcal{F} is a σ -algebra with $\top, \perp, \sqcap, \sqcup, \neg$ interpreted by $\Omega, \emptyset, \cap, \cup, \setminus$.
- (2) \mathcal{J} is a set of functions $a: \Omega \rightarrow M$.
- (3) For each formula $\psi(\vec{x})$ of L and tuple \vec{a} in \mathcal{J} , we have

$$\llbracket \psi(\vec{a}) \rrbracket = \{\omega \in \Omega : \mathcal{M} \models \psi(\vec{a}(\omega))\} \in \mathcal{F}.$$

- (4) \mathcal{F} is equal to the set of all events $\llbracket \psi(\vec{a}) \rrbracket$ where $\psi(\vec{v})$ is a formula of L and \vec{a} is a tuple in \mathcal{J} .
- (5) For each formula $\theta(u, \vec{v})$ of L and tuple \vec{b} in \mathcal{J} , there exists $a \in \mathcal{J}$ such that

$$\llbracket \theta(a, \vec{b}) \rrbracket = \llbracket (\exists u \theta)(\vec{b}) \rrbracket.$$

- (6) On \mathcal{J} , the distance predicate $d_{\mathbb{K}}$ defines the pseudo-metric

$$d_{\mathbb{K}}(a, b) = \mu \llbracket a \neq b \rrbracket.$$

- (7) On \mathcal{F} , the distance predicate $d_{\mathbb{B}}$ defines the pseudo-metric

$$d_{\mathbb{B}}(\mathbf{B}, \mathbf{C}) = \mu(\mathbf{B} \Delta \mathbf{C}).$$

Fact 3.2. (Lemma 2.1.8 in [AGK1]) In the big model $\mathcal{N} = (\mathcal{K}, \mathcal{E})$ of T^R , for each $\mathbf{a}, \mathbf{b} \in \mathcal{K}$ and $\mathbf{B} \in \mathcal{E}$, there is an element $\mathbf{c} \in \mathcal{K}$ that agrees with \mathbf{a} on \mathbf{B} and agrees with \mathbf{b} on $\neg \mathbf{B}$, that is, $\mathbf{B} \sqsubseteq \llbracket \mathbf{c} = \mathbf{a} \rrbracket$ and $(\neg \mathbf{B}) \sqsubseteq \llbracket \mathbf{c} = \mathbf{b} \rrbracket$.

Definition 3.3. A first order or continuous theory has $\text{acl} = \text{dcl}$ if $\text{acl}(A) = \text{dcl}(A)$ for every set A in every model of the theory.

For example, any first order theory T with a definable linear ordering has $\text{acl} = \text{dcl}$.

Fact 3.4. ([AGK1], Proposition 3.3.7, see also [Be2])

In the big model \mathcal{N} of T^R , $\text{acl}_{\mathbb{B}}(A) = \text{dcl}_{\mathbb{B}}(A)$ and $\text{acl}(A) = \text{dcl}(A)$. Thus T^R has $\text{acl} = \text{dcl}$.

As a corollary, we obtain the following characterization of algebraic independence in \mathcal{N} .

Corollary 3.5. In the big model \mathcal{N} of T^R , $A \downarrow_C^a B$ if and only if

$$[\text{dcl}(AC) \cap \text{dcl}(BC) = \text{dcl}(C)] \wedge [\text{dcl}_{\mathbb{B}}(AC) \cap \text{dcl}_{\mathbb{B}}(BC) = \text{dcl}_{\mathbb{B}}(C)].$$

Proof. By the definition of algebraic independence in the two-sorted metric structure \mathcal{N} and Fact 3.4. \square

From now on we will work within the big model $\mathcal{N} = (\mathcal{K}, \mathcal{E})$ of T^R . By saturation, \mathcal{K} and \mathcal{E} are large. Hereafter, A, B, C will always denote small subsets of \mathcal{K} , and \mathcal{N}_A will denote the expansion of \mathcal{N} formed by adding a constant symbol for each $\mathbf{a} \in A$. We will write dcl, acl for $\text{dcl}^{\mathcal{N}}, \text{acl}^{\mathcal{N}}$, and \downarrow^a will denote the algebraic independence relation in \mathcal{N} .

For each element $\mathbf{b} \in \mathcal{K}$, we will also choose once and for all a *representative* $b \in \mathcal{J}$ such that the image of b under the reduction map is \mathbf{b} . It follows that for each first order formula $\varphi(\vec{v})$, $\llbracket \varphi(\vec{a}) \rrbracket$ in \mathcal{N} is the image of $\llbracket \varphi(\vec{a}) \rrbracket$ in \mathcal{P} under the reduction map. Note that any two representatives of an element $\mathbf{b} \in \mathcal{K}$ agree except on a set of measure zero.

For any small $A \subseteq \mathcal{K}$ and each $\omega \in \Omega$, we define

$$A(\omega) = \{a(\omega) \mid \mathbf{a} \in A\},$$

and let $\text{cl}(A)$ denote the closure of A in the metric $d_{\mathbb{K}}$. When $\mathcal{A} \subseteq \mathcal{E}$, $\text{cl}(\mathcal{A})$ denotes the closure of \mathcal{A} in the metric $d_{\mathbb{B}}$, and $\sigma(\mathcal{A})$ denotes the smallest

σ -subalgebra of \mathcal{E} containing \mathcal{A} . Since the cardinality ν of \mathcal{N} is inaccessible, whenever $A \subseteq \mathcal{K}$ is small, the closure $\text{cl}(A)$ and the set of n -types over A is small. Also, whenever $\mathcal{A} \subseteq \mathcal{E}$ is small, the closure $\text{cl}(\mathcal{A})$ is small.

3.2. Definability in T^R . In this section we review some notions and results about definability that we will need from the paper [AGK1]. We write $\text{dcl}_{\mathbb{B}}(A)$ for the set of elements of sort \mathbb{B} that are definable over A in \mathcal{N} , and write $\text{dcl}(A)$ for the set of elements of sort \mathbb{K} that are definable over A in \mathcal{N} . Similarly for $\text{acl}_{\mathbb{B}}(A)$ and $\text{acl}(A)$.

Definition 3.6. We say that an event E is *first order definable over A* , in symbols $E \in \text{fdcl}_{\mathbb{B}}(A)$, if $E = \llbracket \theta(\vec{a}) \rrbracket$ for some formula θ of L and some tuple $\vec{a} \in A^{<\mathbb{N}}$.

Definition 3.7. We say that \mathbf{b} is *first order definable over A* , in symbols $\mathbf{b} \in \text{fdcl}(A)$, if there is a functional formula $\varphi(u, \vec{v})$ and a tuple $\vec{a} \in A^{<\mathbb{N}}$ such that $\llbracket \varphi(\mathbf{b}, \vec{a}) \rrbracket = \top$.

Fact 3.8. ([AGK1], Theorems 3.1.2 and 3.3.6)

$$\text{dcl}_{\mathbb{B}}(A) = \text{cl}(\text{fdcl}_{\mathbb{B}}(A)) = \sigma(\text{fdcl}_{\mathbb{B}}(A)) \subseteq \mathcal{E}, \quad \text{dcl}(A) = \text{cl}(\text{fdcl}(A)) \subseteq \mathcal{K}.$$

It follows that whenever A is small, $\text{dcl}(A)$ and $\text{dcl}_{\mathbb{B}}(A)$ are small.

Remark 3.9. For each small A ,

$$\text{fdcl}_{\mathbb{B}}(\text{fdcl}(A)) = \text{fdcl}_{\mathbb{B}}(A), \quad \text{dcl}_{\mathbb{B}}(\text{dcl}(A)) = \text{dcl}_{\mathbb{B}}(A).$$

We will sometimes use the $\llbracket \dots \rrbracket$ notation in a general setting. Given a property $P(\omega)$, we write

$$\llbracket P \rrbracket = \{\omega \in \Omega : P(\omega)\}.$$

Definition 3.10. We say that \mathbf{b} is *pointwise definable over A* , in symbols $\mathbf{b} \in \text{dcl}^{\omega}(A)$, if

$$\mu(\llbracket \mathbf{b} \in \text{dcl}^{\mathcal{M}}(A_0) \rrbracket) = 1$$

for some countable $A_0 \subseteq A$.

We say that \mathbf{b} is *pointwise algebraic over A* , in symbols $\mathbf{b} \in \text{acl}^{\omega}(A)$, if

$$\mu(\llbracket \mathbf{b} \in \text{acl}^{\mathcal{M}}(A_0) \rrbracket) = 1$$

for some countable $A_0 \subseteq A$.

Remark 3.11. dcl^{ω} and acl^{ω} have countable character, that is, $\mathbf{b} \in \text{dcl}^{\omega}(A)$ if and only if $\mathbf{b} \in \text{dcl}^{\omega}(A_0)$ for some countable $A_0 \subseteq A$, and similarly for acl^{ω} .

The next result is a useful characterization of $\text{dcl}(A)$.

Fact 3.12. ([AGK1], Corollary 3.3.5) For any element $\mathbf{b} \in \mathcal{K}$, \mathbf{b} is definable over A if and only if:

- (1) \mathbf{b} is pointwise definable over A ;
- (2) $\text{fdcl}_{\mathbb{B}}(\mathbf{b}A) \subseteq \text{dcl}_{\mathbb{B}}(A)$.

Corollary 3.13. *In \mathcal{N} we always have*

$$\text{acl}(A) = \text{dcl}(A) \subseteq \text{dcl}^\omega(A) = \text{dcl}^\omega(\text{dcl}^\omega(A)) \subseteq \text{acl}^\omega(A) = \text{acl}^\omega(\text{acl}^\omega(A)).$$

3.3. Algebraic Independence in the Event Sort. The ternary relation $\downarrow^{\text{a}\mathbb{B}}$ on the big model \mathcal{N} of T^R was introduced in the paper [AGK2] and will be useful here. It is the analogue of algebraic independence obtained by restricting the algebraic closures of sets to the event sort.

Definition 3.14. For small $A, B, C \subseteq \mathcal{K}$, define

$$A \downarrow_C^{\text{a}\mathbb{B}} B \Leftrightarrow \text{acl}_{\mathbb{B}}(AC) \cap \text{acl}_{\mathbb{B}}(BC) = \text{acl}_{\mathbb{B}}(C).$$

Remark 3.15. *By Fact 3.4, for small $A, B, C \subseteq \mathcal{K}$, we have*

$$A \downarrow_C^{\text{a}\mathbb{B}} B \Leftrightarrow \text{dcl}_{\mathbb{B}}(AC) \cap \text{dcl}_{\mathbb{B}}(BC) = \text{dcl}_{\mathbb{B}}(C).$$

By Corollary 3.5, we also have

$$A \downarrow_C^{\text{a}} B \Leftrightarrow (\text{dcl}(AC) \cap \text{dcl}(BC) = \text{dcl}(C)) \wedge A \downarrow_C^{\text{a}\mathbb{B}} B.$$

Fact 3.16. *(Proposition 6.2.4 in [AGK2]). In T^R , the relation $\downarrow^{\text{a}\mathbb{B}}$ satisfies all the axioms for a countable independence relation except base monotonicity. It also has symmetry and small local character.*

We will also need the following fact, which is given by Lemma 6.1.6, Corollary 6.1.7, and Lemma 6.2.3 of [AGK2], and is a consequence of a result in [Be].

Fact 3.17. *There is a countable independence relation $\downarrow^{\text{d}\mathbb{B}}$ (dividing independence in the event sort) that has small local character over \mathcal{N} and is such that $\downarrow^{\text{d}\mathbb{B}} \Rightarrow \downarrow^{\text{a}\mathbb{B}}$.*

4. NEGATIVE RESULTS: FINITE CHARACTER AND BASE MONOTONICITY

In this section we show that for every T , algebraic independence in T^R satisfies neither finite character nor base monotonicity. The following lemmas and notation will be useful for these results.

Lemma 4.1. *There exists a pair $Z = \{\mathbf{0}, \mathbf{1}\} \subseteq \mathcal{K}$ such that $\llbracket \mathbf{0} \neq \mathbf{1} \rrbracket = \top$, and $\text{dcl}_{\mathbb{B}}(Z) = \{\top, \perp\}$.*

Proof. This follows from Fact 3.1 and the fact that \mathcal{N} is saturated. \square

By Fact 3.2, for each event $\mathbf{B} \in \mathcal{E}$, there is a unique element $\mathbf{1}_{\mathbf{B}} \in \mathcal{K}$ that agrees with 1 on \mathbf{B} and agrees with 0 on $\neg\mathbf{B}$. Given a set $\mathcal{A} \subseteq \mathcal{E}$, let $\mathbf{1}_{\mathcal{A}} = \{\mathbf{1}_{\mathbf{B}} \mid \mathbf{B} \in \mathcal{A}\}$.

Lemma 4.2. *If $\mathcal{A} \subseteq \mathcal{E}$, then $\text{dcl}_{\mathbb{B}}(\mathbf{1}_{\mathcal{A}}Z) = \sigma(\mathcal{A})$.*

Proof. By definition, $\mathbf{E} \in \text{fdcl}_{\mathbb{B}}(\mathbf{1}_{\mathcal{A}}Z)$ if and only if $\mathbf{E} = \llbracket \theta(\vec{\mathbf{a}}, \mathbf{0}, \mathbf{1}) \rrbracket$ for some formula $\theta \in L$ and tuple $\vec{\mathbf{a}} \subseteq \mathbf{1}_{\mathcal{A}}$. For each $\mathbf{b} \in \mathbf{1}_{\mathcal{A}}$, we have $\mu(\llbracket \mathbf{b} \in Z \rrbracket) = 1$. It follows that $\llbracket \theta(\vec{\mathbf{a}}, \mathbf{0}, \mathbf{1}) \rrbracket \in \sigma(\mathcal{A})$. By Fact 3.8, $\text{dcl}_{\mathbb{B}}(\mathbf{1}_{\mathcal{A}}Z) \subseteq \sigma(\mathcal{A})$. For each $\mathbf{B} \in \mathcal{A}$ we have $\mathbf{B} = \llbracket \mathbf{1}_{\mathbf{B}} = \mathbf{1} \rrbracket$, so $\mathcal{A} \subseteq \text{fdcl}_{\mathbb{B}}(\mathbf{1}_{\mathcal{A}}Z)$. Then by Fact 3.8 again, $\sigma(\mathcal{A}) \supseteq \text{dcl}_{\mathbb{B}}(\mathbf{1}_{\mathcal{A}}Z)$. \square

4.1. Finite Character.

Proposition 4.3. *For every T , \downarrow^a in T^R does not satisfy finite character.*

Proof. Since μ is atomless, there is an event \mathbf{B} and a sequence of events $\langle \mathbf{B}_n \rangle_{n \in \mathbb{N}}$ such that for each n ,

$$\mathbf{B}_n \sqsubseteq \mathbf{B}_{n+1} \sqsubseteq \mathbf{B}, \quad \mu(\mathbf{B} \setminus \mathbf{B}_n) = 2^{-(n+1)}, \quad \mu(\mathbf{B}) = 1/2.$$

Let $\mathbf{b} = \mathbf{1}_{\mathbf{B}}$, $\mathcal{A}_n = \{\mathbf{B}_m \mid m < n\}$, $\mathcal{A} = \{\mathbf{B}_m \mid m \in \mathbb{N}\}$. Then $\mathbf{1}_{\mathcal{A}} = \bigcup_n \mathbf{1}_{\mathcal{A}_n}$. By Lemma 4.2,

$$\text{dcl}_{\mathbb{B}}(\mathbf{1}_{\mathcal{A}_n}Z) = \sigma(\mathcal{A}_n), \quad \text{dcl}_{\mathbb{B}}(\mathbf{b}Z) = \sigma(\{\mathbf{B}\}) \subseteq \sigma(\mathbf{1}_{\mathcal{A}}Z) = \text{dcl}_{\mathbb{B}}(\mathcal{A}).$$

Note that every element of \mathcal{K} that is pointwise definable from $\mathbf{1}_{\mathcal{A}}Z$ is pointwise definable from Z . Then by Fact 3.12, we have

$$\begin{aligned} \text{dcl}(Z) &= \{\mathbf{x} \in \text{dcl}^{\omega}(Z) \mid \text{fdcl}_{\mathbb{B}}(\mathbf{x}Z) \subseteq \{\top, \perp\}\}, \\ \text{dcl}(\mathbf{1}_{\mathcal{A}}Z) &= \{\mathbf{x} \in \text{dcl}^{\omega}(Z) \mid \text{fdcl}_{\mathbb{B}}(\mathbf{x}\mathbf{1}_{\mathcal{A}}Z) \subseteq \sigma(\mathcal{A})\}, \\ \text{dcl}(\mathbf{1}_{\mathcal{A}_n}Z) &= \{\mathbf{x} \in \text{dcl}^{\omega}(Z) \mid \text{fdcl}_{\mathbb{B}}(\mathbf{x}\mathbf{1}_{\mathcal{A}_n}Z) \subseteq \sigma(\mathcal{A}_n)\}, \\ \text{dcl}(\mathbf{b}Z) &= \{\mathbf{x} \in \text{dcl}^{\omega}(Z) \mid \text{fdcl}_{\mathbb{B}}(\mathbf{x}\mathbf{b}Z) \subseteq \sigma(\mathbf{B})\}. \end{aligned}$$

But $\sigma(\mathcal{A}_n) \cap \sigma(\{\mathbf{B}\}) = \{\top, \perp\}$, so by Fact 3.4,

$$\text{acl}(\mathbf{1}_{\mathcal{A}_n}Z) \cap \text{acl}(\mathbf{b}Z) = \text{dcl}(\mathbf{1}_{\mathcal{A}_n}Z) \cap \text{dcl}(\mathbf{b}Z) = \text{dcl}(Z) = \text{acl}(Z),$$

and hence $\mathbf{1}_{\mathcal{A}_n} \downarrow_Z^a \mathbf{b}$.

However, $\mathbf{B} \in \sigma(\mathcal{A})$, so by Lemma 4.2 we have $\text{dcl}_{\mathbb{B}}(\mathbf{b}\mathbf{1}_{\mathcal{A}}Z) = \sigma(\mathcal{A})$. Moreover, $\mathbf{b} \in \text{dcl}^{\omega}(Z)$. Therefore

$$\mathbf{b} \in \text{acl}(\mathbf{1}_{\mathcal{A}}Z) \cap \text{acl}(\mathbf{b}) \setminus \text{acl}(Z),$$

so $\mathbf{1}_{\mathcal{A}} \not\downarrow_Z^a \mathbf{b}$ and finite character fails. \square

4.2. Base Monotonicity. By Proposition 1.5 (3) in [Ad1], for any complete first order theory T , \downarrow^a satisfies base monotonicity if and only if the lattice of algebraically closed sets is modular. The argument there shows that the same result holds for any complete continuous theory. We show that for T^R , \downarrow^a never satisfies base monotonicity, and thus is never modular and is never a countable independence relation.

Proposition 4.4. *For every T , \downarrow^a in T^R does not satisfy base monotonicity.*

Proof. Since μ is atomless, there are two independent events D, F in \mathcal{E} of probability $1/2$. Let $E = D \cap F$. $\mathbf{a} = 1_D$, $\mathbf{b} = 1_E$, and $\mathbf{c} = 1_F$. Then

$$\begin{aligned} \text{dcl}_{\mathbb{B}}(\mathbf{a}) &= \sigma(\{D\}), & \text{dcl}_{\mathbb{B}}(\mathbf{c}) &= \sigma(\{F\}), \\ \text{dcl}_{\mathbb{B}}(\mathbf{ac}) &= \sigma(\{D, F\}), & \text{dcl}_{\mathbb{B}}(\mathbf{bc}) &= \sigma(\{E, F\}). \end{aligned}$$

As in the proof of Proposition 4.3, we have

$$\begin{aligned} \text{dcl}(Z) &= \{\mathbf{x} \in \text{dcl}^\omega(Z) \mid \text{fdcl}_{\mathbb{B}}(\mathbf{x}Z) \subseteq \{\top, \perp\}\}, \\ \text{dcl}(\mathbf{a}Z) &= \{\mathbf{x} \in \text{dcl}^\omega(Z) \mid \text{fdcl}_{\mathbb{B}}(\mathbf{x}\mathbf{a}Z) \subseteq \sigma(\{D\})\}, \\ \text{dcl}(\mathbf{c}Z) &= \{\mathbf{x} \in \text{dcl}^\omega(Z) \mid \text{fo}_{\mathbb{B}}(\mathbf{x}\mathbf{c}Z) \subseteq \sigma(\{F\})\}, \\ \text{dcl}(\mathbf{ac}Z) &= \{\mathbf{x} \in \text{dcl}^\omega(Z) \mid \text{fdcl}_{\mathbb{B}}(\mathbf{x}\mathbf{ac}Z) \subseteq \sigma(\{D, F\})\}, \\ \text{dcl}(\mathbf{bc}Z) &= \{\mathbf{x} \in \text{dcl}^\omega(Z) \mid \text{fdcl}_{\mathbb{B}}(\mathbf{x}\mathbf{bc}Z) \subseteq \sigma(\{E, F\})\}. \end{aligned}$$

It follows that $\mathbf{a} \downarrow_Z^{\mathbf{a}} \mathbf{bc}Z$. But

$$E \in \sigma(\{D, F\}) \cap \sigma(\{E, F\}) \setminus \sigma(\{F\}),$$

so

$$\mathbf{b} \in \text{acl}(\mathbf{ac}Z) \cap \text{acl}(\mathbf{bc}Z) \setminus \text{acl}(\mathbf{c}Z).$$

Therefore $\mathbf{a} \not\downarrow_{\mathbf{c}Z}^{\mathbf{a}} \mathbf{bc}Z$, and base monotonicity fails. \square

Recall from [Ad1] that the ternary relation \downarrow^M is defined by

$$A \downarrow_C^M B \Leftrightarrow (\forall D \in [C, \text{acl}(BC)]) A \downarrow_D^{\mathbf{a}} B.$$

\downarrow^M is the weakest ternary relation that implies $\downarrow^{\mathbf{a}}$ and satisfies base monotonicity. So in both first order and continuous model theory, if $\downarrow^{\mathbf{a}}$ satisfies base monotonicity then $\downarrow^M = \downarrow^{\mathbf{a}}$ and hence \downarrow^M satisfies symmetry. Thus the following corollary is an improvement of Proposition 4.4.

Corollary 4.5. *For every T , \downarrow^M in T^R does not satisfy symmetry.*

Proof. We use the notation introduced in the proof of 4.4. Since $\mathbf{a} \not\downarrow_{\mathbf{c}Z}^{\mathbf{a}} \mathbf{bc}Z$ and $\downarrow^M \Rightarrow \downarrow^{\mathbf{a}}$, we have $\mathbf{a} \not\downarrow_{\mathbf{c}Z}^M \mathbf{bc}Z$. However, it follows from the proof of 4.4 that $\mathbf{bc}Z \downarrow_Z^M \mathbf{a}$, so \downarrow^M does not satisfy symmetry. \square

As an example, we look at the relations $\downarrow^{\mathbf{a}}$ and \downarrow^M in the continuous theory DLO^R , the randomization of the theory of dense linear order without endpoints. We will see that these relations are much more complicated in DLO^R than they are in DLO .

Example 4.6. Let $T = \text{DLO}$. In the big model \mathcal{M} of DLO , we have $\text{acl}(A) = \text{dcl}(A) = A$ for every set A . Thus in \mathcal{M} the lattice of algebraically closed sets is modular, $\downarrow^{\mathbf{a}} = \downarrow^M$, and $\downarrow^{\mathbf{a}}$ is a strict independence relation.

In the big model \mathcal{N} of DLO^R , $\downarrow^{\mathbf{a}}$ does not satisfy base monotonicity by Proposition 4.4, and \downarrow^M does not satisfy symmetry by Corollary 4.5. Proposition 4.2.3 of [AGK1] shows that for every finite set $A \subseteq \mathcal{K}$, $\text{dcl}(A)$ is the

smallest set $D \supseteq A$ such that whenever $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d} \in D$, the element of \mathcal{K} that agrees with \mathbf{c} on $\llbracket \mathbf{a} < \mathbf{b} \rrbracket$ and agrees with \mathbf{d} on $\neg \llbracket \mathbf{a} < \mathbf{b} \rrbracket$ belongs to D .

Let $\mathbf{a} \vee \mathbf{b}$ and $\mathbf{a} \wedge \mathbf{b}$ denote the pointwise maximum and minimum, respectively. We leave it to the reader to work out the following characterizations of $A \downarrow_C^{\mathbf{a}} B$ and $A \downarrow_C^M B$ in the simple case that A, B, C are singletons in \mathcal{N} .

- (1) $\mathbf{a} \downarrow_{\emptyset}^{\mathbf{a}} \mathbf{b} \Leftrightarrow \mathbf{a} \neq \mathbf{b}$.
- (2) $\text{acl}(\mathbf{a}\mathbf{b}) = \{\mathbf{a}, \mathbf{b}, \mathbf{a} \vee \mathbf{b}, \mathbf{a} \wedge \mathbf{b}\}$.
- (3) $\mathbf{a} \downarrow_c^{\mathbf{a}} \mathbf{b} \Leftrightarrow \{\mathbf{a}, \mathbf{c}, \mathbf{a} \vee \mathbf{c}, \mathbf{a} \wedge \mathbf{c}\} \cap \{\mathbf{b}, \mathbf{c}, \mathbf{b} \vee \mathbf{c}, \mathbf{b} \wedge \mathbf{c}\} = \{\mathbf{c}\}$.

To see where base monotonicity fails for $\downarrow^{\mathbf{a}}$, let \mathbf{E} be an event with $0 < \mu(\mathbf{E}) < 1$ and take $\mathbf{a}, \mathbf{b}, \mathbf{c}$ so that $\mathbf{a} = \mathbf{b} < \mathbf{c}$ on \mathbf{E} and $\mathbf{c} < \mathbf{a} < \mathbf{b}$ on $\neg \mathbf{E}$. Then use (1) and (3) to show that $\mathbf{a} \downarrow_{\emptyset}^{\mathbf{a}} \mathbf{b}$ but $\mathbf{a} \not\downarrow_c^{\mathbf{a}} \mathbf{b}$.

- (4) If $\mathbf{b} \in \{\mathbf{b} \vee \mathbf{c}, \mathbf{b} \wedge \mathbf{c}\}$, then $\mathbf{a} \downarrow_c^M \mathbf{b} \Leftrightarrow \mathbf{a} \downarrow_c^{\mathbf{a}} \mathbf{b}$.
- (5) If $\mathbf{b} \notin \{\mathbf{b} \vee \mathbf{c}, \mathbf{b} \wedge \mathbf{c}\}$, then $\mathbf{a} \downarrow_c^M \mathbf{b}$ if and only if:

$$\mathbf{a} \downarrow_c^{\mathbf{a}} \mathbf{b}, \quad \mathbf{b} \notin \text{dcl}(\{\mathbf{a}, \mathbf{c}, \mathbf{b} \wedge \mathbf{c}\}), \quad \mathbf{b} \notin \text{dcl}(\{\mathbf{a}, \mathbf{c}, \mathbf{b} \vee \mathbf{c}\}).$$

To see where symmetry fails for \downarrow^M , partition Ω into three events $\{\mathbf{D}, \mathbf{E}, \mathbf{F}\}$ of positive measure. Take $\mathbf{a}, \mathbf{b}, \mathbf{c}$ so that $\mathbf{a} = \mathbf{b} < \mathbf{c}$ in \mathbf{D} , $\mathbf{a} < \mathbf{c} < \mathbf{b}$ in \mathbf{E} , and $\mathbf{c} < \mathbf{a} < \mathbf{b}$ in \mathbf{F} . Use (5) to show that $\mathbf{a} \downarrow_c^M \mathbf{b}$ but $\mathbf{b} \not\downarrow_c^M \mathbf{a}$.

5. FULL EXISTENCE AND EXTENSION

By Proposition 2.11, $\downarrow^{\mathbf{a}}$ in continuous model theory satisfies symmetry and all axioms for a strict countable independence relation except for base monotonicity and extension.

Remark 5.1. *If T is stable, then the relation $\downarrow^{\mathbf{a}}$ in the theory T^R satisfies full existence and extension.*

Proof. by Theorem 5.1.4 in [BK], T^R is stable, so it has a unique strict independence relation. This relation satisfies full existence and is stronger than $\downarrow^{\mathbf{a}}$. Then by Remark 2.7, $\downarrow^{\mathbf{a}}$ satisfies full existence. By Fact 2.4, $\downarrow^{\mathbf{a}}$ in T^R satisfies extension. \square

Our main result in this section is another sufficient condition for algebraic independence in T^R to satisfy full existence and extension

Theorem 5.2. *Suppose T has $\text{acl} = \text{dcl}$. Then the relation $\downarrow^{\mathbf{a}}$ in T^R satisfies full existence and extension.*

Proof. By Fact 2.4 and Proposition 2.11, if $\downarrow^{\mathbf{a}}$ over \mathcal{N} has full existence, then it has extension. By Remark 3.15, to prove full existence we must show that for all small A, B, C , there is $A' \equiv_C A$ such that

$$[\text{dcl}(A'C) \cap \text{dcl}(BC) = \text{dcl}(C)] \wedge A' \downarrow_C^{\mathbf{a}\mathbb{B}} B.$$

In view of Fact 2.10 and Remark 3.9, we may assume without loss of generality that $C = \text{acl}(C)$, $A = \text{acl}(AC) \setminus \text{acl}(C)$, and $B = \text{acl}(BC) \setminus \text{acl}(C)$.

Then $C = \text{dcl}(C)$, $A = \text{dcl}(AC) \setminus \text{dcl}(C)$, and $B = \text{dcl}(BC) \setminus \text{dcl}(C)$. By Fact 3.16, the relation \downarrow_C^{aB} over \mathcal{N} has full existence. Therefore we may also assume that $A \downarrow_C^{\text{aB}} B$. By Remark 3.15,

$$\text{dcl}_{\mathbb{B}}(AC) \cap \text{dcl}_{\mathbb{B}}(BC) = \text{dcl}_{\mathbb{B}}(C).$$

So it suffices to show that there is $A' \equiv_C A$ such that

$$A' \cap B = \emptyset \wedge \text{dcl}_{\mathbb{B}}(A'C) = \text{dcl}_{\mathbb{B}}(AC).$$

For each element $\mathbf{a} \in A$, we define $\varepsilon(\mathbf{a})$ as the infimum of all the values $1 - \mu(\llbracket a \in \text{dcl}^{\mathcal{M}}(D) \rrbracket)$ over all countable $D \subseteq C$. Note that $\varepsilon(\mathbf{a}) = 0$ if and only if \mathbf{a} is pointwise definable over some countable subset of C . Add a constant symbol for each $\mathbf{a} \in A$, $\mathbf{b} \in B$, and $\mathbf{c} \in C$. For each $\mathbf{a} \in A$, add a variable \mathbf{a}' . Consider the set Γ of all conditions of the form

$$\llbracket \theta(\vec{\mathbf{a}}, \vec{\mathbf{c}}) \rrbracket = \llbracket \theta(\vec{\mathbf{a}}', \vec{\mathbf{c}}) \rrbracket \wedge \bigwedge_{i \leq |\vec{\mathbf{a}}|} d_{\mathbb{K}}(\mathbf{a}'_i, \mathbf{b}) \geq \varepsilon(\mathbf{a}_i)$$

where θ is an L -formula, $\vec{\mathbf{a}} \in A^{<\mathbb{N}}$, $\vec{\mathbf{c}} \in C^{<\mathbb{N}}$, and $\mathbf{b} \in B$.

Claim 1. For every finite subset Γ_0 of Γ , there is a set $A' = \{\mathbf{a}' : \mathbf{a} \in A\}$ that satisfies Γ_0 in \mathcal{N}_{ABC} .

Proof of Claim 1: Let A_0, B_0, C_0 be the set of elements of A, B, C respectively that occur in Γ_0 . Then A_0, B_0, C_0 are finite. If A_0 is empty, then Γ_0 is trivially satisfiable in \mathcal{N}_{ABC} , so we may assume that A_0 is non-empty. Let

$$A_0 = \{\mathbf{a}_0, \dots, \mathbf{a}_n\}, \vec{\mathbf{a}} = \langle \mathbf{a}_0, \dots, \mathbf{a}_n \rangle, C_0 = \{\mathbf{c}_0, \dots, \mathbf{c}_k\}, \vec{\mathbf{c}} = \langle \mathbf{c}_0, \dots, \mathbf{c}_k \rangle.$$

Let Θ_0 be the set of all sentences that occur on the left side of an equation in Γ_0 . Then Θ_0 is finite. By combining tuples, we may assume that each sentence in Θ_0 has the form $\theta(\vec{\mathbf{a}}, \vec{\mathbf{c}})$.

Since the algebraic independence relation over \mathcal{M} satisfies full existence, and T has $\text{acl} = \text{dcl}$, for each $\omega \in \Omega$ there exists

$$G_0(\omega) = \{g_0(\omega), \dots, g_n(\omega)\} \subseteq M$$

such that

$$\text{tp}^{\mathcal{M}}(G_0(\omega)/C_0(\omega)) = \text{tp}^{\mathcal{M}}(A_0(\omega)/C_0(\omega))$$

and

$$G_0(\omega) \cap B_0(\omega) \subseteq \text{dcl}^{\mathcal{M}}(C_0(\omega)).$$

Let $i \leq n$. Whenever $a_i(\omega) \notin \text{dcl}^{\mathcal{M}}(C_0(\omega))$, we have $g_i(\omega) \notin \text{dcl}^{\mathcal{M}}(C_0(\omega))$, and hence $g_i(\omega) \notin B_0(\omega)$.

Let $Z = \{\mathbf{0}, \mathbf{1}\}$ be as in Lemma 4.1. For each $i \leq n$ let

$$\mathbf{E}_i = \llbracket a_i \in \text{dcl}^{\mathcal{M}}(C_0) \rrbracket.$$

By Fact 3.2, for each i there exists a unique element $\mathbf{1}_{\mathbf{E}_i} \in \mathcal{K}$ that agrees with $\mathbf{1}$ on \mathbf{E}_i and agrees with $\mathbf{0}$ on $\neg \mathbf{E}_i$. By applying Condition (5) in Fact

3.1 to the formula

$$\bigwedge_{\theta \in \Theta_0} (\theta(\vec{u}, \vec{c}) \leftrightarrow \theta(\vec{a}, \vec{c})) \wedge \bigwedge_{i=0}^n \bigwedge_{\mathbf{b} \in B_0} (1_{E_i} = \tilde{0} \rightarrow u_i \neq \mathbf{b}),$$

we see that there exists a set

$$G_0 = \{\mathbf{g}_0, \dots, \mathbf{g}_n\} \subseteq \mathcal{K}$$

such that for each $\omega \in \Omega$, $\theta(\vec{a}, \vec{c}) \in \Theta_0$, $i \leq n$, and $\mathbf{b} \in B_0$:

- $\mathcal{M} \models \theta(\vec{g}(\omega), \vec{c}(\omega)) \leftrightarrow \theta(\vec{a}(\omega), \vec{c}(\omega))$;
- if $a_i(\omega) \notin \text{dcl}^{\mathcal{M}}(C_0(\omega))$, then $g_i(\omega) \neq b(\omega)$.

It follows that $\llbracket \theta(\vec{g}, \vec{c}) \rrbracket = \llbracket \theta(\vec{a}, \vec{c}) \rrbracket$ for each $\theta(\vec{a}, \vec{c}) \in \Theta_0$, and that $d_{\mathbb{K}}(\mathbf{g}_i, \mathbf{b}) \geq \varepsilon(\mathbf{a}_i)$ for each $i \leq n$ and $\mathbf{b} \in B_0$. Therefore Γ_0 is satisfied by G_0 in \mathcal{N}_{ABC} , and Claim 1 is proved.

By saturation, Γ is satisfied in \mathcal{N}_{ABC} by some set A' . Γ guarantees that $A' \equiv_C A$ and $\text{dcl}_{\mathbb{B}}(A'C) = \text{dcl}_{\mathbb{B}}(AC)$. It remains to show that for each $\mathbf{a} \in A$, $\mathbf{a}' \notin B$. Let $\mathbf{a} \in A$. By hypothesis we have $\mathbf{a} \notin \text{dcl}(C)$. By Fact 3.12, either \mathbf{a} is not pointwise definable over a countable subset of C and thus $\varepsilon(\mathbf{a}) > 0$, or there is a formula $\theta(u, \vec{v})$ and a tuple $\vec{c} \in C^{<\mathbb{N}}$ such that

$$\llbracket \theta(\mathbf{a}, \vec{c}) \rrbracket \in \text{fdcl}_{\mathbb{B}}(\{\mathbf{a}\} \cup C) \setminus \text{dcl}_{\mathbb{B}}(C).$$

Γ guarantees that $d_{\mathbb{K}}(\mathbf{a}', B) \geq \varepsilon(\mathbf{a})$, so in the case that $\varepsilon(\mathbf{a}) > 0$ we have $\mathbf{a}' \notin B$. Γ also guarantees that

$$\llbracket \theta(\mathbf{a}', \vec{c}) \rrbracket = \llbracket \theta(\mathbf{a}, \vec{c}) \rrbracket,$$

so in the case that $\varepsilon(\mathbf{a}) = 0$, we have

$$\llbracket \theta(\mathbf{a}', \vec{c}) \rrbracket = \llbracket \theta(\mathbf{a}, \vec{c}) \rrbracket \in \text{dcl}_{\mathbb{B}}(AC) \setminus \text{dcl}_{\mathbb{B}}(C).$$

But we are assuming that

$$\text{dcl}_{\mathbb{B}}(AC) \cap \text{dcl}_{\mathbb{B}}(BC) = \text{dcl}_{\mathbb{B}}(C),$$

so

$$\llbracket \theta(\mathbf{a}', \vec{c}) \rrbracket \notin \text{dcl}_{\mathbb{B}}(BC),$$

and hence $\mathbf{a}' \notin B$. This completes the proof. \square

6. SMALL LOCAL CHARACTER

In this section we show that if T has $\text{acl} = \text{dcl}$, then algebraic independence in T^R has small local character. In order to do this, we need the pointwise algebraic independence relation $\downarrow^{a\omega}$, which is of interest in its own right and will be studied further in the next section.

In the following, $\forall^c D$ means “for all countable D ”, and $\exists^c D$ means “there exists a countable D ”.

Definition 6.1. Let I be a ternary relation over \mathcal{M} that has monotonicity. The ternary relation $\downarrow^{I\omega}$ over \mathcal{N} (called *pointwise I -independence*) is defined as follows. For all small A, B, C , $A \downarrow_C^{I\omega} B$ if and only if

$$(\forall^c A' \subseteq A)(\forall^c B' \subseteq B)(\forall^c C' \subseteq C)(\exists^c D \in [C', C]) A' \downarrow_D^{I\omega} B'.$$

Fact 6.2. (Consequence of Lemma 4.1.4 in [AGK2].) If I be a ternary relation over \mathcal{M} that has monotonicity, then for all countable A, B, C ,

$$A \downarrow_C^{I\omega} B \Leftrightarrow \mu(\llbracket A \downarrow_C^I B \rrbracket) = 1.$$

We recall a definition from [AGK2].

Definition 6.3. In T , $A \downarrow_C^c B$ (read “ C covers A in B ”), is the relation that holds if and only if for every first order formula $\varphi(\bar{x}, \bar{y}, \bar{z}) \in [L]$ and all tuples $\bar{a} \in A^{|\bar{x}|}$, $\bar{b} \in B^{|\bar{y}|}$ and $\bar{c} \in C^{|\bar{z}|}$, there exists $\bar{d} \in C^{|\bar{y}|}$ such that

$$\mathcal{M} \models \varphi(\bar{a}, \bar{b}, \bar{c}) \Rightarrow \varphi(\bar{a}, \bar{d}, \bar{c}).$$

Fact 6.4. (Lemma 7.2.4 in [AGK2].) In T^R , the relation $\downarrow^{c\omega}$ has small local character.

Lemma 6.5. In T , $\downarrow^c \Rightarrow \downarrow^a$.

Proof. Suppose A, B, C are small and $A \downarrow_C^c B$ in \mathcal{M} . Let $e \in \text{acl}^{\mathcal{M}}(AC) \cap \text{acl}^{\mathcal{M}}(BC)$. Then there are algebraical formulas $\varphi(u, \vec{x}, \vec{z}), \psi(u, \vec{y}, \vec{w})$ and tuples $\vec{a} \in A^{<\mathbb{N}}, \vec{b} \in B^{<\mathbb{N}}, \vec{c}, \vec{c}' \in C^{<\mathbb{N}}$ such that

$$\mathcal{M} \models \varphi(e, \vec{a}, \vec{c}) \wedge \psi(e, \vec{b}, \vec{c}')$$

and

$$(\forall u \in M)[\mathcal{M} \models \varphi(u, \vec{a}, \vec{c}) \Rightarrow \text{tp}(u/AC) = \text{tp}(e/AC)].$$

Then

$$\mathcal{M} \models (\exists u)[\varphi(u, \vec{a}, \vec{c}) \wedge \psi(u, \vec{b}, \vec{c}')].$$

Since $A \downarrow_C^c B$, there exists $\vec{d} \in C^{<\mathbb{N}}$ such that

$$\mathcal{M} \models (\exists u)[\varphi(u, \vec{a}, \vec{c}) \wedge \psi(u, \vec{d}, \vec{c}')].$$

Therefore

$$\mathcal{M} \models \psi(e, \vec{d}, \vec{c}'),$$

so $e \in \text{acl}^{\mathcal{M}}(C)$. □

Proposition 6.6. In T^R , $\downarrow^{a\omega}$ has small local character.

Proof. By Lemma 6.5, for all countable $A, B, C \subseteq \mathcal{K}$, we have

$$\llbracket A \downarrow_C^c B \rrbracket \subseteq \llbracket A \downarrow_C^a B \rrbracket.$$

It follows easily that $\downarrow^{c\omega} \Rightarrow \downarrow^{a\omega}$. $\downarrow^{c\omega}$ has small local character by Fact 6.4, so by Remark 2.7, $\downarrow^{a\omega}$ has small local character. □

Proposition 6.7. *In T^R , $\downarrow^{a\omega} \wedge \downarrow^{a\mathbb{B}}$ has small local character.*

Proof. By Fact 3.17, $\downarrow^{d\mathbb{B}} \Rightarrow \downarrow^{a\mathbb{B}}$, so $\downarrow^{a\omega} \wedge \downarrow^{d\mathbb{B}} \Rightarrow \downarrow^{a\omega} \wedge \downarrow^{a\mathbb{B}}$. Then by Remark 2.7, it suffices to show that $\downarrow^{a\omega} \wedge \downarrow^{d\mathbb{B}}$ has small local character.

Let A, B, C_0 be small subsets of \mathcal{K} such that $C_0 \subseteq B$ and $|C_0| \leq |A| + \aleph_0$. By Fact 3.17, $\downarrow^{d\mathbb{B}}$ has small local character, so there is a set $C_1 \in [C_0, B]$ such that $|C_1| \leq |A| + \aleph_0$ and $A \downarrow_{C_1}^{d\mathbb{B}} B$. By Proposition 6.6, there is a set $C_2 \in [C_1, B]$ such that $|C_2| \leq |A| + \aleph_0$ and $A \downarrow_{C_2}^{a\omega} B$. By Fact 3.17, $\downarrow^{d\mathbb{B}}$ has base monotonicity, so $A \downarrow_{C_2}^{d\mathbb{B}} B$. Therefore $\downarrow^{a\omega} \wedge \downarrow^{d\mathbb{B}}$ has small local character. \square

Proposition 6.8. *The following are equivalent:*

- (i) T has $\text{acl} = \text{dcl}$.
- (ii) In T^R , $\downarrow^{a\omega} \wedge \downarrow^{a\mathbb{B}} \Rightarrow \downarrow^a$.

Proof. Suppose that (i) fails. Then in \mathcal{M} there is a finite set C and an element $a \in \text{acl}^{\mathcal{M}}(C) \setminus \text{dcl}^{\mathcal{M}}(C)$. By Fact 3.1 and saturation, there is an element \mathbf{b} and a finite set D in \mathcal{K} such that for each first order formula $\varphi(u, \vec{v})$, if $\mathcal{M} \models \varphi(a, C)$ then $\mu(\llbracket \varphi(\mathbf{b}, D) \rrbracket) = 1$ in \mathcal{N} . Therefore in \mathcal{N} we have $\mathbf{b} \downarrow_D^{a\omega} \mathbf{b} \wedge \mathbf{b} \downarrow_D^{a\mathbb{B}} \mathbf{b}$, but $\mathbf{b} \notin \text{acl}(D)$. Then $\mathbf{b} \not\downarrow_D^a \mathbf{b}$, so (ii) fails.

Now suppose (i) holds, and assume that $A \downarrow_C^{a\omega} B \wedge A \downarrow_C^{b\mathbb{B}} B$. We prove that $A \downarrow_C^a B$. By Remark 3.15, it suffices to show that $\text{dcl}(AC) \cap \text{dcl}(BC) \subseteq \text{dcl}(C)$. Let $\mathbf{d} \in \text{dcl}(AC) \cap \text{dcl}(BC)$. By Fact 3.12,

$$(6.1) \quad \mathbf{d} \in \text{dcl}^\omega(AC), \quad \mathbf{d} \in \text{dcl}^\omega(BC),$$

and

$$\text{fdcl}_{\mathbb{B}}(\mathbf{d}AC) \subseteq \text{dcl}_{\mathbb{B}}(AC), \quad \text{fdcl}_{\mathbb{B}}(\mathbf{d}BC) \subseteq \text{dcl}_{\mathbb{B}}(BC).$$

By Fact 3.8,

$$\text{dcl}_{\mathbb{B}}(\mathbf{d}AC) \subseteq \text{dcl}_{\mathbb{B}}(AC), \quad \text{dcl}_{\mathbb{B}}(\mathbf{d}BC) \subseteq \text{dcl}_{\mathbb{B}}(BC).$$

Then

$$\text{dcl}_{\mathbb{B}}(\mathbf{d}C) \subseteq \text{dcl}_{\mathbb{B}}(AC) \cap \text{dcl}_{\mathbb{B}}(BC).$$

Since $A \downarrow_C^{a\mathbb{B}} B$, we have

$$(6.2) \quad \text{dcl}_{\mathbb{B}}(\mathbf{d}C) \subseteq \text{dcl}_{\mathbb{B}}(C).$$

We next show that

$$(6.3) \quad \mathbf{d} \in \text{dcl}^\omega(C).$$

By Fact 3.12, it will then follow that $\mathbf{d} \in \text{dcl}(C)$, as required.

By (6.1), there are countable sets $A_0 \subseteq A, B_0 \subseteq B, C_0 \subseteq C$ such that

$$\mu(\llbracket \mathbf{d} \in \text{dcl}^{\mathcal{M}}(A_0C_0) \rrbracket) = \mu(\llbracket \mathbf{d} \in \text{dcl}^{\mathcal{M}}(B_0C_0) \rrbracket) = 1.$$

Since $A \downarrow_{C_0}^{a\omega} B$, there is a countable set $C_1 \in [C_0, C]$ such that

$$\mu(\llbracket A_0 \downarrow_{C_1}^a B_0 \rrbracket) = 1.$$

Then

$$\mu(\llbracket \text{dcl}^{\mathcal{M}}(A_0C_0) \cap \text{dcl}^{\mathcal{M}}(B_0C_0) \subseteq \text{acl}^{\mathcal{M}}(C_1) \rrbracket) = 1,$$

so $\mu(\llbracket \mathbf{d} \in \text{acl}^{\mathcal{M}}(C_1) \rrbracket) = 1$, and hence $\mathbf{d} \in \text{acl}^{\omega}(C)$. By (i), $\text{acl}^{\omega}(C) = \text{dcl}^{\omega}(C)$, so (6.3) holds. \square

Theorem 6.9. *Suppose T has $\text{acl} = \text{dcl}$. Then the relation \Downarrow^a in T^R has small local character.*

Proof. By Proposition 6.7, $\Downarrow^{a\omega} \wedge \Downarrow^{a\mathbb{B}}$ has small local character. By Remark 2.7, Proposition 6.8, and the hypothesis that T has $\text{acl} = \text{dcl}$, it follows that \Downarrow^a in T^R has small local character. \square

Here is a summary of our results about algebraic independence in T^R : For any T , algebraic independence in T^R does not satisfy finite character and does not satisfy base monotonicity. If T has $\text{acl} = \text{dcl}$, then algebraic independence in T^R satisfies all the axioms for a strict countable independence relation except base monotonicity, and also satisfies finite character and small local character.

7. POINTWISE ALGEBRAIC INDEPENDENCE

In the preceding sections we obtained results about the algebraic independence relation \Downarrow^a in T^R under the assumption that the underlying first order theory T has $\text{acl} = \text{dcl}$. In the general case where T is not assumed to have $\text{acl} = \text{dcl}$, the pointwise algebraic independence relation $\Downarrow^{a\omega}$ may be an attractive alternative to the algebraic independence relation \Downarrow^a in T^R . In this section we investigate the properties of $\Downarrow^{a\omega}$ in T^R when the underlying first order theory T is an arbitrary complete theory with models of cardinality > 1 . We first recall some results from [AGK2].

Fact 7.1. *(Special case of Proposition 7.1.4 in [AGK2].) In T^R , $\Downarrow^{a\omega}$ satisfies symmetry and all the axioms for a countable independence relation except perhaps base monotonicity and extension. Also, if \Downarrow^a in T has base monotonicity, then so does $\Downarrow^{a\omega}$ in T^R .*

Fact 7.2. *(Corollary 7.2.5 in [AGK2].) In T^R , $\Downarrow^{a\omega}$ has small local character.*

Definition 7.3. A ternary relation \Downarrow^I has the *countable union property* if whenever A, B, C are countable, $C = \bigcup_n C_n$, and $C_n \subseteq C_{n+1}$ and $A \Downarrow_{C_n}^I B$ for each n , we have $A \Downarrow_C^I B$.

Fact 7.4. *(Special case of Proposition 7.1.6 in [AGK2].) If the relation \Downarrow^a in T has monotonicity, finite character, and the countable union property, then the relation $\Downarrow^{a\omega}$ in T^R has finite character.*

Theorem 7.5. *In T^R , the relation $\Downarrow^{a\omega}$ has finite character.*

Proof. It is well-known that \downarrow^a in T has monotonicity and finite character. We show that \downarrow^a in T has the countable union property. Suppose A, B, C are countable, $C = \bigcup_n C_n$, and $C_n \subseteq C_{n+1}$ and $A \downarrow_{C_n}^I B$ for each n . Let $d \in \text{acl}^{\mathcal{M}}(AC) \cap \text{acl}^{\mathcal{M}}(BC)$. Then for some n we have $d \in \text{acl}^{\mathcal{M}}(AC_n) \cap \text{acl}^{\mathcal{M}}(BC_n)$. Since $A \downarrow_{C_n}^a B$, $d \in \text{acl}^{\mathcal{M}}(C_n)$, so $d \in \text{acl}^{\mathcal{M}}(C)$. Therefore $A \downarrow_C^a B$, and hence \downarrow^a has the countable union property. So by Fact 7.4, $\downarrow^{a\omega}$ has finite character. \square

$\llbracket A \downarrow_C^I B \rrbracket \in \mathcal{F}$ for all countable $A, B, C \subseteq \mathcal{K}$.

Lemma 7.6. *For all countable sets $A, B, C \subseteq \mathcal{K}$, the set $\llbracket A \downarrow_C^a B \rrbracket$ belongs to \mathcal{F} , and thus is measurable in the underlying probability space $(\Omega, \mathcal{F}, \mu)$.*

Proof. Let $\{\varphi_i(u, \vec{x}) \mid i \in \mathbb{N}\}$, $\{\psi_j(u, \vec{y}) \mid j \in \mathbb{N}\}$, and $\{\chi_k(u, \vec{z}) \mid k \in \mathbb{N}\}$ enumerate all algebraical formulas over the indicated variables. Then the set $\llbracket A \downarrow_C^a B \rrbracket$ is equal to

$$\bigcap_{i \in \mathbb{N}} \bigcap_{\vec{a} \subseteq AC} \bigcap_{j \in \mathbb{N}} \bigcap_{\vec{b} \subseteq BC} \bigcup_{k \in \mathbb{N}} \bigcup_{\vec{c} \subseteq C} \llbracket \forall u [\varphi_i(u, \vec{a}) \wedge \psi_j(u, \vec{b}) \Rightarrow \chi_k(u, \vec{c})] \rrbracket.$$

\square

Theorem 7.7. *The relation $\downarrow^{a\omega}$ over \mathcal{N} satisfies extension and full existence for all countable sets A, B, \widehat{B}, C .*

Proof. We first prove full existence for countable sets. Let A, B, C be countable subsets of \mathcal{K} . By Fact 3.16, the relation $\downarrow^{a\mathbb{B}}$ over \mathcal{N} has full existence. Therefore we may assume that $A \downarrow_C^{a\mathbb{B}} B$. By Fact 3.4,

$$\text{dcl}_{\mathbb{B}}(AC) \cap \text{dcl}_{\mathbb{B}}(BC) = \text{dcl}_{\mathbb{B}}(C).$$

Since \downarrow^a has full existence in \mathcal{M} , for each $\omega \in \Omega$ there exists a set $A'_0 \subseteq M$ such that $A'_0 \equiv_{C(\omega)} A(\omega)$ and $A'_0 \downarrow_{C(\omega)}^a B(\omega)$ in \mathcal{M} .

Let $\varphi_i(u, A, C)$, $\psi_i(u, B, C)$, and $\chi_i(u, C)$ be enumerations of all algebraical formulas over the indicated sets (with repetitions) such that for each pair of algebraical formulas $\varphi(u, A, C)$ and $\psi(u, B, C)$ there exists an i such that $(\varphi_i, \psi_i) = (\varphi, \psi)$. Whenever $\omega \in \Omega$, $A'_0 \subseteq M$, and $A'_0 \downarrow_{C(\omega)}^a B(\omega)$ in \mathcal{M} , for each $i \in \mathbb{N}$ there exists $j \in \mathbb{N}$ such that

$$(7.1) \quad \mathcal{M} \models \forall u [\varphi_i(u, A'_0, C(\omega)) \wedge \psi_i(u, B(\omega), C(\omega)) \rightarrow \chi_j(u, C(\omega))].$$

Let $\mathbb{N}^0 = \{\emptyset\}$ and $\mathbb{E}_\emptyset = \Omega$. For each $n > 0$ and n -tuple $s = \langle s(0), \dots, s(n-1) \rangle$ in \mathbb{N}^n , let \mathbb{E}_s be the set of all $\omega \in \Omega$ such that for some set $A'_0 \subseteq M$, $A'_0 \equiv_{C(\omega)} B(\omega)$ and (7.1) holds whenever $i < n$ and $j = s(i)$.

Let L' be the signature formed by adding to L the constant symbols

$$\{k_a, k_b, k_c : a \in A, b \in B, c \in C\}.$$

For each $\omega \in \Omega$, $(\mathcal{M}, A(\omega), B(\omega), C(\omega))$ will be the L' -structure where k_a, k_b, k_c are interpreted by $a(\omega), b(\omega), c(\omega)$. Form L'' by adding to L' countably many

additional constant symbols $\{k'_a : a \in A\}$ that will be used for elements of a countable subset A'_0 of \mathcal{M} .

Then for each $n > 0$ and $s \in \mathbb{N}^n$, there is a countable set of sentences Γ_s of L'' such that for each ω , $\omega \in \mathbf{E}_s$ if and only if Γ_s is satisfiable in $(\mathcal{M}, A(\omega), B(\omega), C(\omega))$. Since \mathcal{M} is \aleph_1 -saturated, Γ_s is satisfiable if and only if it is finitely satisfiable in $(\mathcal{M}, A(\omega), B(\omega), C(\omega))$. It follows that the set \mathbf{E}_s belongs to the σ -algebra \mathcal{F} . Moreover, since \downarrow^a has full existence in \mathcal{M} , for each n and $s \in \mathbb{N}^n$ we have

$$\Omega \doteq \bigcup \{E_t : t \in \mathbb{N}^n\}, \quad \mathbf{E}_s \doteq \bigcup \{E_{sk} : k \in \mathbb{N}\},$$

where sk is the $(n+1)$ -tuple formed by adding k to the end of s . We now cut down the sets \mathbf{E}_s to sets $\mathbf{F}_s \in \mathcal{F}$ such that:

- (a) $\mathbf{F}_\emptyset = \Omega$;
- (b) $\mathbf{F}_s \subseteq \mathbf{E}_s$ whenever $s \in \mathbb{N}^n$;
- (c) $\mathbf{F}_s \cap \mathbf{F}_t = \emptyset$ whenever $s, t \in \mathbb{N}^n$ and $s \neq t$;
- (d) $\mathbf{F}_s \doteq \bigcup \{\mathbf{F}_{sk} : k \in \mathbb{N}\}$ whenever $s \in \mathbb{N}^n$.

This can be done as follows. Assuming \mathbf{F}_s has been defined for each $s \in \mathbb{N}^n$, we let

$$\mathbf{F}_{sk} = \mathbf{F}_s \cap (\mathbf{E}_{sk} \setminus \bigcup_{j < k} \mathbf{F}_{sj}).$$

Now let $\theta_i(A, C)$ enumerate all first order sentences with constants for the elements of AC . Let Σ and Δ be the following countable sets of sentences of $(L'')^R$:

$$\Sigma = \{[\theta_i(A', C)] \doteq [\theta_i(A, C)] : i \in \mathbb{N}\}.$$

$$\Delta = \{\mathbf{F}_s \subseteq [\forall u[\varphi_i(u, A', C) \wedge \psi_i(u, B, C)] \rightarrow \chi_{s(i)}(u, C)] : s \in \mathbb{N}^{<\mathbb{N}}, i < |s|\}.$$

It follows from Fact 3.1 (5) and conditions (a)–(d) above that $\Sigma \cup \Delta$ is finitely satisfiable in \mathcal{N}_{ABC} . Then by saturation, there is a set A' that satisfies $\Sigma \cup \Delta$ in \mathcal{N}_{ABC} . Since A' satisfies Σ , we have $A' \equiv_C A$. The sentences Δ guarantee that $A' \downarrow_C^{a\omega} B$.

By the proof of Fact 2.4 (1) (see the Appendix of [Ad1]), invariance, monotonicity, transitivity, normality, symmetry, and full existence for all countable sets implies extension for all countable sets. Then by the preceding paragraphs and Fact 7.1, $\downarrow^{a\omega}$ satisfies extension for all countable sets. \square

Question 7.8. Does $\downarrow^{a\omega}$ satisfy extension for countable A, B, C and small \widehat{B} ?

Question 7.9. Does $\downarrow^{a\omega}$ satisfy full existence and/or extension?

We conclude by showing that the relations \downarrow^a and $\downarrow^{a\omega}$ are incomparable except in trivial cases.

Proposition 7.10.

- (i) $\downarrow^{a\omega}$ is not anti-reflexive.
- (ii) $\downarrow^{a\omega} \Rightarrow \downarrow^a$ always fails in \mathcal{N} .

(iii) $\downarrow^a \Rightarrow \downarrow^{a\omega}$ holds in \mathcal{N} if and only if the models of T are finite.

Proof. (i) and (ii) Let $Z = \{\mathbf{0}, \mathbf{1}\}$ be as in Lemma 4.1. Let \mathbf{D} be a set in \mathcal{F} of measure $\mu(\mathbf{D}) = 1/2$, and let $\mathbf{1}_{\mathbf{D}}$ agree with $\mathbf{1}$ on \mathbf{D} and agree with $\mathbf{0}$ on $\neg\mathbf{D}$. Then $\mathbf{1}_{\mathbf{D}} \downarrow_Z^{a\omega} \mathbf{1}_{\mathbf{D}}$, but $\mathbf{1}_{\mathbf{D}} \notin \text{acl}(Z)$. Therefore $\mathbf{1}_{\mathbf{D}} \not\downarrow_Z^a \mathbf{1}_{\mathbf{D}}$ and \downarrow^a is not anti-reflexive.

(iii) If \mathcal{M} is finite, then $\text{acl}^{\mathcal{M}}(\emptyset) = M$, so $A \downarrow_C^a B$ always holds in \mathcal{M} . Therefore $A \downarrow_C^{a\omega} B$ always holds in \mathcal{N} , and hence $\downarrow^a \Rightarrow \downarrow^{a\omega}$ holds in \mathcal{N} .

For the other direction, assume \mathcal{M} is infinite. By saturation of \mathcal{M} , there exist elements $0, 1, a, b \in M$ such that

$$0 \neq 1, \quad a \notin \text{acl}^{\mathcal{M}}(01), \quad \text{tp}(a/\text{acl}^{\mathcal{M}}(01)) = \text{tp}(b/\text{acl}^{\mathcal{M}}(01)), \quad a \downarrow_{01}^a b.$$

By a routine transfinite induction using Fact 3.1 and the saturation of \mathcal{N} , there is a mapping $a \mapsto \tilde{a}$ from M into \mathcal{K} such that for each tuple a_0, a_1, \dots in M and formula $\varphi(v_0, v_1, \dots)$ of L , if $\mathcal{M} \models \varphi(a_0, a_1, \dots)$ then $\mu(\llbracket \varphi(\tilde{a}_0, \tilde{a}_1, \dots) \rrbracket) = 1$ in \mathcal{N} . Let $\tilde{M} = \{\tilde{a} \mid a \in M\}$.

To simplify notation, suppose first that T already has a constant symbol for each element of $\text{acl}(01)$. Then $\text{acl}^{\mathcal{M}}(01) = \text{acl}^{\mathcal{M}}(\emptyset)$, so

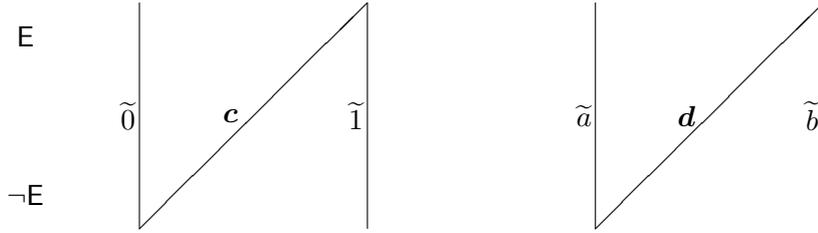
$$0 \neq 1, \quad a \notin \text{acl}^{\mathcal{M}}(\emptyset), \quad \text{tp}(a) = \text{tp}(b), \quad a \downarrow_{\emptyset}^a b \quad \text{in } \mathcal{M},$$

$$\mu(\llbracket \tilde{0} \neq \tilde{1} \rrbracket) = 1, \quad \tilde{a} \notin \text{dcl}(\emptyset), \quad \text{tp}(\tilde{a}) = \text{tp}(\tilde{b}), \quad \tilde{a} \downarrow_{\emptyset}^a \tilde{b} \quad \text{in } \mathcal{N}.$$

By Results 3.4 and 3.8, for each $A \subseteq \tilde{M}$,

$$\text{acl}(A) = \text{dcl}(A) = \text{cl}(\text{fdcl}(A)) = \text{fdcl}(A) \subseteq \tilde{M}.$$

Let $\mathbf{E} \in \mathcal{E}$ be an event of measure $\mu(\mathbf{E}) = 1/2$. Let \mathbf{c} agree with $\tilde{1}$ on \mathbf{E} and $\tilde{0}$ on $\neg\mathbf{E}$. Let \mathbf{d} agree with \tilde{a} on $\neg\mathbf{E}$ and with \tilde{b} on \mathbf{E} (see the figure).



Claim 1: $\tilde{a} \downarrow_{\emptyset}^a \mathbf{cd}$ in \mathcal{N} .

Proof of Claim 1: Suppose $x \in \text{acl}(\tilde{a}) \cap \text{acl}(\mathbf{cd})$ in \mathcal{N} . Then $x \in \text{dcl}(\tilde{a})$, so $x = \tilde{z}$ for some $z \in \text{dcl}^{\mathcal{M}}(a)$. Moreover, $x \in \text{dcl}(\mathbf{cd})$, so $x \in \text{dcl}^{\omega}(\mathbf{cd})$, and hence $x(\omega) \in \text{dcl}^{\mathcal{M}}(1b) = \text{dcl}^{\mathcal{M}}(b)$ for all $\omega \in \mathbf{E}$. Therefore $z \in \text{dcl}^{\mathcal{M}}(b)$. Since $\tilde{a} \downarrow_{\emptyset}^a \tilde{b}$ in \mathcal{N} , we have $x \in \text{acl}(\tilde{a}) \cap \text{acl}(\tilde{b}) = \text{acl}(\emptyset)$.

Claim 2: $\tilde{a} \not\downarrow_{\emptyset}^{a\omega} \mathbf{cd}$ in \mathcal{N} .

Proof of Claim 2: For all $\omega \in \neg\mathbf{E}$ we have $\mathbf{d}(\omega) = \tilde{a}(\omega)$, so

$$\tilde{a}(\omega) \in \text{acl}^{\mathcal{M}}(\tilde{a}(\omega)) \cap \text{acl}^{\mathcal{M}}(\mathbf{cd}(\omega)) \setminus \text{acl}^{\mathcal{M}}(\emptyset),$$

and hence $\omega \notin \llbracket \tilde{a} \downarrow_{\emptyset}^a \mathbf{cd} \rrbracket$. Therefore $\mu(\llbracket \tilde{a} \downarrow_{\emptyset}^a \mathbf{cd} \rrbracket) \leq 1/2$, so $\tilde{a} \not\downarrow_{\emptyset}^{a\omega} \mathbf{cd}$.

By Claims 1 and 2, $\downarrow^a \Rightarrow \downarrow^{a\omega}$ fails in \mathcal{N} .

We now turn to the general case where T need not have a constant symbol for each element of $\text{acl}(01)$. Our argument above shows that $\tilde{a} \downarrow_{01}^a \mathbf{cd}$ but $\tilde{a} \not\downarrow_{01}^{a\omega} \mathbf{cd}$ in \mathcal{N} , so $\downarrow^a \Rightarrow \downarrow^{a\omega}$ still fails in \mathcal{N} . \square

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