Derivations on operator algebras and their non associative counterparts

Model Theory of Operator Algebras
University of California, Irvine

Bernard Russo

University of California, Irvine

September 23, 2017

Motivation

"A veritable army of researchers took the theory of derivations of operator algebras to dizzying heights—producing a theory of cohomology of operator algebras as well as much information about automorphisms of operator algebras."

—Dick Kadison (Which Singer is that? 2000)

It is conjectured that all of the Hochschild cohomology groups $H^n(A, A)$ of a von Neumann algebra A vanish and that this is known to be true for most of them. In addition to associative algebras, cohomology groups are defined for Lie algebras and to some extent, for Jordan algebras. Since the structures of Jordan derivations and Lie derivations on von Neumann algebras are well understood, isn't it time to study the higher dimensional non associative cohomology of a von Neumann algebra? This section will be an introduction to the first and second Jordan cohomology groups of a von Neumann algebra. (Spoiler alert: Very little is known in this context about even the second Jordan cohomology group.)

Outline

1. Local and 2-local derivations on von Neumann algebras
2. Nonassociative structures on C*-algebras
3. Local and 2-local triple derivations on von Neumann algebras
4. Why study triple derivations?
5. Other contexts
6. Derivations into a module
7. Jordan cohomology

(Note: Only sections 1-5 were covered in the talk)

1. Local and 2-local <u>derivations</u> on von Neumann algebras
    - 1A Local derivations on von Neumann algebras
    - 1B 2-local derivations on von Neumann algebras
2. Nonassociative structures on C*-algebras
    - 2A Nonassociative structures on C*-algebras
    - 2B Nonassociative derivations on C*-algebras
    - 2C Some positive results on non associative derivations
3. Local and 2-local <u>triple derivations</u> on von Neumann algebras
    - 3A Local triple derivations on C*-algebras
    - 3B 2-local triple derivations on von Neumann algebras
4. Why study <u>triple derivations</u>?
    - 4A The contractive projection principle
    - 4B The Tits-Kantor-Koecher (TKK) Lie algebra

1A Local derivations on von Neumann algebras

A **derivation** is a linear map D from an algebra A to a two sided A-module M satisfying the Leibniz identity: $D(ab) = a \cdot D(b) + D(a) \cdot b$ for all $a, b \in A$.

It is **inner** if there is an element $m \in M$ such that $D(a) = m \cdot a - a \cdot m$.

Kaplansky 1953 Amer. J. Math., Kadison, Sakai 1966 Ann. Math.

Every derivation of a von Neumann algebra (into itself) is inner.

A **local derivation** from an algebra into a module is a linear mapping T whose value at each point a in the algebra coincides with the value of some derivation D_a at that point.

Kadison 1990 J. Alg., Johnson 2001 Trans. AMS

Every local derivation of a C*-algebra (into a Banach module) is a derivation.

Larson and **Sourour** showed in 1990 that a local derivation on the algebra of all bounded linear operators on a Banach space is a derivation. (+non self-adjoint)

Kaplansky's role in mid-20th century functional analysis

(Richard Kadison, Notices AMS 2008)

Gerd Pedersen "The density theorem is Kaplansky's gift to mankind. It can be used every day and twice on Sundays"

Bill Arveson's response "I use it twice on Saturdays too."

Gerd Pedersen On Kaplansky's introduction of AW*-algebras: "the subject refuses to die."

Roger Godement's reaction "What is the point of this generalization from W* to AW*, except, perhaps, to offer simplified proofs?"

Irving Kaplansky's response "I am pleased that he noticed the simplified proofs."

Richard Kadison "It was important to separate what was algebraic from what required analytic (measure-theoretic) considerations."

Algebraic and Analytic Aspects of Operator Algebras

CBMS Regional Conference Series in Mathematics-number 1 1970
University of Hawaii 1969
Irving Kaplansky

Irving Kaplansky
(1917–2006)

Richard V. Kadison
(b. 1925)

Soichiro Sakai
(b. 1928)

More from "Kaplansky's role in mid-20th century functional analysis" (Richard Kadison, Notices AMS 2008)

"In a paper that had a great influence in the development of operator algebras, Kaplansky proved that automorphisms of a type I AW*-algebra that leave the center element-wise fixed, are inner."

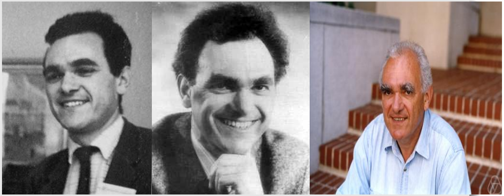
"For the purposes of that proof, Kaplansky introduces and develops the basics of the concept of 'Hilbert C*-module' (over commutative C*-algebras)."

"That concept has come to play an important role in the theory of operator algebras (Morita equivalence, for example)."

" A companion to the automorphism result is his proof that every derivation of a type I AW*-algebra is inner."

"Ten years later, the fact that all derivations on a von Neumann algebra are inner was proved, after which a torrent of work on derivations and the extension to higher Hochschild cohomology groups flowed through the literature of operator algebra theory and mathematical physics."

Barry E. Johnson
(1937-2002)



David R. Larson (b.)

Ahmed R. Sourour (b.)

Example 1 (attributed to C. U. Jensen)

Let $\mathbb{C}(x)$ denote the algebra of all rational functions (quotients of polynomials). There exists a local derivation of $\mathbb{C}(x)$ which is not a derivation.

Exercise 1

The derivations of $\mathbb{C}(x)$ are the mappings of the form $f \mapsto gf'$ for some g in $\mathbb{C}(x)$, where f' is the usual derivative of f.

Exercise 2

The local derivations of $\mathbb{C}(x)$ are the mappings which annihilate the constants.

Exercise 3

Write $\mathbb{C}(x) = S + T$ where S is the 2-dimensional space generated by 1 and x. Define $\alpha : \mathbb{C}(x) \to \mathbb{C}(x)$ by $\alpha(a + b) = b$. Then α is a local derivation which is not a derivation.

Example 2 (attributed to I. Kaplansky 1990)

There exists a local derivation of $\mathbb{C}(x)/[x^3]$ which is not a derivation (dim $= 3$) .

1B 2-local derivations on von Neumann algebras

A **2-local derivation** from an algebra into a module is a mapping T (not necessarily linear) whose values at each pair of points a, b in the algebra coincides with the values of some derivation $D_{a,b}$ at those two points.

Semrl 1997, Ayupov-Kudaybergenov 2015

Every 2-local derivation of a von Neumann algebra A is a derivation.

- Semrl 1997 (Proc. AMS) $A = B(H)$, H separable infinite dimensional
- Kim and Kim 2004 (Proc. AMS) $A = B(H)$, H finite dimensional
- Ayupov-Kudaybergenov 2012 (J. Math. Anal. Appl) $A = B(H)$, H arbitrary
- Ayupov-Kudaybergenov 2015 (Positivity) A arbitrary

Peter Semrl (b.)

Shavkat Ayupov (b.)

Karimbergen Kudaybergenov (b.)

Example: Zhang-Li 2006 (Acta Math. Sinica)

Consider the algebra of all upper-triangular complex 2×2-matrices

$$A = \left\{ x = \left(\begin{array}{cc} \lambda_{11} & \lambda_{12} \\ 0 & \lambda_{22} \end{array} \right) : \lambda_{ij} \in \mathbb{C} \right\}.$$

Define an operator Δ on A by

$$\Delta(x) = \begin{cases} 0, & \text{if } \lambda_{11} \neq \lambda_{22}, \\ \left(\begin{array}{cc} 0 & 2\lambda_{12} \\ 0 & 0 \end{array} \right), & \text{if } \lambda_{11} = \lambda_{22}. \end{cases}$$

Then Δ is a 2-local derivation, which is not a derivation.

2A Nonassociative structures on C*-algebras

Let A be a C*-algebra (or any associative *-algebra). A becomes

- A **Lie algebra** in the binary product $[a, b] = ab - ba$
  (reflects the differential geometric structure)
- A **Jordan algebra** in the binary product $a \circ b = (ab + ba)/2$
  (prototype of $\underline{JB^*\text{-algebra}}$, reflects the order structure)
- A **Jordan triple system** in the triple product $\{abc\} := (ab^*c + cb^*a)/2$
  (prototype of $\underline{JB^*\text{-triple}}$)

Jordan triple system reflects the metric geometric and holomorphic structure

- surjective isometry = triple isomorphism
- open unit ball is bounded symmetric domain

Lesser siblings:

- **associative triple systems** ab^*c (Ternary ring of operators, or TRO)
- **Lie triple systems** $[a, [b, c]]$

2B Nonassociative derivations on C*-algebras

Derivations

- **Lie derivation** $D([a,b]) = [a, D(b)] + [D(a), b]$
- **Jordan derivation** $D(a \circ b) = a \circ D(b) + D(a) \circ b$
  - simplifies to $D(a^2) = 2 \circ D(a)$
- **(Jordan) triple derivation** $D\{abc\} = \{D(a)bc\} + \{aD(b)c\} + \{abD(c)\}$

$$D(ab^*c)+D(cb^*a) = (Da)b^*c+cb^*(Da)+a(Db)^*c+c(Db)^*a+ab^*(Dc)+(Dc)b^*a$$

- simplifies to $D(ab^*a) = (Da)b^*a + a(Db)^*a + ab^*(Da)$

Inner derivations

- Inner Lie derivation: $x \mapsto [a, x]$ for some $a \in A$
- Inner Jordan derivation: $x \mapsto a \circ (b \circ x) - b \circ (a \circ x)$ for some $a, b \in A$
- Inner triple derivation: $x \mapsto \{abx\} - \{bax\}$ for some $a, b \in A$

Outer (Lie) derivation Any linear map into the center which vanishes on commutators is a Lie derivation which is not inner. Example:
$x \mapsto \mathrm{tr}\,(x)I \neq xy - yx$

2C Some positive results on non associative derivations

Sinclair 1970 (Proc. AMS)
Johnson 1996 (Math. Proc. Camb. Ph. Soc.)

Every Jordan derivation of a C*-algebra into a module is a derivation.

Upmeier 1980 (Math. Scand.)

Every Jordan derivation of a von Neumann algebra is an inner Jordan derivation.

Ho-Martinez-Peralta-R 2002 (J. Lon. Math. Soc.)

- Every triple derivation on a von Neumann algebra is an inner triple derivation.
- $B(H, K)$ has outer derivations if and only if $\dim H < \dim K = \infty$

Note: $B(H, K)$ is not an algebra, but closed under $\{xyz\} = (xy^*z + zy^*x)/2$.

Alan M. Sinclair (b.)

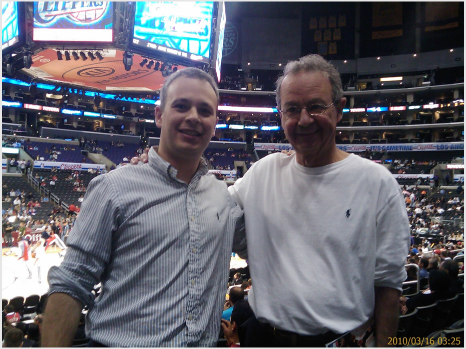
Harald Upmeier (b. 1950)

Juan Martinez Moreno (b.)

Jordan Algebras in Analysis, Operator Theory, and Quantum Mechanics

CBMS Regional Conference Series in Mathematics-number 67 1987
University of California, Irvine 1985
Harald Upmeier

Antonio Peralta (b. 1974) Bernard Russo (b. 1939)



C. Robert Miers (b.)

Theorem. Miers Duke 1973

Every Lie derivation $D : M \to M$ where M is a von Neumann algebra, has the form $D(x) = [a, x] + \lambda(x)$ for some $a \in M$ and all $x, y \in M$, where λ is a center valued linear map which annihilates commutators,

Johnson 1996 (Math. Proc. Camb. Ph. Soc.)

Every Lie derivation of a C*-algebra into a module is an inner Lie derivation plus a center valued linear map which vanishes on commutators.
(automatic continuity: Alaminos-Bresar-Villena 2004 MPCPS)

Theorem. Miers PAMS 1978

Every Lie triple derivation $D : M \to M$ where M is a von Neumann algebra without central abelian projections, has the form $D(x) = [a, x] + \lambda(x)$ for some $a \in M$ and all $x, y \in M$, where λ is a center valued linear map which annihilates commutators.

Definition of Lie triple derivation

$$D([x[y, z]]) = [Dx[y, z]] + [x, [Dy, z]] + [x, [y, Dz]]$$

3A Local triple derivations on C*-algebras

A **local triple derivation** from a triple system into itself is a linear mapping whose value at each point coincides with the value of some triple derivation at that point.

Mackey 2012, Burgos-Polo-Peralta 2014 (Bull. Lon. Math. Soc.)

Every local triple derivation on a C*-algebra is a triple derivation.

More precisely,

- Mackey 2012: <u>continuous</u> local triple derivation on a JBW*-triple .
- Burgos-Polo-Peralta 2014: local triple derivation on a JB*-triple

Michael Mackey

Maria Burgos

Francisco Fernandez Polo (b.)

Example (Variant of C. U. Jensen's example)

Let $\mathbb{C}(x)$ denote the algebra of all rational functions (quotients of polynomials). There exists a local triple derivation of $\mathbb{C}(x)$ which is not a triple derivation.

Exercise $1'$

The triple derivations of $\mathbb{C}(x)$ are the mappings of the form $\delta_{u,v}$, where $\delta_{u,v}f = uf' + ivf$, for f in $\mathbb{C}(x)$ and u, v fixed self adjoint elements of $\mathbb{C}(x)$.

Exercise $2'$

The local triple derivations of $\mathbb{C}(x)$ are the mappings which take 1 to iv for some fixed self adjoint v in $\mathbb{C}(x)$..

Exercise $3'$

The linear map $f \mapsto i(xf)'$ is a local triple derivation which is not a triple derivation.

3B 2-local triple derivations on von Neumann algebras

A **2-local triple derivation** from a triple system into itself is a mapping (not necessarily linear) whose values at each pair of points coincides with the values of some triple derivation at those two points.

Kudaybergenov-Oikhberg-Peralta-R 2014 (Ill. J. Math.)

Every 2-local triple derivation on a von Neumann algebra (considered as a Jordan triple system) is a triple derivation.

Ingredients

- The 2-local triple derivation is weak*-completely additive on projections
- Dorofeev-Shertsnev boundedness theorem
- Mackey-Gleason-Bunce-Wright theorem

Negative example? What about C*-algebras?

4. Why study triple derivations?

4A **The contractive projection principle**

Choi-Effros 1977 (J. Funct. Anal.)

The range of a unital completely contractive projection on a C*-algebra A is isometrically isomorphic to a C*-algebra

Effros-Stormer 1979 (Trans. Amer. Math. Soc.)

The range of a unital contractive projection on a C*-algebra A is isometrically isomorphic to a Jordan C*-algebra, that is, a JC*-algebra

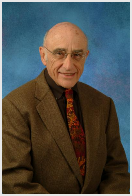
Friedman-R 1985 (J. Funct. Anal.)

The range of a contractive projection on a C*-algebra A is isometric to a subspace of A^{**} closed under $ab^*c + cb^*a$, that is, a JC*-triple.

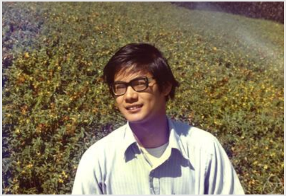
Kaup 1984 (Math. Scand.)

The range of a contractive projection on a JB*-triple is isometrically isomorphic to a JB*-triple. (Stacho: nonlinear projections on bounded symmetric domains)

Edward G. Effros

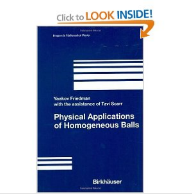


Man-Duen Choi

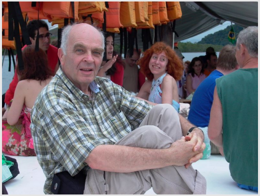


Erling Størmer

Yaakov Friedman



Wilhelm Kaup



4B **The Tits-Kantor-Koecher (TKK) Lie algebra**

Given two elements a, b in a Jordan triple V (for example, a von Neumann algebra under $\{abc\} = (ab^*c + cb^*a)/2$), we define the *box operator* $a \square b : V \to V$ by

$$a \square b(\cdot) = \{a, b, \cdot\}.$$

Since $[a \square b, c \square d] = [\{abc\}, d] - [c, \{bad\}]$,

$$V_0 = \{h = \sum a_j \square b_j : a_j, b_j \in V\}$$

is a Lie algebra with involution $(a \square b)^\natural = b \square a$.

The TKK Lie algebra $\mathfrak{L}(V)$ of V is $\mathfrak{L}(V) = V \oplus V_0 \oplus V$, with Lie product

$$[(x, h, y), (u, k, v)] = (hu - kx,\ [h, k] + x \square v - u \square y,\ k^\natural y - h^\natural v),$$

and involution $\theta : \mathfrak{L}(V) \to \mathfrak{L}(V), \quad \theta(x, h, y) = (y, -h^\natural, x)$

Let $\omega : V \to V$ be a triple derivation. Then $\mathfrak{L}(\omega) : \mathfrak{L}(V) \to \mathfrak{L}(V)$ defined by

$$\mathfrak{L}(\omega)(x, a\square b, y) = (\omega(a), \omega(a)\square b + a\square \omega(b), \omega(y))$$

is a θ-invariant Lie derivation, that is, $\mathfrak{L}(\omega)\circ\theta = \theta\circ\mathfrak{L}(\omega)$, and ω is an inner triple derivation if and only if $\mathfrak{L}(\omega)$ is an inner Lie derivation. Hence,

Theorem (Chu-R 2016, Cont. Math.)

If V is any von Neumann algebra, then every (Lie) derivation of $\mathfrak{L}(V)$ is inner.

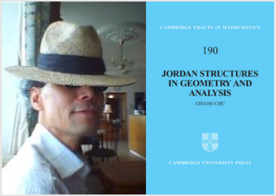
Proposition (Chu-R 2016, Cont. Math.)

If V is a finite von Neumann algebra, then $\mathfrak{L}(V) = [M_2(V), M_2(V)]$
(Lie isomorphism)

In particular, if $V = M_n(\mathbb{C})$, $\mathfrak{L}(V) = sl(2n, \mathbb{C})$ (classical Lie algebra of type A)

Timur Oikhberg

Cho-Ho Chu



5. Other contexts

5A **JB*-triples**

Burgos-Polo-Peralta 2014 (Bull. Lon. Math. Soc.)

Every local triple derivation on a JB*-triple is a triple derivation.

JB*-triples include C*-algebras, JC*-triples (=J* algebras of Harris), Jordan C*-algebras (=JB*-algebras), Cartan factors

Cartan factors: $B(H,K), A(H), S(H)$,spin factors + 2 exceptional

Gelfand-Naimark Theorem. Friedman-R 1986 (Duke Math. J.)

Every JB*-triple is isomorphic to a subtriple of a direct sum of Cartan factors.

- second dual is JB*-triple (Dineen 1984): contractive projection (Kaup 1984) and local reflexivity
- atomic decomposition of JBW*-triples (Friedman-R 1985): $(\oplus C_\alpha) \oplus \{$no extreme functionals$\}$

An application of ultra products. Proof of the Gelfand-Naimark Theorem.

Proposition (Heinrich Crelles Journal 1979)

For any Banach space E, there is a set I, an ultrafilter $\mathcal{U}$ on I, an isometry $J : E^{**} \to E^{\mathcal{U}}$, such that the range of J is the image of the contractive projection P on $E^{\mathcal{U}}$ defined by

$$P(x_i)_{\mathcal{U}} = J(w^*\text{-}\lim_{\mathcal{U}} x_i)$$

.

If E is a JB*-triple, $E^{\mathcal{U}}$ is also a JB*-triple, so by the contractive projection principle, we have

Theorem (Dineen 1984)

The bidual of a JB*-triple E is a JB*-triple containing E as a JB*-subtriple. The triple product on E^{**} is given by

$$\{xyz\} = w^*\text{-}\lim_{\mathcal{U}}\{Jx, Jy, Jz\}, \text{ for } x, y, z \in E^{**}.$$

Proof of Gelfand-Naimark: $E \to E^{**} \to \oplus C_\alpha$ is an isometry.

JBW*-triples (=JB*-triples with predual)

Hamhalter-Kudaybergenov-Peralta-R 2016 (J. Math. Phys.)

Every 2-local triple derivation on a continuous JBW*-triple is a triple derivation.

Structure of JBW*-triples. Horn 1987 (Math. Zeit.) Horn-Neher 1988 (Trans. AMS)

$$\underbrace{\oplus_\alpha L^\infty(\Omega_\alpha, C_\alpha)}_{\textit{Type I}} \oplus \underbrace{pM \oplus H(N, \beta)}_{\textit{Continuous}},$$

where each C_α is a Cartan factor, M and N are continuous von Neumann algebras, p is a projection in M, and β is a *-antiautomorphism of N of order 2 with fixed points $H(N, \beta)$.

No one has done the type I case yet. (derivations, local derivations, 2-local derivations)

Jan Hamhalter

Erhard Neher (on the right)

Ingredients of the Proof

pM

- p properly infinite: reduces to local derivation
- p finite:
  - $D = L_a + R_b + D_1,\ D_1|_{pMp} = 0, a, b$ skew hermitian
  - $\tau(D(x)y^*) = -\tau(xD(y)^*)$

$H(M, \beta)$

- Jordan version of Dorofeev boundedness theorem (Matveichuk 1995)
- a completely additive complex measure on $P(H(M, \beta))$ is bounded
- $D|_{P(H(M,\beta))}$ is completely additive

Ayupov-Kudaybergenov 2016 (Lin. Alg. Appl.)

A local Lie derivation on finite dimensional semisimple Lie algebra is Lie derivation.

Ayupov-Kudaybergenov-Rakhimov 2015 (Lin. Alg. Appl.)

A 2-local Lie derivation on finite dimensional semisimple Lie algebra is derivation.

Examples

• Let L be a finite-dimensional <u>nilpotent</u> Lie algebra with $\dim L \geq 2$. Then L admits a 2-local derivation which is not a derivation.
•Let L be a finite-dimensional <u>filiform</u> Lie algebra with $\dim L \geq 3$. Then L admits a local derivation which is not a derivation. (The case of general nilpotent finite-dimensional Lie algebras is still open.)

A Lie algebra L is called <u>filiform</u> if $\dim L^k = n - k - 1$ for $1 \leq k \leq n-1$, where $L^0 = L$, $L^k = [L^{k-1}, L]$, $k \geq 1$

Theorem (automatic continuity)

Every derivation on the algebra of locally measurable operators affiliated with a properly infinite von Neumann algebra is continuous in the local measure topology.

Proof

- Type I_∞ **Albeverio-Ayupov-Kudaybergenov** J. Funct. Anal. 2009
- Type III **Ayupov-Kudaybergenov**
  Inf. Dim. Anal. Quant. Prob. Rel. Top. 2010
- Type II_∞ **Ber-Chilin-Sukochev** Int. Eq. Op. Th. 2013

Ber-Chilin-Sukochev 2004 (Math. Notes) 2006 (Extracta Math.) (outer derivation)

If M is a commutative von Neumann algebra, then the algebra of measurable operators ($=$ measurable functions) admits a non-inner derivation if and only if the projection lattice of M is atomic.

I. S. Rakhimov

Sergio Albeverio

Fedor Sukochev

Inner derivations

Ber-Chilin-Sukochev Proc. Lon. Math. Soc. 2014

- **Theorem** Every continuous derivation on the algebra of locally measurable operators is inner.
- **Corollary** Every derivation on the algebra of locally measurable operators affiliated with a properly infinite von Neumann algebra is inner.

Proof

- Type I_∞ **Albeverio-Ayupov-Kudaybergenov** Siberian Adv. Math. 2008, Positivity 2008, J. Funct. Anal. 2009, Extracta Math. 2009
- Type III **Ayupov-Kudaybergenov**
  J. Op. Thy. 2012, J. Math. Anal. Appl. 2013
- Type II_∞ **Ber-Chilin-Sukochev** Proc. Lon. Math. Soc. 2014

Continuity for the II_1 case remains open.

5D **Local and 2-local derivations on algebras of measurable operators**

Local derivations

Albeverio-Ayupov-Kudaybergenov-Nurjanov Comm. Contem. Math. 2011

- If M is a von Neumann algebra with a faithful normal semifinite trace τ, then every local derivation on the τ-measurable operators, which is continuous in the τ-measure topology, is a derivation, inner if M is of Type I.
- If M is finite of type I, all local derivations are derivations.

Example

If M is a commutative von Neumann algebra, then the algebra of measurable operators (= measurable functions) admits a local derivation which is not a derivation if and only if the projection lattice of M is atomic.

Ayupov-Kudaybergenov-Alauadinov
Ann. Funct. Anal. 2013, Lin. Alg. Appl. 2013

• If M is a von Neumann algebra of Type I_∞, then every 2-local derivation on the algebra of locally measurable operators is a derivation.
• If M is a finite von Neumann algebra of Type I without abelian direct summands, then every 2-local derivation on the algebra of locally measurable operators ($=$ measurable operators) is a derivation.

Example

If M is a commutative von Neumann algebra, then the algebra of measurable operators ($=$ measurable functions) admits a 2-local derivation which is not a derivation if and only if the projection lattice of M is atomic.

An application of local derivation

Ayupov-Kudaybergenov (ArXiv 1604.07147 April 25,2016)

Let $S(M)$ be the algebra of measurable operators affiliated with a von Neumann algebra M. The following conditions are equivalent:
(a) M is abelian;
(b) For every derivation D on $S(M)$ its square D^2 is a local derivation

Example

Consider the algebra A of all upper-triangular complex 2×2-matrices.
- All derivations of A are inner. (Coelho-Milies 1993 Lin. Alg. Appl.)
- If $a = [a_{ij}]$ and $D = \mathrm{ad}\, a$, then D is a derivation and D^2 is the inner derivation $\mathrm{ad}\, b$, where

$$b = \begin{pmatrix} a_{11}^2 + a_{22}^2 & (a_{11} - a_{22})a_{12} \\ 0 & 2a_{11}a_{22} \end{pmatrix}$$

- So the square of every derivation is a derivation, hence a local derivation
- A is not commutative.

5E **Ternary Rings of Operators**

Derivations on C^*-algebras have suitable counterparts in a more general setting of ternary rings of operators, or TROs for short.

If X is a Banach subspace of a C^*-algebra and $xy^*z + zy^*x \in X$ for every x, y, z in X, then X is called a JC*-triple and a *triple derivation* on X is an operator $\tau \in B(X)$ satisfying

$$\tau(\{xy^*z\}) = \{\tau(x)y^*z\} + \{x\tau(y)^*z\} + \{xy^*\tau(z)\}$$

for every x, y, z in X, where $\{xyz\} = (xy^*z + zy^*x)/2$.

We shall use the term TRO-derivation, as follows: If X is a Banach subspace of a C^*-algebra and $xy^*z \in X$ for every x, y, z in X, then X is called a TRO and a *TRO-derivation* on X is an operator $\tau \in B(X)$ satisfying

$$\tau(xy^*z) = \tau(x)y^*z + x\tau(y)^*z + xy^*\tau(z)$$

for every x, y, z in X.

If A is a C^*-algebra and $p \in A$ is a projection, we let $D_p^*(A)$ denote the (real) Banach Lie algebra of self-adjoint p-derivations on A. To be more precise $D_p^*(A)$ consists of all derivations $\delta \in D(A)$ that satisfy $\delta(p) = 0$ and $\delta = \delta^*$.

If X is a TRO, we use $D_{TRO}(X)$ to denote the (real) Banach Lie algebra of all TRO-derivations on X.

Remark

Let A be a unital C^*-algebra and let $p \in A$ be a projection. Then the map

$$\Delta\colon D_p^*(A) \to D_{TRO}(pA(1-p)), \qquad \Delta(\delta) = \delta_{|pA(1-p)}$$

is a homomorphism of Banach Lie algebras.

If A is a unital C*-algebra and e is a projection in A, then $X := eA(1-e)$ is a TRO.

Conversely if $X \subset B(K,H)$ is a TRO, then with $X^* = \{x^* : x \in X\} \subset B(H,K)$, $XX^* = \text{span}\,\{xy^* : x, y \in X\} \subset B(H)$, $X^*X = \text{span}\,\{z^*w : z, w \in X\} \subset B(K)$, $K_l(X) = \overline{XX^*}^n$, $K_r(X) = \overline{X^*X}^n$, we let

$$A_X = \begin{bmatrix} K_l(X) + \mathbb{C}1_H & X \\ X^* & K_r(X) + \mathbb{C}1_K \end{bmatrix} \subset B(H \oplus K)$$

denote the (unital) linking C*-algebra of X. Then we have a TRO-isomorphism $X \simeq eA_X(1-e)$, where $e = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right]$.

Lemma

Let X be a TRO and let $D : X \to X$ be a TRO-derivation of X. If $A_0 = \left(\begin{smallmatrix} XX^* & X \\ X^* & X^*X \end{smallmatrix}\right)$, then the map $\delta_0 : A_0 \to A_0$ given by

$$\begin{pmatrix} \sum_i x_i y_i^* & x \\ y^* & \sum_j z_j^* w_j \end{pmatrix} \mapsto \begin{pmatrix} \sum_i (x_i (Dy_i)^* + (Dx_i) y_i^*) & Dx \\ (Dy)^* & \sum_j (z_j^* (Dw_j) + (Dz_j)^* w_j) \end{pmatrix}$$

is well defined and a bounded *-derivation of A_0, which extends D (when X is embedded in A_X via $x \mapsto \left(\begin{smallmatrix} 0 & x \\ 0 & 0 \end{smallmatrix}\right)$), and which itself extends to a *-derivation δ of A_X. Thus, the Lie algebra homomorphism $\Delta : \delta \mapsto \delta_{|X}$ given in the above Remark is onto.

Theorem (Pluta-R)

Every TRO-derivation of a TRO X is spatial in the sense that there exist $\alpha \in K_l(X)''$ and $\beta \in K_r(X)''$ such that $\alpha^* = -\alpha,\ \beta^* = -\beta$, and $Dx = \alpha x + x\beta$ for every $x \in X$.

Corollary

Every TRO derivation of a C*-algebra A is of the form $A \ni x \mapsto \alpha x + x\beta$ with elements $\alpha, \beta \in \overline{A}^w$ with $\alpha^* = -\alpha, \beta^* = -\beta$. In particular, every TRO derivation of a von Neumann algebra is an inner TRO derivation.

Thus, every W*-TRO which is TRO-isomorphic to a von Neumann algebra has only inner TRO derivations. For example, this is the case for the stable W*-TROs of Ruan and the weak*-closed right ideals in certain continuous von Neumann algebras acting on separable Hilbert spaces .

A triple derivation δ of a JC*-triple X is said to be an *inner triple derivation* if there exist finitely many elements $a_i, b_i \in X$, $1 \leq i \leq n$, such that $\delta x = \sum_{i=1}^n (\{a_i b_i x\} - \{b_i a_i x\})$ for $x \in X$, where $\{xyz\} = (xy^*z + zy^*x)/2$. For convenience, we denote the inner triple derivation $x \mapsto \{abx\} - \{bax\}$ by $\delta(a, b)$. Thus

$$\delta(a, b)(x) = (ab^*x + xb^*a - ba^*x - xa^*b)/2.$$

Let X be a TRO. Then X is a JC*-triple in the triple product $(xy^*z + zy^*x)/2$, and every TRO-derivation of X is obviously a triple derivation.

On the other hand, every inner triple derivation is an inner TRO-derivation. Indeed, if $\delta(x) = \{abx\} - \{bax\}$, for some $a, b \in X$, then $\delta(x) = Ax + xB$, where $A = ab^* - ba^* \in XX^*$, $B = b^*a - a^*b \in X^*X$ with A, B skew-hermitian.

Moreover, since every triple derivation δ on X is the strong operator limit of a net δ_α of inner triple derivations, hence TRO-derivations, we have (i) and (ii) in the following proposition.

Proposition

Let X be a TRO.

(i) Every TRO-derivation is the strong operator limit of inner TRO-derivations.

(ii) The triple derivations on X coincide with the TRO-derivations.

(iii) The inner triple derivations on X coincide with the inner TRO-derivations

(iv) All TRO derivations of X are inner, if and only if, all triple derivations of X are inner.

6. Derivations into a module

6A **Automatic Continuity and 1-Cohomology**

Two fundamental questions concerning derivations from a Banach algebra A into a Banach A-bimodule M are:

- Is an everywhere defined derivation automatically continuous?
- Are all continuous derivations inner? If not, can every continuous derivation be approximated by inner derivations?

One can ask the same questions in the setting of Jordan Banach algebras (and Jordan modules), and more generally for Jordan Banach triple systems (and Jordan Banach triple modules). Significant special cases occur in each context when $M = A$ or when $M = A^*$.

In order to obtain a better perspective on the objectives of this program, we shall give here a comprehensive review of the major existing results on these two problems in the contexts in which we will be interested, namely, C*-algebras, JB*-algebras, and JB*-triples.

C^*-algebras

A derivation on a Banach algebra, A, into a Banach A-bimodule, M, is a linear mapping $D : A \to M$ such that $D(ab) = a \cdot D(b) + D(a) \cdot b$. An inner derivation, in this context, is a derivation of the form: $\mathrm{ad}_x(a) = x \cdot a - a \cdot x$ ($x \in M$, $a \in A$).

In the context of C^*-algebras, automatic continuity results were initiated by Kaplansky before 1950 and culminated in the following results:

- Every derivation from a C^*-algebra into itself is continuous (Sakai, 1960)
- Every derivation from a C^*-algebra A into a Banach A-bimodule is continuous (Ringrose, 1972).

The major results for C^*-algebras regarding inner derivations read as follows:

- Every derivation from a C^*-algebra on a Hilbert space H into itself is of the form $x \mapsto ax - xa$ for some a in the weak closure of the C^*-algebra in $L(H)$ (Sakai, Kadison 1966)
- Every amenable C^*-algebra is nuclear (Connes, 1976)
- Every nuclear C^*-algebra is amenable (Haagerup, 1983)
- Every C^*-algebra is weakly amenable (Haagerup, 1983).

JB*-triples

A (triple or ternary) derivation on a Jordan Banach triple A into a Jordan Banach triple module M is a *conjugate* linear mapping $D : A \to M$ such that $D\{a,b,c\} = \{Da,b,c\} + \{a,Db,c\} + \{a,b,Dc\}$. (Jordan Banach triple and Jordan Banach triple module will be defined below)

In the context of JB*-triples, automatic continuity results are as follows:
• Every triple derivation of a JB*-triple is continuous (Barton-Friedman 1990)
• Peralta and Russo in 2013 gave necessary and sufficient conditions under which a derivation of a JB*-triple into a Jordan Banach triple module is continuous.

These conditions are automatically satisfied in the case that the JB*-triple is actually a C*-algebra, with the triple product $(xy^*z + zy^*x)/2$, leading to a new proof of the theorem of Ringrose quoted above.

These conditions are also satisfied in the case of a derivation of a JB*-triple into its dual space leading Ho, Peralta and Russo to the study of weak amenability for JB*-triples.

Statement of Main Automatic Continuity Result

For each submodule S of a triple E-module X, we define its *quadratic annihilator*, $\text{Ann}_E(S)$, as the set $\{a \in E : Q(a)(S) = \{a, S, a\} = 0\}$.

The *separating space*, $\sigma_Y(T)$, of $T : X \to Y$ is defined as the set of all z in Y for which there exists a sequence $(x_n) \subseteq X$ with $x_n \to 0$ and $T(x_n) \to z$.

Theorem

Let E be a complex JB*-triple, X a Banach triple E-module, and let $\delta : E \to X$ be a triple derivation. Then δ is continuous if and only if $\text{Ann}_E(\sigma_X(\delta))$ is a (norm-closed) linear subspace of E and

$$\{\text{Ann}_E(\sigma_X(\delta)), \text{Ann}_E(\sigma_X(\delta)), \sigma_X(\delta)\} = 0.$$

Corollary

Let E be a real or complex JB*-triple. Then
(a) Every derivation $\delta : E \to E$ is continuous. (Barton-Friedman)
(b) Every derivation $\delta : E \to E^*$ is continuous (suggests ternary weak amenability).

Associative modules

Let A be an associative algebra. Let us recall that an A-**bimodule** is a vector space X, equipped with two bilinear products $(a,x) \mapsto ax$ and $(a,x) \mapsto xa$ from $A \times X$ to X satisfying the following axioms:

$$a(bx) = (ab)x, \quad a(xb) = (ax)b, \text{ and, } (xa)b = x(ab),$$

for every $a, b \in A$ and $x \in X$.

The space $A \oplus X$ is an associative algebra with respect to the product

$$(a,x)(b,y) := (ab, ay + xb).$$

Jordan modules

Let A be a Jordan algebra. A **Jordan A-module** is a vector space X, equipped with two bilinear products $(a,x) \mapsto a \circ x$ and $(x,a) \mapsto x \circ a$ from $A \times X$ to X, satisfying:

$$a \circ x = x \circ a, \quad a^2 \circ (x \circ a) = (a^2 \circ x) \circ a, \text{ and,}$$

$$2((x \circ a) \circ b) \circ a + x \circ (a^2 \circ b) = 2(x \circ a) \circ (a \circ b) + (x \circ b) \circ a^2,$$

for every $a, b \in A$ and $x \in X$.

The space $A \oplus X$ is a Jordan algebra with respect to the product

$$(a,x) \circ (b,y) := (a \circ b, a \circ y + b \circ x).$$

Jordan module—We must first define Jordan triple system

A complex (resp., real) **Jordan triple** is a complex (resp., real) vector space E equipped with a triple product $E \times E \times E \to E \quad , \quad (x, y, z) \mapsto \{x, y, z\}$ which is bilinear and symmetric in the outer variables and conjugate linear (resp., linear) in the middle one and satisfying the so-called *"Jordan Identity"*:

$$L(a, b)L(x, y) - L(x, y)L(a, b) = L(L(a, b)x, y) - L(x, L(b, a)y),$$

for all a, b, x, y in E, where $L(x, y)z := \{x, y, z\}$.

The Jordan identity is equivalent to

$$\{a, b, \{c, d, e\}\} = \{\{a, b, c\}, d, e\} - \{c, \{b, a, d\}, e\} + \{c, d, \{a, b, e\}\},$$

which asserts that the map $iL(a, a)$ is a *triple derivation* (to be defined shortly).

It also shows that the span of the "multiplication" operators $L(x, y)$ is a Lie algebra. (We won't use this fact in this talk)

Jordan triple module

Let E be a complex (resp. real) Jordan triple. A **Jordan triple E-module** is a vector space X equipped with three mappings

$$\{.,.,.\}_1 : X \times E \times E \to X, \quad \{.,.,.\}_2 : E \times X \times E \to X$$

$$\text{and } \{.,.,.\}_3 : E \times E \times X \to X$$

in such a way that the space $E \oplus X$ becomes a real Jordan triple with respect to the triple product $\{a_1 + x_1, a_2 + x_2, a_3 + x_3\} =$

$$\{a_1, a_2, a_3\}_E + \{x_1, a_2, a_3\}_1 + \{a_1, x_2, a_3\}_2 + \{a_1, a_2, x_3\}_3.$$

(PS: we don't really need the subscripts on the triple products)

The Jordan identity

$\{a, b, \{c, d, e\}\} = \{\{a, b, c\}, d, e\} - \{c, \{b, a, d\}, e\} + \{c, d, \{a, b, e\}\}$, holds whenever exactly one of the elements belongs to X.

In the complex case we have the unfortunate technical requirement that $\{x, a, b\}_1$ (which $=\{b, a, x\}_3$) is linear in a and x, and **conjugate** linear in b; and $\{a, x, b\}_2$ is **conjugate** linear in a, b, x.

Every (associative) Banach A-bimodule (resp., Jordan Banach A-module) X over an associative Banach algebra A (resp., Jordan Banach algebra A) is a real Banach triple A-module (resp., A-module) with respect to the "*elementary*" product

$$\{a, b, c\} := \frac{1}{2}(abc + cba)$$

(resp., $\{a, b, c\} = (a \circ b) \circ c + (c \circ b) \circ a - (a \circ c) \circ b$), where one element of a, b, c is in X and the other two are in A. In particular, this holds if $X = A$.

The dual space, E^*, of a complex (resp., real) Jordan Banach triple E is a complex (resp., real) triple E-module with respect to the products:

$$\{a, b, \varphi\}(x) = \{\varphi, b, a\}(x) := \varphi\{b, a, x\} \tag{1}$$

and

$$\{a, \varphi, b\}(x) := \overline{\varphi\{a, x, b\}}, \tag{2}$$

$\forall x \in X, a, b \in E, \varphi \in E^*$.

Derivations

Let X be a Banach A-bimodule over an (associative) Banach algebra A. A linear mapping $D : A \to X$ is said to be a **derivation** if $D(ab) = D(a)b + aD(b)$, for every a, b in A. For emphasis we call this a **binary (or associative) derivation**.

We denote the set of all continuous binary derivations from A to X by $\mathcal{D}_b(A, X)$.

When X is a Jordan Banach module over a Jordan Banach algebra A, a linear mapping $D : A \to X$ is said to be a **derivation** if $D(a \circ b) = D(a) \circ b + a \circ D(b)$, for every a, b in A. For emphasis we call this a **Jordan derivation**.

We denote the set of continuous Jordan derivations from A to X by $\mathcal{D}_J(A, X)$.

In the setting of Jordan Banach triples, a **triple** or **ternary derivation** from a (real or complex) Jordan Banach triple, E, into a Banach triple E-module, X, is a conjugate linear mapping $\delta : E \to X$ satisfying

$$\delta \{a, b, c\} = \{\delta(a), b, c\} + \{a, \delta(b), c\} + \{a, b, \delta(c)\}, \tag{3}$$

for every a, b, c in E.

We denote the set of all continuous ternary derivations from E to X by $\mathcal{D}_t(E, X)$.

Inner derivations

Let X be a Banach A-bimodule over an associative Banach algebra A. Given x_0 in X, the mapping $D_{x_0} : A \to X$, $D_{x_0}(a) = x_0 a - a x_0$ is a bounded (associative or binary) derivation. Derivations of this form are called **inner**.

The set of all inner derivations from A to X will be denoted by $\mathcal{I}nn_b(A, X)$.

Banach algebra A is **weakly amenable** if $\mathcal{I}nn_b(A, A^*) = \mathcal{D}(A, A^*)$

When x_0 is an element in a Jordan Banach A-module, X, over a Jordan Banach algebra, A, for each $b \in A$, the mapping $\delta_{x_0,b} : A \to X$,

$$\delta_{x_0,b}(a) := (x_0 \circ a) \circ b - (b \circ a) \circ x_0, \ (a \in A),$$

is a bounded derivation. Finite sums of derivations of this form are called **inner**.

The set of all inner Jordan derivations from A to X is denoted by $\mathcal{I}nn_J(A, X)$

Jordan Banach algebra A is **(Jordan) weakly amenable** if $\mathcal{I}nn_J(A, A^*) = \mathcal{D}_J(A, A^*)$

Let E be a complex (resp., real) Jordan triple and let X be a triple E-module. For each $b \in E$ and each $x_0 \in X$, we conclude, via the main identity for Jordan triple modules, that the mapping $\delta = \delta(b, x_0) : E \to X$, defined by

$$\delta(a) = \delta(b, x_0)(a) := \{b, x_0, a\} - \{x_0, b, a\} \quad (a \in E), \tag{4}$$

is a ternary derivation from E into X. Finite sums of derivations of the form $\delta(b, x_0)$ are called **inner triple derivations**.

The set of all inner ternary derivations from E to X is denoted by $\mathcal{I}nn_t(E, X)$.

Jordan Banach triple E is **ternary weakly amenable** if $\mathcal{I}nn_t(E, E^*) = \mathcal{D}_t(E, E^*)$

Ternary Weak Amenability (Ho-Peralta-R)

Proposition

Let A be a unital Banach *-algebra equipped with the ternary product given by $\{a, b, c\} = \frac{1}{2}\,(ab^*c + cb^*a)$ and the Jordan product $a \circ b = (ab + ba)/2$. Then

$$\mathcal{D}_t(A, A^*) = \mathcal{D}_J^*(A, A^*) \circ * + \mathcal{I}nn_t(A, A^*).$$

Proposition

Every commutative (real or complex) C*-algebra A is **ternary weakly amenable**, that is $\mathcal{D}_t(A, A^*) = \mathcal{I}nn_t(A, A^*)$ ($\neq 0$ btw).

Proposition

The C*-algebra $A = M_n(\mathbb{C})$ is ternary weakly amenable (Hochschild 1945) and **Jordan weakly amenable** (Jacobson 1951).

Question

Is $C_0(X, M_n(\mathbb{C}))$ ternary weakly amenable?

Negative results

Proposition

The C*-algebra $A = K(H)$ of all compact operators on an infinite dimensional Hilbert space H is **not** ternary weakly amenable.

Proposition

The C*-algebra $A = B(H)$ of all bounded operators on an infinite dimensional Hilbert space H is **not** ternary weakly amenable.

Non algebra results

Theorem

Let H and K be two complex Hilbert spaces with $\dim(H) = \infty > \dim(K)$. Then the rectangular complex Cartan factor of type I, $L(H, K)$, and all its real forms are **not** ternary weakly amenable.

Theorem

Every commutative (real or complex) JB*-triple E is **approximately ternary weakly amenable**, that is, $\mathcal{I}nn_t(E, E^*)$ is a norm-dense subset of $\mathcal{D}_t(E, E^*)$.

Commutative Jordan Gelfand Theory (Kaup,Friedman-R)

Given a commutative (complex) JB*-triple E, there exists a principal $\mathbb{T}$-bundle $\Lambda = \Lambda(E)$, i.e. a locally compact Hausdorff space Λ together with a continuous mapping $\mathbb{T} \times \Lambda \to \Lambda$, $(t, \lambda) \mapsto t\lambda$ such that $s(t\lambda) = (st)\lambda$, $1\lambda = \lambda$ and $t\lambda = \lambda \Rightarrow t = 1$, satisfying that E is JB*-triple isomorphic to

$$\mathcal{C}_0^{\mathbb{T}}(\Lambda) := \{f \in \mathcal{C}_0(\Lambda) : f(t\lambda) = tf(\lambda), \forall t \in \mathbb{T}, \lambda \in \Lambda\}.$$

6C **Normal Ternary Weak Amenability**

Building on earlier work of Kadison, Sakai proved that every derivation of a von Neumann algebra into itself is inner. Building on earlier work of Bunce and Paschke, Haagerup, on his way to proving that every C*-algebra is weakly amenable, showed that every derivation of a von Neumann algebra into its predual is inner.

Thus the first Hochschild cohomology groups $H^1(M, M)$ and $H^1(M, M_*)$ vanish for any von Neumann algebra M.

It is known that every commutative (real or complex) C*-algebra A is **ternary weakly amenable**, that is, every triple derivation of A into its dual A^* is an inner triple derivation, but the C*-algebras $K(H)$ of all compact operators and $B(H)$ of all bounded operators on an infinite dimensional Hilbert space H fail this property.

It follows that finite dimensional von Neumann algebras and abelian von Neumann algebras have the property that every triple derivation into the predual is an inner triple derivation, analogous to the Haagerup result. We call this property **normal ternary weak amenability**

Sums of commutators in von Neumann algebras

If M is a finite von Neumann algebra, then every element of M of central trace zero is a finite sum of commutators (Fack-delaHarpe 1980, Pearcy-Topping 1969.

If M is properly infinite (no finite central projections), then every element of M is a finite sum of commutators (Halpern 1969, Halmos 1952, 1954, Brown-Pearcy-Topping 1968).

Thus for any von Neumann algebra, we have $M = Z(M) + [M, M]$, where $Z(M)$ is the center of M and $[M, M]$ is the set of finite sums of commutators in M.

Theorem 1 (Pluta-R)

Let M be a von Neumann algebra.

(a) If every triple derivation of M into M_* is approximated in norm by inner triple derivations, then M is finite.

(b) If M is a finite von Neumann algebra acting on a separable Hilbert space or if M is a finite factor, then every triple derivation of M into M_* is approximated in norm by inner triple derivations.

Corollary

If M acts on a separable Hilbert space, or if M is a factor, then M is finite if and only if every triple derivation of M into M_* is approximated in norm by inner triple derivations.

Theorem 2 (Pluta-R)

If M is a properly infinite factor, then the real vector space of triple derivations of M into M_*, modulo the norm closure of the inner triple derivations, has dimension 1.

Corollary

If M is a factor, the linear space of triple derivations into the predual, modulo the norm closure of the inner triple derivations, has dimension 0 or 1: It is zero if the factor is finite; and it is 1 if the factor is infinite.

Summary: If M is a factor,

M is infinite $\Leftrightarrow \dfrac{\text{Jordan Derivations into } M_*}{\textbf{Norm closure} \text{ of inner } \textbf{Jordan} \text{ derivations into } M_*} \sim \mathbb{C}$

M is finite $\Leftrightarrow \dfrac{\text{Jordan Derivations into } M_*}{\textbf{Norm closure} \text{ of inner } \textbf{Jordan} \text{ derivations into } M_*} = 0$

M is infinite $\Leftrightarrow \dfrac{\text{Jordan triple Derivations into } M_*}{\textbf{Norm closure} \text{ of inner } \textbf{triple} \text{ derivations into } M_*} \sim \mathbb{R}$

M is finite $\Leftrightarrow \dfrac{\text{Jordan triple Derivations into } M_*}{\textbf{Norm closure} \text{ of inner } \textbf{triple} \text{ derivations into } M_*} = 0$

Sums of commutators in the predual

For any von Neumann algebra M, we shall write $(M_*)_0$ for the set of elements $\psi \in M_*$ such that $\psi(1) = 0$. If M is finite and admits a faithful normal finite trace tr, which therefore extends to a trace on M_*, then $(M_*)_0 = \mathrm{tr}^{-1}(0)$.

Recall that a finite factor M of type I is both normally ternary weakly amenable and satisfies $(M_*)_0 = [M_*, M]$. For a finite factor of type II, the corresponding statements with $[M_*, M]$ and $\mathcal{I}nn_t(M, M_*)$ replaced by their norm closures are also both true. More is true in the type II case.

Proposition

Let M be a finite von Neumann algebra.

(a) If M acts on a separable Hilbert space or is a factor (hence admits a faithful normal finite trace), and if $(M_*)_0 = [M_*, M]$, then M is normally ternary weakly amenable.

(b) If M is a factor and M is normally ternary weakly amenable, then $(M_*)_0 = [M_*, M]$.

Corollary

Let M be a factor of type II_1. Then M is normally ternary weakly amenable if and only if $(M_*)_0 = [M_*, M]$.

Corollary

A factor of type II_1 is never normally ternary weakly amenable.

No infinite factor can be approximately normally ternary weakly amenable by Theorem, much less normally ternary weakly amenable.

As for the case of a factor $M = B(H)$ of type I_∞, we also have $(M_*)_0 \neq [M_*, M]$, due to the work of Gary Weiss (1980, 1986, 2004), and others.

7. Jordan Cohomology

"A veritable army of researchers took the theory of derivations of operator algebras to dizzying heights—producing a theory of cohomology of operator algebras as well as much information about automorphisms of operator algebras."
—Dick Kadison (Which Singer is that? 2000)

It is conjectured that all of the Hochschild cohomology groups $H^n(A, A)$ of a von Neumann algebra A vanish and that this is known to be true for most of them. In addition to associative algebras, cohomology groups are defined for Lie algebras and to some extent, for Jordan algebras. Since the structures of Jordan derivations and Lie derivations on von Neumann algebras are well understood, isn't it time to study the higher dimensional non associative cohomology of a von Neumann algebra? This section will be an introduction to the first and second Jordan cohomology groups of a von Neumann algebra. (Spoiler alert: Very little is known about the second Jordan cohomology group. So this is a good place to stop.)

7A **Brief survey of cohomology theories**

The starting point for the cohomology theory of associative algebras is the paper of Hochschild from 1945. The standard reference of the theory is the book by Cartan and Eilenberg of 1956. Two other useful references are due to Weibel (1994,1995)

- G. Hochschild, *On the cohomology groups of an associative algebra.* Ann. of Math. (2) **46** (1945), 58–67.
- H. Cartan and S. Eilenberg, *Homological algebra.* Princeton University Press, Princeton, N. J., 1956. xv+390 pp.
- C. A. Weibel, *An introduction to homological algebra.* Cambridge Studies in Advanced Mathematics, 38. Cambridge University Press, Cambridge, 1994. xiv+450 pp.
- C. A. Weibel, *History of homological algebra.* History of topology, 797–836, North-Holland, Amsterdam, 1999.

Shortly after the introduction of cohomology for associative algebras, there appeared a corresponding theory for Lie algebras (Chevalley-Eilenberg 1948). A standard reference is the book of Jacobson 1962. Applications can be found in the books by Fuks 1986 and Knapp 1988.

- Chevalley, Claude; Eilenberg, Samuel Cohomology theory of Lie groups and Lie algebras. Trans. Amer. Math. Soc. 63, (1948). 85–124.
- N. Jacobson, *Lie algebras.* Interscience Tracts in Pure and Applied Mathematics, No. 10 Interscience Publishers (a division of John Wiley & Sons), New York-London 1962 ix+331 pp.
- D. B. Fuks, *Cohomology of infinite-dimensional Lie algebras.* Translated from the Russian by A. B. Sosinski. Contemporary Soviet Mathematics. Consultants Bureau, New York, 1986. xii+339 pp.
- A. W. Knapp, *Lie groups, Lie algebras, and cohomology.* Mathematical Notes, 34. Princeton University Press, Princeton, NJ, 1988. xii+510 pp.

The cohomology theory for Jordan algebras is less well developed than for associative and Lie algebras. A starting point would seem to be the papers of Gerstenhaber in 1964 and Glassman in 1970, which concern arbitrary nonassociative algebras. A study focussed primarily on Jordan algebras is another paper by Glassman in 1970.

- M. Gerstenhaber, *A uniform cohomology theory for algebras.* Proc. Nat. Acad. Sci. U.S.A. **51** (1964), 626–629.
- N. D. Glassman, *Cohomology of nonassociative algebras.* Pacific J. Math. **33** (1970), 617–634.
- N. D. Glassman, *Cohomology of Jordan algebras.* J. Algebra **15** (1970), 167–194.

Two fundamental results appeared earlier, namely, the Jordan analogs of the first and second Whitehead lemmas (1947,1951).

Jordan analog of first Whitehead lemma (Jacobson 1951)

Let J be a finite dimensional semisimple Jordan algebra over a field of characteristic 0 and let M be a J-module. Let f be a linear mapping of J into M such that $f(ab) = f(a)b + af(b)$. Then there exist $v_i \in M, b_i \in J$ such that

$$f(a) = \sum_i ((v_i a)b - v_i(ab_i)).$$

- N. Jacobson, *General representation theory of Jordan algebras.* Trans. Amer. Math. Soc. **70** (1951), 509–530.

Jordan analog of second Whitehead lemma (Albert 1947, Penico 1951)

Let J be a finite dimensional separable[a] Jordan algebra and let M be a J-module. Let f be a bilinear mapping of $J \times J$ into M such that $f(a,b) = f(b,a)$ and

$$f(a^2, ab) + f(a,b)a^2 + f(a,a)ab = f(a^2 b, a) + f(a^2, b)a + (f(a,a)b)a$$

Then there exist a linear mapping g from J into M such that

$$f(a,b) = g(ab) - g(b)a - g(a)b$$

[a] Separable, in this context, means that the algebra remains semisimple with respect to all extensions of the ground field. For algebraically closed fields, this is the same as being semisimple

- A. A. Albert, The Wedderburn principal theorem for Jordan algebras, Ann. of Math. vol. 48 (1947) pp. 1–7.
- A. J. Penico, *The Wedderburn principal theorem for Jordan algebras.* Trans. Amer. Math. Soc. **70** (1951). 404–420.

Jordan 2-cocycles

Let M be an associative algebra. A **Hochschild 2-cocycle** is a bilinear map $f : M \times M \to M$ satisfying

$$af(b,c) - f(ab,c) + f(a,bc) - f(a,b)c = 0 \tag{5}$$

EXAMPLE: **Hochschild 2-coboundary**

$$f(a,b) = a\mu(b) - \mu(ab) + \mu(a)b \quad , \qquad \mu : M \to M \text{ linear}$$

A **Jordan 2-cocycle** is a bilinear map $f : M \times M \to M$ satisfying

$$f(a,b) = f(b,a) \text{ (symmetric)}$$

$$\begin{aligned} f(a^2, a\circ b) + f(a,b)\circ a^2 + f(a,a)\circ(a\circ b) \\ -f(a^2\circ b, a) - f(a^2,b)\circ a - (f(a,a)\circ b)\circ a = 0 \end{aligned} \tag{6}$$

EXAMPLE: **Jordan 2-coboundary**

$$f(a,b) = a\circ\mu(b) - \mu(a\circ b) + \mu(a)\circ b, \qquad , \quad \mu : M \to M \text{ linear}$$

$$H^1(M,M)=\frac{\text{1-cocycles}}{\text{1-coboundaries}}=\frac{\text{derivations}}{\text{inner derivations}}\quad (=0 \text{ for } M \text{ von Neumann})$$

$$H^1_J(M,M)=\frac{\text{Jordan 1-cocycles}}{\text{Jordan 1-coboundaries}}=\frac{\text{Jordan derivations}}{\text{inner Jordan derivations}}\quad (=0 \text{ for } M \text{ vN})$$

$$H^2(M,M)=\frac{\text{2-cocycles}}{\text{2-coboundaries}}\quad , \quad H^2_J(M,M)=\frac{\text{Jordan 2-cocycles}}{\text{Jordan 2-coboundaries}}$$

For almost all von Neumann algebras, $H^2(M,M)=0$. How about $H^2_J(M,M)$?

FD char 0: **Albert** 1947, **Penico** 1951; char $\neq 2$: **Taft** 1957

Two approaches: Jordan classification; TKK Lie algebra

- O. Kühn and A. Rosendahl, *Wedderburnzerlegung für Jordan-Paare.* (German. English summary) Manuscripta Math. **24** (1978), no. 4, 403–435.

Examples

Every symmetric Hochschild 2-cocycle is a Jordan 2-cocycle.

Recall that (at least for C*-algebras) every Jordan derivation (Jordan 1-cocycle) is a derivation (Hochschild 1-cocycle)

Let M be an associative algebra. Let $f : M \times M \to M$ be defined by $f(a, b) = a \circ b$. Then f is a Jordan 2-cocycle with values in M, which is not a Hochschild 2-cocycle unless M is commutative.

If M is a von Neumann algebra with faithful normal finite trace tr, then $f : M \times M \to M_*$ defined by $f(a, b)(x) = \mathrm{tr}\,((a \circ b)x)$ is a Jordan 2-cocycle with values in M_* which is not a Hochschild 2-cocycle unless M is commutative.

Conjecture

If M is a von Neumann algebra, then $H^2_J(M, M) = 0$

A study of low dimensional cohomology for quadratic Jordan algebras was initiated by McCrimmon, in 1971.

- K. McCrimmon, *Representations of quadratic Jordan algebras.* Trans. Amer. Math. Soc. **153** (1971), 279–305.

This paper, which is mostly concerned with representation theory, proves, for the only cohomology groups defined, the linearity of the functor H^n:

$$H^n(J, \oplus_i M_i) = \oplus_i H^n(J, M_i), \quad n = 1, 2.$$

Although this paper is about Jordan algebras, the concepts are phrased in terms of the associated triple product $\{abc\} = (ab)c + (cb)a - (ac)b$.

Quadratic Jordan algebras can be considered as a bridge from Jordan algebras to Jordan triple systems

A subsequent paper, by McCrimmon in 1982, which is mostly concerned with compatibility of tripotents in Jordan triple systems, proves versions of the linearity of the functor H^n, $n = 1, 2$, corresponding to the Jordan triple structure.

- K. McCrimmon, *Compatible Peirce decompositions of Jordan triple systems.* Pacific J. Math. **103** (1982), no. 1, 57–102.

However, both papers stop short of defining higher cohomology groups.

7B **Jordan triples and TKK Lie algebras**

By a *Jordan triple*, we mean a real or complex vector space V, equipped with a Jordan triple product $\{\cdot,\cdot,\cdot\} : V^3 \to V$ which is linear and symmetric in the outer variables, conjugate linear in the middle variable, and satisfies the Jordan triple identity

$$\{x,y,\{a,b,c\}\} = \{\{x,y,a\},b,c\} - \{a,\{y,x,b\},c\} + \{a,b,\{x,y,c\}\} \qquad (7)$$

for $a,b,c,x,y \in V$.

Let V be a Jordan triple. A vector space M over the same scalar field is called a *Jordan triple V-module* if it is equipped with three mappings $\{\cdot,\cdot,\cdot\}_1 : M\times V\times V \to M$, $\{\cdot,\cdot,\cdot\}_2 : V\times M\times V \to M$, $\{\cdot,\cdot,\cdot\}_3 : V\times V\times M \to M$ such that (i) $\{a,b,c\}_1 = \{c,b,a\}_3$;

(ii) $\{\cdot,\cdot,\cdot\}_1$ is linear in the first two variables and conjugate linear in the last variable, $\{\cdot,\cdot,\cdot\}_2$ is conjugate linear in all variables;

(iii) denoting by $\{\cdot,\cdot,\cdot\}$ any of the products $\{\cdot,\cdot,\cdot\}_j$ ($j = 1,2,3$), the identity (7) is satisfied whenever one of the above elements is in M and the rest are in V.

Given two elements a, b in a Jordan triple V, we define the *box operator* $a\square b : V \to V$ by $a\square b(\cdot) = \{a, b, \cdot\}$.

If M is a Jordan triple V-module, the box operator $a\square b : V \to V$ can also be considered as a mapping from M to M. Similarly, for $u \in V$ and $m \in M$, the "*box operators*"

$$u\square m,\ m\square u : V \longrightarrow M$$

are defined in a natural way as $v \mapsto \{u, m, v\}$ and $v \mapsto \{m, u, v\}$ respectively.

Given $a, b, u \in V$ and $m \in M$, the identity (7) implies

$$[a\square b, u\square m] = \{a, b, u\}\square m - u\square \{m, a, b\} \tag{8}$$

and

$$[a\square b, m\square u] = \{a, b, m\}\square u - m\square \{u, a, b\}. \tag{9}$$

We also have $[u\square m, a\square b] = \{u, m, a\}\square b - a\square \{b, u, m\}$ and similar identity for $[m\square u, a\square b]$.

A Jordan triple is called *nondegenerate* if for each $a \in V$, the condition $\{a, a, a\} = 0$ implies $a = 0$.

Given that V is nondegenerate facilitates a simple definition of the TKK Lie algebra $\mathfrak{L}(V) = V \oplus V_0 \oplus V$ of V, with an involution θ, where $V_0 = \{\sum_j a_j \square b_j : a_j, b_j \in V\}$, the Lie product is defined by

$$[(x, h, y), (u, k, v)] = (hu - kx,\ [h, k] + x \square v - u \square y,\ k^\natural y - h^\natural v),$$

and for each $h = \sum_i a_i \square b_i$ in the Lie subalgebra V_0 of $\mathfrak{L}(V)$, the map $h^\natural : V \to V$ is well defined by $h^\natural = \sum_i b_i \square a_i$.

The involution $\theta : \mathfrak{L}(V) \to \mathfrak{L}(V)$ is given by

$$\theta(x, h, y) = (y, -h^\natural, x) \qquad ((x, h, y) \in \mathfrak{L}(V)).$$

Let M_0 be the linear span of $\{u \Box m, n \Box v : u, v \in V, m, n \in M\}$ in the vector space $L(V, M)$ of linear maps from V to M. [a]

[a] M_0 is the space of *inner structural transformations* $\operatorname{Instrl}(V, M)$ (see the 1982 paper of McCrimmon quoted above—more about structural transformations later)

Extending the above commutator products (8) and (9) by linearity, we can define an action of V_0 on M_0 by

$$(h, \varphi) \in V_0 \times M_0 \mapsto [h, \varphi] \in M_0,$$

where $h = \sum a_i \Box b_i$ and $\varphi = \sum_j (m_j \Box u_j + v_j \Box n_j)$.

Given a Lie algebra $\mathfrak{L}$ and a module X over $\mathfrak{L}$, we denote the action of $\mathfrak{L}$ on X by

$$(\ell, x) \in \mathfrak{L} \times X \mapsto \ell.x \in X \text{ so that } [\ell, \ell'].x = \ell'.(\ell.x) - \ell.(\ell'.x).$$

M_0 is a V_0-module of the Lie algebra V_0, that is,

$$[[h, k], \varphi] = [h, [k, \varphi]] - [k, [h, \varphi]].$$

Let V be a Jordan triple and $\mathfrak{L}(V)$ its TKK Lie algebra. Given a triple V-module M, let $\mathcal{L}(M) = M \oplus M_0 \oplus M$ and define the action

$$((a,h,b),(m,\varphi,n)) \in \mathfrak{L}(V) \times \mathcal{L}(M) \mapsto (a,h,b).(m,\varphi,n) \in \mathcal{L}(M)$$

by $$(a,h,b).(m,\varphi,n) = (hm - \varphi a,\ [h,\varphi] + a \,\square\, n - m \,\square\, b,\ \varphi^\natural b - h^\natural(n)\,),$$

where, for $h = \sum_i a_i \,\square\, b_i$ and $\varphi = \sum_i u_i \,\square\, m_i + \sum_j n_j \,\square\, v_j$, we have the following natural definitions

$$hm = \sum_i \{a_i, b_i, m\}, \quad \varphi a = \sum_i \{u_i, m_i, a\} + \sum_j \{n_j, v_j, a\},$$

and

$$\varphi^\natural = \sum_i m_i \,\square\, u_i + \sum_j v_j \,\square\, n_j.$$

Theorem

Let V be a Jordan triple and let $\mathfrak{L}(V)$ be its TKK Lie algebra. Let M be a triple V-module. Then $\mathcal{L}(M)$ is a Lie $\mathfrak{L}(V)$-module.

7C **Cohomology of Lie algebras with (or without) involution**

Given an involutive Lie algebra $(\mathfrak{L}, \theta)$, an $(\mathfrak{L}, \theta)$-*module* is a (left) $\mathfrak{L}$-module $\mathfrak{M}$, equipped with an involution $\widetilde{\theta} : \mathfrak{M} \to \mathfrak{M}$ satisfying

$$\widetilde{\theta}(\ell.\mu) = \theta(\ell).\widetilde{\theta}(\mu) \qquad (\ell \in \mathfrak{L}, \mu \in \mathfrak{M}).$$

A k-linear map $\psi : \mathfrak{L}^k \to \mathfrak{M}$ is called θ-*invariant* if

$$\psi(\theta x_1, \cdots, \theta x_k) = \widetilde{\theta}\psi(x_1, \cdots, x_k) \quad \text{for} \quad (x_1, \cdots, x_k) \in \mathfrak{L} \times \cdots \times \mathfrak{L}.$$

Let $(\mathfrak{L},\theta)$ be an involutive Lie algebra and $\mathfrak{M}$ an $(\mathfrak{L},\theta)$-module. We define

$$A^0(\mathfrak{L},\mathfrak{M}) = \mathfrak{M} \text{ and } A^0_\theta(\mathfrak{L},\mathfrak{M}) = \{\mu \in \mathfrak{M} : \widetilde{\theta}\mu = \mu\}.$$

For $k = 1, 2, \ldots$, we let

$$A^k(\mathfrak{L},\mathfrak{M}) = \{\psi : \mathfrak{L}^k \to \mathfrak{M} \mid \psi \text{ is k-linear and } \text{ alternating}\} \quad \text{and}$$

$$A^k_\theta(\mathfrak{L},\mathfrak{M}) = \{\psi \in A^k(\mathfrak{L},\mathfrak{M}) : \mid \psi \text{ is } \theta\text{-invariant}\}.$$

For $k = 0, 1, 2, \ldots$, we define the *coboundary operator* in the usual way: $d_k : A^k(\mathfrak{L},\mathfrak{M}) \to A^{k+1}(\mathfrak{L},\mathfrak{M})$ by $d_0 m(x) = x.m$ and for $k \geq 1$,

$$\begin{aligned}(d_k\psi)(x_1,\ldots,x_{k+1}) &= \sum_{\ell=1}^{k+1}(-1)^{\ell+1}x_\ell.\psi(x_1,\ldots,\widehat{x}_\ell,\ldots,x_{k+1}) \\ &\quad + \sum_{1\leq i<j\leq k+1}(-1)^{i+j}\psi([x_i,x_j],\ldots,\widehat{x}_i,\ldots,\widehat{x}_j,\ldots,x_{k+1})\end{aligned}$$

where the symbol $\widehat{z}$ indicates the omission of z.

The restriction of d_k to the subspace $A^k_\theta(\mathfrak{L},\mathfrak{M})$, which by an abuse of notation, we still denote by d_k, has range in $A^k_\theta(\mathfrak{L},\mathfrak{M})$. Also, as usual, we have $d_k d_{k-1} = 0$ for $k = 1, 2, \ldots$ and two cochain complexes

$$A^0(\mathfrak{L},\mathfrak{M}) \quad \xrightarrow{d_0} \quad A^1(\mathfrak{L},\mathfrak{M}) \quad \xrightarrow{d_1} \quad A^2(\mathfrak{L},\mathfrak{M}) \quad \xrightarrow{d_2} \quad \cdots$$

$$A^0_\theta(\mathfrak{L},\mathfrak{M}) \quad \xrightarrow{d_0} \quad A^1_\theta(\mathfrak{L},\mathfrak{M}) \quad \xrightarrow{d_1} \quad A^2_\theta(\mathfrak{L},\mathfrak{M}) \quad \xrightarrow{d_2} \quad \cdots .$$

As usual, we define the *k-th cohomology group of $\mathfrak{L}$ with coefficients in $\mathfrak{M}$* to be

$$H^k(\mathfrak{L},\mathfrak{M}) = \ker d_k / d_{k-1}(A^{k-1}(\mathfrak{L},\mathfrak{M})) = \ker d_k / \operatorname{im} d_{k-1}$$

for $k = 1, 2, \ldots$ and define $H^0(\mathfrak{L},\mathfrak{M}) = \ker d_0$.

We define the *k-th involutive cohomology group of $(\mathfrak{L},\theta)$ with coefficients in an $(\mathfrak{L},\theta)$-module* $\mathfrak{M}$ to be the quotient

$$H^k_\theta(\mathfrak{L},\mathfrak{M}) = \ker d_k / d_{k-1}(A^{k-1}_\theta(\mathfrak{L},\mathfrak{M})) = \ker d_k / \operatorname{im} d_{k-1}$$

for $k = 1, 2, \ldots$ and define $H^0_\theta(\mathfrak{L},\mathfrak{M}) = \ker d_0 \subset H^0(\mathfrak{L},\mathfrak{M})$.

7D **Cohomology of Jordan triples**

Let V be a Jordan triple and let $\mathfrak{L}(V) = V \oplus V_0 \oplus V$ be its TKK Lie algebra with the involution $\theta(a, h, b) = (b, -h^\natural, a)$. Given a V-module M, we have shown that $\mathfrak{L}(M) = M \oplus M_0 \oplus M$ is an $\mathfrak{L}(V)$-module. We define an induced involution $\widetilde{\theta} : \mathfrak{L}(M) \to \mathfrak{L}(M)$ by

$$\widetilde{\theta}(m, \varphi, n) = (n, -\varphi^\natural, m)$$

for $(m, \varphi, n) \in M \oplus M_0 \oplus M$.

Lemma

$\mathfrak{L}(M)$ is an $(\mathfrak{L}(V), \theta)$-module, that is, we have $\widetilde{\theta}(\ell.\mu) = \theta(\ell).\widetilde{\theta}(\mu)$ for $\ell \in \mathfrak{L}(V)$ and $\mu \in \mathfrak{L}(M)$.

We will construct cohomology groups of a Jordan triple V with coefficients in a V-module M using the cohomology groups of $\mathfrak{L}(V)$ with coefficients $\mathfrak{L}(M)$.

Let V be a Jordan triple. V is identified as the subspace $\{(v, 0, 0) : v \in V\}$ of the TKK Lie algebra $\mathfrak{L}(V)$.

For a triple V-module M, there is a natural embedding of M into $\mathfrak{L}(M) = M \oplus M_0 \oplus M$ given by $\iota : m \in M \mapsto (m, 0, 0) \in \mathfrak{L}(M)$ and we will identify M with $\iota(M)$.

We denote by $\iota_p : \mathfrak{L}(M) \to \iota(M)$ the natural projection $\iota_p(m, \varphi, n) = (m, 0, 0)$.

We define $A^0(V, M) = M$ and for $k = 1, 2, \ldots$, we denote by $A^k(V, M)$ the vector space of all alternating k-linear maps $\omega : V^k = \overbrace{V \times \cdots \times V}^{k-\text{times}} \to M$.

To motivate the definition of an extension $\mathfrak{L}_k(\omega) \in A^k(\mathfrak{L}(V), \mathfrak{L}(M))$ of an element $\omega \in A^k(V, M)$, for $k \geq 1$, we first consider the case $k = 1$ and note that $\omega \in A^1(V, M)$ is a Jordan triple derivation if and only if

$$\omega \circ (a \square b) - (a \square b) \circ \omega = \omega(a) \square b + a \square \omega(b).$$

Let us call a linear transformation $\omega : V \to M$ *extendable* if the following condition holds:

$$\sum_i a_i \square b_i = 0 \Rightarrow \sum_i (\omega(a_i) \square b_i + a_i \square \omega(b_i)) = 0.$$

Thus a Jordan triple derivation is extendable, and if ω is any extendable transformation in $A^1(V, M)$, then the map

$$\mathfrak{L}_1(\omega)(x_1 \oplus a_1 \square b_1 \oplus y_1) := (\,\omega(x_1),\, \omega(a_1) \square b_1 + a_1 \square \omega(b_1),\ \omega(y_1)\,)$$

is well defined and extends linearly to an element $\mathfrak{L}_1(\omega) \in A^1(\mathfrak{L}(V), \mathfrak{L}(M))$, in which case we call $\mathfrak{L}_1(\omega)$ the *Lie extension* of ω on the Lie algebra $\mathfrak{L}(V)$.

For $k > 1$, given a k-linear mapping $\omega : V^k \to M$, we say that ω is *extendable* if it satisfies the following condition under the assumption $\sum_i u_i \Box v_i = 0$:

$$\sum_i (\omega(u_i, a_2, \ldots, a_k) \Box (v_i + b_2 + \cdots + b_k))$$

$$+ \sum_i ((u_i + a_2 + \cdots + a_k) \Box \omega(v_i, b_2, \ldots, b_k)) = 0,$$

for all $a_2, \ldots, a_k, b_2, \ldots, b_k \in V$.

For an extendable ω, we can unambiguously define a k-linear map $\mathfrak{L}_k(\omega) : \mathfrak{L}(V)^k \to \mathfrak{L}(M)$ as the linear extension (in each variable) of

$$\mathfrak{L}_k(\omega)(x_1 \oplus a_1 \Box b_1 \oplus y_1, x_2 \oplus a_2 \Box b_2 \oplus y_2, \cdots, x_k \oplus a_k \Box b_k \oplus y_k) =$$
$$(\omega(x_1, \ldots, x_k), \sum_{j=1}^{k} \omega(a_1, \ldots, a_k) \Box b_j + \sum_{j=1}^{k} a_j \Box \omega(b_1, \ldots, b_k), \omega(y_1, \ldots, y_k)).$$

We call $\mathfrak{L}_k(\omega)$ the *Lie extension* of ω and $\mathfrak{L}_k(\omega) \in A^k(\mathfrak{L}(V), \mathfrak{L}(M))$.

We thus have the following extension map[a] on the subspace $A^k(V,M)'$ of extendable maps in $A^k(V,M)$:

$$\mathfrak{L}_k : \omega \in A^k(V,M)' \mapsto \mathfrak{L}_k(\omega) \in A^k(\mathfrak{L}(V), \mathfrak{L}(M)).$$

[a] Note that $\mathfrak{L}_k(\omega) \in A^k_\theta(\mathfrak{L}(V), \mathfrak{L}(M))$ if and only if k is odd.

Conversely, given $\psi \in A^k(\mathfrak{L}(V), \mathfrak{L}(M))$ for $k = 1, 2, \ldots$, one can define an alternating map

$$J^k(\psi) : V^k \to M$$

by

$$J^k(\psi)(x_1, \ldots, x_k) = \iota_p \psi(\,(x_1, 0, 0),\, \ldots\,, (x_k, 0, 0)\,)$$

for $(x_1, \ldots, x_k) \in V^k$.

We define $J^0 : \mathfrak{L}(M) \to \iota(M) \approx M = A^0(V,M)$ by

$$J^0(m, \varphi, n) = (m, 0, 0) \qquad ((m, \varphi, n) \in \mathfrak{L}(M)).$$

We call $J^k(\psi)$ the *Jordan restriction* of ψ in $A^k(V,M)$.

We now define the two classes of cohomology groups for a Jordan triple V with coefficients M. First, $H^k(V,M)$, and then $H^k_\theta(V,M)$:

$$\begin{array}{ccccccc}
A^0(\mathfrak{L}(V),\mathfrak{L}(M)) & \xrightarrow{d_0} & A^1(\mathfrak{L}(V),\mathfrak{L}(M)) & \xrightarrow{d_1} & A^2(\mathfrak{L}(V),\mathfrak{L}(M)) & \xrightarrow{d_2} & \cdots \\
\downarrow J^0 & & \downarrow J^1 & & \downarrow J^2 & & \cdots \\
M = A^0(V,M) & & A^1(V,M) & & A^2(V,M) & & \cdots
\end{array}$$

For $k=0,1,2,\ldots$, the *k-th cohomology groups* $H^k(V,M)$ are defined by

$$\begin{aligned}
H^0(V,M) &= J^0(\ker d_0) = J^0\{(m,\varphi,n) : (u,h,v).(m,\varphi,n) = 0,\ \forall (u,h,v)\in\mathfrak{L}(V) \\
&= \{m \in M : m\,\square\, v = 0,\ \forall v\in V\} = \{0\}
\end{aligned}$$

$$H^k(V,M) = Z^k(V,M)/B^k(V,M) \qquad (k=1,2,\ldots)$$

$$Z^k(V,M) = J^k(Z^k(\mathfrak{L}(V),\mathfrak{L}(M))), \quad Z^k(\mathfrak{L}(V),\mathfrak{L}(M)) = \ker d_k$$

$$B^k(V,M) = J^k(B^k(\mathfrak{L}(V),\mathfrak{L}(M))), \quad B^k(\mathfrak{L}(V),\mathfrak{L}(M)) = \mathrm{im}\, d_{k-1}.$$

$$\begin{array}{ccccccc}
A^0_\theta(\mathfrak{L}(V),\mathfrak{L}(M)) & \xrightarrow{d_0} & A^1_\theta(\mathfrak{L}(V),\mathfrak{L}(M)) & \xrightarrow{d_1} & A^2_\theta(\mathfrak{L}(V),\mathfrak{L}(M)) & \xrightarrow{d_2} & \cdots \\
\downarrow J^0 & & \downarrow J^1 & & \downarrow J^2 & & \cdots \\
M = A^0(V,M) & & A^1(V,M) & & A^2(V,M) & & \cdots
\end{array}$$

For $k = 0, 1, 2 \ldots$, the *k-th involutive cohomology groups* $H^k_\theta(V, M)$ are

$$H^0(V,M) = J^0(\ker d_0) = \{0\}$$

$$H^k_\theta(V,M) = Z^k_\theta(V,M)/B^k_\theta(V,M) \qquad (k = 1, 2, \ldots)$$

$$Z^k_\theta(V,M) = J^k(Z^k_\theta(\mathfrak{L}(V),\mathfrak{L}(M))), \quad Z^k_\theta(\mathfrak{L}(V),\mathfrak{L}(M))) = \ker d_k|_{A^k_\theta(\mathfrak{L}(V),\mathfrak{L}(M))}$$

$$B^k_\theta(V,M) = J^k(B^k_\theta(\mathfrak{L}(V),\mathfrak{L}(M))), B^k_\theta(\mathfrak{L}(V),\mathfrak{L}(M)) = d_{k-1}(A^{k-1}_\theta(\mathfrak{L}(V),\mathfrak{L}(M))).$$

We call elements in $H^k(V, M)$ the *Jordan triple k-cocycles*, and the ones in $H^k_\theta(V, M)$ the *involutive Jordan triple k-cocycles*. Customarily, elements in $B^k(V, M)$ and $B^k_\theta(V, M)$ are called the *coboundaries*.

Some immediate consequences of the construction

Let V be a Jordan triple with TKK Lie algebra $(\mathfrak{L}(V), \theta)$. If the k-th Lie cohomology group $H^k(\mathfrak{L}(V), \mathfrak{L}(M))$ vanishes, then $H^k(V, M) = \{0\}$ and $H^k_\theta(V, M) = \{0\}$.

From Whitehead's lemmas for semisimple Lie algebras, we have

Let V be a finite dimensional Jordan triple with semisimple TKK Lie algebra $\mathfrak{L}(V)$. Then for any finite dimensional V-module M, we have $H^1(V, M) = H^2(V, M) = \{0\}$.

In particular, if V is a finite dimensional semisimple Jordan triple system, then every derivation of V into M is inner. (Meyberg notes)

In fact, we have $H^k(\mathfrak{L}(V), \mathfrak{L}(M)) = \{0\}$ for all $k \geq 3$ if $\mathfrak{L}(M)$ is a nontrivial irreducible module over $\mathfrak{L}(V)$.

7E **Examples of Jordan triple cocycles**

If $\omega \in A^2(V, M)$ is extendable with $\mathfrak{L}_2(\omega) \in Z^2(\mathfrak{L}(V), \mathfrak{L}(M))$, then $\omega = 0$.

Proof

For $x, y, z \in V$,

$$\begin{aligned} 0 &= d_2\mathfrak{L}_2(\omega)((x,0,0),(y,0,0),(0,0,z)) \\ &= (x,0,0)\cdot(\mathfrak{L}(\omega)((y,0,0),(0,0,z) - (y,0,0)\cdot(\mathfrak{L}(\omega)((x,0,0),(0,0,z) \\ &\quad +(0,0,z)\cdot(\mathfrak{L}(\omega)((x,0,0),(y,0,0) - \mathfrak{L}(\omega)([(x,0,0),(y,0,0],(0,0,z)) \\ &\quad +\mathfrak{L}(\omega)([(x,0,0),(0,0,z],(y,0,0)) - \mathfrak{L}(\omega)([(y,0,0),(0,0,z],(x,0,0)) \\ &= -(0,\omega(x,y)\,\Box\, z,0), \end{aligned}$$

hence $\omega(x,y)\,\Box\, z = 0$ for all x, y, z and $\omega = 0$.

We have seen in the first example that there are no non-zero extendable elements $\omega \in Z^2(V, M)$ with $\mathfrak{L}_2(\omega) \in Z^2(\mathfrak{L}(V), \mathfrak{L}(M))$. The next example examines this phenomenon for extendable $\omega \in A^3(V, M)$ with $\mathfrak{L}_3(\omega) \in Z^3_\theta(\mathfrak{L}(V), \mathfrak{L}(M))$.

For $a, b \in V$ and $m \in M$, $[a, b] := a\square\, b - b\square\, a$ and $[m, a] := m\square\, a - a\square\, m$.

Let ω be an extendable element of $A^3(V, M)$. Then its Lie extension $\mathfrak{L}_3(\omega)$ is a Lie 3-cocycle in $A^3_\theta(\mathfrak{k}(V), \mathfrak{k}(M))$ if and only if ω satisfies the following three conditions for all $a, b, c, d, x, y, z \in V$

$$[a, b]\omega(x, y, z) = \omega([a, b]x, y, z) + \omega(x, [a, b]y, z) + \omega(x, y, [a, b]z) \tag{10}$$

$$[\omega(a, b, c), d] = [\omega(d, b, c), a] = [\omega(a, b, d), c] = [\omega(a, d, c), b] \tag{11}$$

$$[\omega(x, y, [a, b]z), c] = 0. \tag{12}$$

Note that (10)-(12) involve 5,4 and 6 variables respectively.

Let $\omega \in A^3(V,M)$ be extendable and let $\psi = d_3\mathfrak{L}_3(\omega)$ (ψ is θ-invariant since 3 is odd). Write $X_j = (x_j, a_j \square b_j - b_j \square a_j, x_j) \in \mathfrak{k}(V)$ as $X_j = (x_j, 0, x_j) + (0, [a_j, b_j], 0)$. By the alternating character of ψ, it is a Lie 3-cocycle, that is, $\psi(X_1, X_2, X_3, X_4) = 0$ for $X_j \in \mathfrak{k}(V)$, if and only if the following five equations hold for $a_i, b_i, x_i \in V$.

$$\psi((x_1,0,x_1),(x_2,0,x_2),(x_3,0,x_3),(x_4,0,x_4)) = 0, \tag{13}$$

$$\psi((x_1,0,x_1),(x_2,0,x_2),(x_3,0,x_3),(0,[a_4,b_4],0)) = 0, \tag{14}$$

$$\psi((x_1,0,x_1),(x_2,0,x_2),(0,[a_3,b_3],0),(0,[a_4,b_4],0)) = 0, \tag{15}$$

$$\psi((x_1,0,x_1),(0,[a_2,b_2],0),(0,[a_3,b_3],0),(0,[a_4,b_4],0)) = 0, \tag{16}$$

$$\psi((0,[a_1,b_1],0),(0,[a_2,b_2],0),(0,[a_3,b_3],0),(0,[a_4,b_4],0)) = 0. \tag{17}$$

Since (10)-(12) involve 5,4 and 6 variables respectively, and (13)-(17) involve 4,5,6,7,8 variables respectively, there is an additional amount of redundancy in (13)-(17).

7F **Examples of TKK Lie algebras**

Let A be a unital associative algebra with Lie product the commutator $[x, y] = xy - yx$, Jordan product the anti-commutator $x \circ y = (xy + yx)/2$ and Jordan triple product $\{xyz\} = (xy^*z + zy^*x)/2$ if A has an involution. Denote by $Z(A)$ the center of A and by $[A, A]$ the set of finite sums of commutators.

Suppose $Z(A) \cap [A, A] = \{0\}$. Then the mapping $(x, a\Box b, y) \mapsto \begin{bmatrix} ab & x \\ y & -ba \end{bmatrix}$ is an isomorphism of the TKK Lie algebra $\mathfrak{L}(A)$ onto the Lie subalgebra

$$\left\{ \begin{bmatrix} u + \sum [v_i, w_i] & x \\ y & -u + \sum [v_i, w_i] \end{bmatrix} : u, x, y, v_i, w_i \in A \right\} \tag{18}$$

of the Lie algebra $M_2(A)$ with the commutator product.

- K. Meyberg, *Lectures on algebras and triple systems.* The University of Virginia, Charlottesville, Va., 1972. v+226 pp.
- M. Koecher, *Imbedding of Jordan algebras in Lie algebras.* I, Amer. J. Math **8**9, no. 3 (1967), 787–816.

Proposition

Let V be a finite von Neumann algebra. Then $\mathfrak{L}(V)$ is isomorphic to the Lie algebra $[M_2(V), M_2(V)]$.

In a properly infinite von Neumann algebra, the assumption $Z(A) \cap [A, A] = \{0\}$ fails since $A = [A, A]$.

This assumption also fails in the Murray-von Neumann algebra of measurable operators affiliated with a factor of type II_1 (according to a preprint of Thom in 2013).

For a finite factor of type I_n, the proposition states that the classical Lie algebras $sl(2n, \mathbb{C})$ of type A are TKK Lie algebras.

Similarly, the TKK Lie algebra of a Cartan factor of type 3 on an n-dimensional Hilbert space is the classical Lie algebra $sp(2n, \mathbb{C})$ of type C (Meyberg notes p.131)

More examples of TKK Lie algebras can be found in the book of Chu 2012 (section 1.4) and the lecture notes of Koecher 1969 (chapter III).

- C-H. Chu, *Jordan structures in geometry and analysis*. Cambridge Tracts in Math. **190**, Cambridge Univ. Press, Cambridge, 2012.
- M. Koecher, *An elementary approach to bounded symmetric domains*, Lecture Notes, Rice University, 1969.

7G **Structural transformations**

Let V be a Jordan triple and M a triple V-module. A mapping $\omega : V \to M$ is called an *inner triple derivation* if it is of the form

$$\omega = \sum_{i=1}^{k} (m_i \,\square\, v_i - v_i \,\square\, m_i) \in M_0$$

for some $m_1, \ldots, m_k \in M$ and $v_1, \ldots, v_k \in V$. Note that $\omega^\natural = -\omega$ and $(0, \omega, 0) \in \mathfrak{k}(M)$.

$B^1_\theta(V, M)$ coincides with the space of inner triple derivations from V to M.

$Z^1_\theta(V, M)$ coincides with the set of triple derivations of V

The first involutive cohomology group $H^1_\theta(V, M) = Z^1_\theta(V, M)/B^1_\theta(V, M)$ is the space of triple derivations modulo the inner triple derivations of V into M. This will be generalized shortly.

A (conjugate-) linear transformation $S : V \to M$ is said to be a *structural transformation* if there exists a (conjugate-) linear transformation $S^* : V \to M$ such that

$$S\{xyx\} + \{x(S^*y)x\} = \{xySx\}$$

$$S^*\{xyx\} + \{x(Sy)x\} = \{xyS^*x\}.$$

By polarization, this property is equivalent to

$$S\{xyz\} + \{x(S^*y)z\} = \{zySx\} + \{xySz\}$$

$$S^*\{xyz\} + \{x(Sy)z\} = \{zyS^*x\} + \{xyS^*z\}.$$

- A triple derivation D is a a structural transformation S with $S^* = -S$.
- The space of *inner structural transformations* coincides with the space M_0.
- Triple derivations which are inner structural transformations are inner
- If ω is a structural transformation, then $\omega - \omega^*$ is a triple derivation
- If ω is a triple derivation, $i\omega$ is a structural transformation, inner if ω is inner.

Proposition

Let ψ be a Lie derivation of $\mathfrak{L}(V)$ into $\mathfrak{L}(M)$. Then

(i) $J(\psi) : V \to M$ is a structural transformation with $(J\psi)^* = -J\psi'$ where $\psi' = \widetilde{\theta}\psi\theta$.

(ii) If ψ is θ-invariant, then $\psi' = \psi$ and $J\psi$ is a triple derivation.

(iii) If ψ is an inner derivation then $J\psi$ is an inner structural transformation. In particular, if ψ is a θ-invariant inner derivation then $J\psi$ is an inner triple derivation.

Conversely, let ω be a structural transformation.

(iv) The mapping $D = \frac{1}{2}\mathfrak{L}_1(\omega - \omega^*) : \mathfrak{L}(V) \to \mathfrak{L}(M)$ defined by

$$D(x, a \square b, y) = \frac{1}{2}(\omega(x) - \omega^*(x), \omega(a) \square b - a \square \omega^*(b) - \omega^*(a) \square b + a \square \omega(b), \omega(y)$$

is a derivation of the Lie algebra $\mathfrak{L}(V)$ into $\mathfrak{L}(M)$.

(v) D is θ-invariant if and only if ω is a triple derivation, that is, $\omega^* = -\omega$.

(vi) If ω is an inner structural transformation then D is an inner derivation. In particular, if ω is an inner triple derivation then D is a θ-invariant inner derivation.

The following theorem provides some significant infinite dimensional examples of Lie algebras in which every derivation is inner.

Theorem

Let V be a von Neumann algebra considered as a Jordan triple system with the triple product $\{xyz\} = (xy^*z + zy^*x)/2$. Then every structural transformation on V is an inner structural transformation. Hence, every derivation of the TKK Lie algebra $\mathfrak{L}(V)$ is inner.

- Pluta, Robert; Russo, Bernard Triple derivations on von Neumann algebras. Studia Math. 226 (2015), no. 1, 57–73.

Proof:

See the next page

Let S be a structural transformation on the von Neumann algebra V and to avoid cumbersome notation, denote S^* by $\overline{S}$. From the defining equations, $\overline{S}(1) = S(1)^*$, and if $S(1) = 0$, then S is a Jordan derivation.

For an arbitrary structural transformation S, write $S = S_0 + S_1$ where $S_0 = S - 1 \square \overline{S}(1)$ is therefore a Jordan derivation and $S_1 = 1 \square \overline{S}(1)$ is an inner structural transformation.

By the theorem of Sinclair 1970, S_0 is a derivation and by the theorems of Kadison and Sakai 1966, S_0 is an inner derivation, say $S_0(x) = ax - xa$ for some $a \in V$.

By well known structure of the span of commutators in von Neumann algebras due to Pearcy-Topping, Halmos, Halpern, Fack-de la Harpe, and others $a = z + \sum [c_i, d_i]$, where $c_i, d_i \in V$ and z belongs to the center of V. It follows that

$$S_0 = 2 \sum_i c_i \square d_i^* - 2 \sum_i d_i \square c_i^*$$

and is therefore also an inner structural transformation. The second statement follows from the Proposition.