Introduction to C^*-Algebras

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Model Theory of Operator Algebras
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Hilbert Spaces

- H complex Hilbert space with $(\cdot,\cdot)$ inner product
- $\|\xi\| := (\xi,\xi)^{1/2}$ (complete) norm
- H^* continuous linear functionals $\phi : H \to \mathbb{C}$
- (F. Reisz) $\phi \in H^*$ then $\exists \eta \in H \forall \xi \in H(\phi(\xi) = (\xi,\eta))$

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Bounded Operators

- $\mathcal{B}(H, K) = \{T : H \to K : \text{T linear, continuous}\}$
- operator norm

$$\|T\| := \sup_{\xi \neq 0} \frac{\|T\xi\|}{\|\xi\|} < \infty$$

- $\mathcal{B}(H, K)$ complete
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$$\|T\| = \sup_{\|\xi\|=\|\eta\|=1} |(T\xi, \eta)|$$

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- $\mathcal{B}(H) := \mathcal{B}(H, H)$
- $\|ST\| \leq \|S\| \cdot \|T\|$
- $\mathcal{B}(H) \leftrightarrow \{A(\xi, \eta) \to \mathbb{C} : |A(\xi, \eta)| \leq C\|\xi\| \cdot \|\eta\|, \text{ bilinear}\}$
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Adjoints

- $(T^*)^* = T$
- $(\lambda T)^* = \bar{\lambda} T^*$, $(ST)^* = T^* S^*$
- $\|T^*\| = \|T\|$
- $\|T^* T\| = \|T\|^2$

Proof.

$$\sup_{\|\xi\|=\|\eta\|=1} |(T^* T\xi, \eta)| = \sup_{\|\xi\|=\|\eta\|=1} |(T\xi, T\eta)| = \sup_{\|\xi\|=1} \|T\xi\|^2$$

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Definition

A concrete C^*-algebra is a norm closed, adjoint closed, subalgebra of $\mathcal{B}(H)$ for some Hilbert space H.

Definition

A C^*-algebra is a:

- complex Banach algebra - $\|ab\| \leq \|a\| \cdot \|b\|$
- with involution - $(\lambda a)^* = \bar{\lambda} a^*$, $(a^*)^* = a$
- which satisfies the C^*-identity

$$\|a^* a\| = \|a\|^2.$$

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Theorem (Big Theorem)

Every C^*-algebra is isometrically $*$-isomorphic to a concrete C^*-algebra.

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Examples - Finite Dimension

Here are some examples to finite-dimensional C*-algebras:

- $\mathbb{C}$
- $M_n(\mathbb{C}) = \mathcal{B}(\ell^2_n)$
- $M_{n_1}(\mathbb{C}) \oplus \cdots \oplus M_{n_k}(\mathbb{C})$

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Examples - Abelian

X is a set. For $f \in \ell^\infty(X)$, $M_f \in \mathcal{B}(\ell^2(X))$ by

$$M_f \xi(x) := f(x)\xi(x), \ \forall x \in X$$

Theorem

$f \mapsto M_f$ gives an isometric $*$-embedding $\ell^\infty(X) \hookrightarrow \mathcal{B}(\ell^2(X))$

Proof.

- $M_{\bar{f}} = M_f^*$
- $M_{fg} = M_f M_g = M_g M_f$
- $$\|M_f\| = \sup_{\|\xi\|_2 = \|\eta\|_2 = 1} |(M_f \xi, \eta)| = \sup_{\|\phi\|_1 = 1} |(\phi, f)| = \sup_x |f(x)|$$

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X compact, Hausdorff.

$$C(X) := \{f : X \to \mathbb{C} : f \text{ is continuous}\}$$

Fact

$C(X)$ is an (abstract) C^*-algebra under pointwise multiplication, pointwise conjugation, and "sup-norm".

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- "$\check{X}$" is X considered as a (discrete) set.

Examples - Abelian

X locally compact, Hausdorff. $C_0(X)$ is all $f : X \to \mathbb{C}$ continuous s.t.

$$\forall \epsilon > 0 \exists K \subset X \text{ compact}(|f(x)| \geq \epsilon \Rightarrow x \in K)$$

- $C_0(X)$ is an (abstract) C*-algebra under pointwise multiplication, pointwise conjugation, and "sup-norm".
- $C_0(X) \hookrightarrow C(\beta X)$, βX = Stone–Čech compactification.
- $C_0(X)$ is unital iff X is compact.

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Abelian C*-Algebras

Theorem (Gelfand–Naimark, 1943)

An abelian C^*-algebra is isometrically $*$-isomorphic to $C_0(X)$ for some X locally compact, Hausdorff.

More Examples

Definition

An operator $T \in \mathcal{B}(H)$ is finite rank if

$$T = \sum_{i=1}^{n} (\cdot, \eta_i)\xi_i$$

The compact operators $\mathcal{K}(H) \subset \mathcal{B}(H)$ are the closure of the finite rank operators.

Fact

$\mathcal{K}(H) = \mathcal{B}(H)$ if $\dim H < \infty$. Otherwise $\mathcal{K}(H)$ is a maximal proper closed ideal in $\mathcal{B}(H)$.

More Examples

Fact

$\mathcal{K}(\ell^2) = \overline{\bigcup_{i=1}^{\infty} M_n(\mathbb{C})}$ *where*

$$M_n(\mathbb{C}) \ni A \mapsto \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} \in M_{n+1}(\mathbb{C}).$$

Example

The CAR algebra is $\overline{\bigcup_{i=1}^{\infty} M_{2^n}(\mathbb{C})}$ where

$$M_{2^n}(\mathbb{C}) \ni A \mapsto \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix} \in M_{2^{n+1}}(\mathbb{C}).$$

Some Operator Theory

Definition

An operator $T \in \mathcal{B}(H)$ is said to be:

- self adjoint if $T^* = T$. $\Rightarrow \forall \xi ((T\xi, \xi) \in \mathbb{R}))$
- positive if self adjoint and $\forall \xi ((T\xi, \xi) \geq 0)$.
- normal if $T^*T = TT^*$.

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The Spectrum

Definition

For $T \in \mathcal{B}(H)$, the spectrum

$$\sigma(T) := \{\lambda \in \mathbb{C} : T - \lambda I \notin \mathrm{GL}(H)\}.$$

Fact

$\sigma(T)$ is nonempty compact.

Fact

If $\lambda \in \sigma(T)$ then either:

- $\exists \xi \neq 0 (T\xi = \lambda\xi)$
- $\forall \epsilon > 0 \exists \xi (\|\xi\| = 1 \wedge \|T\xi - \lambda\xi\| < \epsilon)$
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The Spectrum

Fact

If $T \in \mathcal{B}(H)$ is normal and $\lambda \in \sigma(T)$, then there exists (ξ_n), $\|\xi_n\| = 1$ s.t.

$$\|T\xi_n - \xi_n\| \to 0.$$

Proof.

T normal $\Rightarrow \|T\xi - \lambda\xi\| = \|T^*\xi - \bar{\lambda}\xi\|$. □

The Spectrum

Definition

$T \in \mathcal{B}(H)$ the spectral radius $|\sigma|(T)$ is $\max\{|\lambda| : \lambda \in \sigma(T)\}$.

Fact

$|\lambda| > \|T\|$ *then* $T - \lambda I \in \mathrm{GL}(H)$.

Proof.

Use power series to invert $T - \lambda I$.

□

Corollary

$|\sigma|(T) \leq \|T\|$

Spectral Radius

Theorem

If $T \in \mathcal{B}(H)$, $T \geq 0$, *then* $\|T\| = |\sigma|(T)$

Proof.

Wlog $\|T\| = 1$. Choose (ξ_n), $\|\xi_n\| = 1$, $\|T\xi_n\| \to 1$. Define $(\xi, \eta)_T := (T\xi, \eta)$ positive semidefinite. We have $1 = \lim_n(\xi_n, T\xi_n)_T$ and $\lim_n(\xi_n, \xi_n)_T, \lim_n(T\xi_n, T\xi_n)_T \leq 1$ whence $\xi_n = T\xi_n + \eta_n$ where $\|T\eta_n\| \to 0$ (Cauchy-Schwarz). But then $\|\eta_n\| \to 0$ by optimality of norm estimate. Thus $\|T\xi_n - \xi_n\| \to 0$ and $1 \in \sigma(T)$. □

Spectral Mapping

$p(z)$ complex $*$-polynomial,

$$p(z) = (z - a_1) \cdots (z - a_m)(\bar{z} - b_1) \cdots (\bar{z} - b_n).$$

$T \in \mathcal{B}(H)$, normal,

$$p(T) := (T - a_1 I) \cdots (T - a_j I)(T^* - b_1 I) \cdots (T^* - b_k I)$$

Theorem (Spectral Mapping I)

If $T \in \mathcal{B}(H)$, T normal, then $p(\sigma(T)) = \sigma(p(T))$.

Proof.

$\|p(T)\xi_n\| \to 0 \Leftrightarrow$ either $\liminf \|T\xi_n - a_i\xi_n\| = 0$ for some i or $\liminf \|T^*\xi_n - b_j\xi_n\| = 0$ for some j. □

Spectral Mapping

$\{p_n(x)\}$ uniformly convergent on $\sigma(T)$ to f, $T = T^*$.

Theorem (Spectral Mapping II)

$(f(T)\xi, \eta) := \lim_n (p_n(T)\xi, \eta)$ *exists and* $f(\sigma(T)) = \sigma(f(T))$.

Proof.

Uniformly over pairs ξ, η in unit ball $\{(p_n(T)\xi, \eta)\}$ is Cauchy by spectral mapping, so defines a (bounded) bilinear form, thus an operator $f(T)$. Then $\|p_n(T) - f(T)\| \to 0$. □

Spectral Theorem

$T \in \mathcal{B}(H)$, $T = T^*$, $C^*_{\mathbb{R}}(T)$ smallest norm closed, real algebra containing T and I.

Theorem (Spectral Theorem I)

There is an isometric isomorphism

$$C_{\mathbb{R}}(\sigma(T)) \leftrightarrow C^*_{\mathbb{R}}(T)$$

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Spectral Theorem

Corollary

Every element of a C^*-algebra is a linear combination of four positive elements.

Corollary

In a unital C^*-algebra the set of positive elements A_+ is a complete cone

Definition

$x, y \in A_+$, $x \leq y$ if $y - x \geq 0$.

Fact

If $x \in A_+$, then $x \leq \|x\| \cdot 1$.

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Theorem (Spectral Theorem II)

There is an isometric isomorphism

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- How to bootstrap to abelian C*-algebras?
- Need a complete set of "eigenvectors".

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States

$A \subset \mathcal{B}(H)$ unital C*-subalgebra

Definition

$\phi \in A^*$ is positive if $a \geq 0 \Rightarrow \phi(a) \geq 0$ and a state if $\phi \geq 0$ and $\phi(1) = 1$.

$\phi(x) := (x\xi, \xi)$, $\|\xi\| = 1$ is a state on $\mathcal{B}(H)$. Also $\lim_\omega (x\xi_n, \xi_n)$, $\|\xi_n\| = 1$.

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Fact

$\phi \in A^*$ is positive iff $\phi(1) = \|\phi\|$.

States

Fact (Cauchy–Schwarz)

$\phi \in A^*$, $\phi \geq 0$, *then*

$$|\phi(y^*x)| \leq \phi(x^*x)^{1/2}\phi(y^*y)^{1/2}.$$

Corollary

$\phi \geq 0$, $\phi(1) = 0$, *then* $\phi \equiv 0$.

Fact (State extension)

Any state $\phi \in \mathcal{S}(A)$ *extends to a state* $\phi' \in \mathcal{S}(\mathcal{B}(H))$.

Proof.

Hahn–Banach □

Pure States

Definition

A state $\phi \in \mathcal{S}(A)$ is pure if ϕ is an extreme in $\mathcal{S}(A)$. Let $\mathcal{P}(A)$ denote the pure states.

Fact (Krein–Milman)

The convex hull of $\mathcal{P}(A)$ is $\mathcal{S}(A)$.

Fact

$a \in A$, $a \neq 0$, *there is a pure state such that* $\phi(a) \neq 0$

Proof.

If H is a complex Hilbert space, then $(T\xi, \xi) = 0$, $\forall \xi \in H \Rightarrow T \equiv 0$. □

States

Let $A = \ell^\infty(\mathbb{N})$. What is $\mathcal{P}(A)$?

- $\delta_n(f) := f(n)$.

Fact

In general, $\phi \in \mathcal{S}$, $X \subset \mathbb{N}$, $\mu(X) := \phi(\chi_X)$ *defines a finitely additive probability measure. Conversely for any such measure* μ, $f \mapsto \int f d\mu$ *is a state.*

- $\phi \in \mathcal{P}(\ell^\infty(\mathbb{N})) \leftrightarrow \mu(X) \in \{0, 1\}, \ \forall X \subset \mathbb{N}$
- $\mathcal{P}(\ell^\infty(\mathbb{N})) \leftrightarrow \{\text{ultrafilters on } \mathbb{N}\}$!!

Let $A = \ell^\infty(\mathbb{N})$. What is $\mathcal{P}(A)$?

- $\delta_n(f) := f(n)$.

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In general, $\phi \in \mathcal{S}$, $X \subset \mathbb{N}$, $\mu(X) := \phi(\chi_X)$ defines a finitely additive probability measure. Conversely for any such measure μ, $f \mapsto \int f d\mu$ is a state.

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States

Definition

$\phi, \psi \in A^*$, $\psi \leq \phi$ is $\psi(a) \leq \phi(a)$ for all $a \geq 0$.

Fact

$0 \leq \psi \leq \phi$, $\phi \in \mathcal{P}(A)$, *then* $\psi = c \cdot \phi$ *for some* $c \in [0, 1]$.

Proof.

Wlog $\kappa := \psi(1) \in (0, 1)$.

$$\phi = \kappa \cdot [\kappa^{-1}\psi] + (1 - \kappa) \cdot [(1 - \kappa)^{-1}(\phi - \psi)].$$

□

Characters

Definition

A character is a unital $*$-homomorphism $\phi : A \to \mathbb{C}$

- $a \geq 0$, $\phi(a) = \phi(x^*x) = \overline{\phi(x)}\phi(x) \geq 0$, so $\phi \in \mathcal{S}(A)$.
- $\phi \in \mathcal{P}(A)$.
- $A = C(X)$, X compact T_2, $\forall x \in X$, $\delta_x(f) := f(x)$ character.
- Any character on $C(X)$ is of the form δ_x. (Reisz representation theorem)
- May be no characters, e.g., $M_n(\mathbb{C})$, $n \geq 2$.

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Abelian C*-Algebras

Theorem

If A is a unital, abelian C-algebra, then any pure state is a character.*

Proof.

$\phi \in \mathcal{P}(A)$, $a, b \geq 0$, $\|a\|, \|b\| \leq 1$.

- $x \geq 0 \Rightarrow \phi_b(x) := \phi(xb) = \phi(b^{1/2} x b^{1/2}) \geq 0$
- $\phi_b(x) = \phi(xb) = \phi(x^{1/2} b x^{1/2}) \leq \|b\| \phi(x^{1/2} 1 x^{1/2}) \leq \phi(x) \Rightarrow 0 \leq \phi_b \leq \phi$
- $\phi \in \mathcal{P}(A) \Rightarrow \phi_b = \kappa \cdot \phi(x)$
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□

Gel'fand Spectrum

Definition

Let A be a unital, abelian C*-algebra. The Gel'fand spectrum is

$$\Sigma_A := \mathcal{P}(A) = \{\text{all characters of } A\}$$

Σ_A is a weak* closed subset of $\mathcal{S}(A)$, whence is compact, Hausdorff itself.

Fact (Gel'fand–Mazur)

$\Sigma_A \leftrightarrow \{\text{maximal proper closed ideals of } A\}$

Gel'fand Transform

Definition

$\Gamma : A \to C(\Sigma_A)$, $\Gamma(a)(\phi) := \phi(a)$, Gel'fand transform.

Fact

Easy to check that Γ is a unital contractive $*$-homomorphism.

Fact

Γ is an isometry.

Proof.

A, abelian $*$-algebra $\Rightarrow$ every $a \in A$ is normal. Hence there is a state ϕ s.t. $\phi(a) = \|a\|$. □

Gel'fand Transform

Theorem

Γ *is surjective.*

Proof.

Image of Γ is a∗-subalgebra that separates points. The image is closed since $\Gamma(a)$ determines the values of $(a\xi, \xi)$, $\forall \xi \in H$, whence the bilinear form $(a\xi, \eta)$. Γ an isometry, thus $\Gamma(a_n)$ converges uniformly implies a_n converges uniformly and $\lim_n \Gamma(a_n) = \Gamma(\lim_n a_n)$. Stone–Weierstrass finishes the job. □

Gel'fand transform

Fact

$\exists\phi(\Gamma(a)(\phi) = \lambda) \Leftrightarrow \lambda \in \sigma(a)$

Proof.

$\Gamma(\mathrm{GL}(A)) = \mathrm{GL}(C(\Sigma_A))$

□

Corollary

T normal invertible in $\mathcal{B}(H)$ iff invertible in $C^*(T)$.

Corollary (Gel'fand–Naimark)

A an abelian C^*-algebra, then A is $*$-isomorphic to $C(\Sigma_A)$.

Useful Facts

Fact

A unital, then every self-adjoint $a \in A$ is an average of two unitaries $(u^*u = 1 = uu^*)$. A is the span of $\mathcal{U}(A)$, unitary group of A.

Fact

A $*$-homomorphism of C^*-algebras in contractive.

Proof.

Gel'fand–Naimark □

Fact

The image of a C^*-algebra under a $*$-homomorphism is a C^*-algebra, i.e., images are closed.

Definition

A representation is a $*$-homomorphism $\pi : A \to \mathcal{B}(K)$.

Definition

- A rep'n π is faithful if injective (=isometric).
- A rep'n π is cyclic if $\pi(A)\xi$ dense in K for some $\xi \in K$.

- $\pi_1 \oplus \pi_2 : A \to \mathcal{B}(K_1 \oplus K_2)$
- $\pi_1 \otimes \pi_2 : A \to \mathcal{B}(K_1 \otimes K_2)$

Gel'fand–Naimark–Segal Construction

Theorem (GNS Construction)

A a unital C^*-algebra. For every $\phi \in \mathcal{S}(A)$ there is:

- *A cyclic rep'n $\pi_\phi : A \to \mathcal{B}(K_\phi)$*
- *A distinguished cyclic vector ξ_ϕ*
- $(\pi_\phi(a)\xi_\phi, \xi_\phi) = \phi(a)$, $\forall a \in A$

- $\pi : A \to \mathcal{B}(K)$, $\xi \in K$, $\|\xi\| = 1$, then $(\pi(a)\xi, \xi)$ is a state.
- $\pi_\phi \ncong \pi \oplus \pi'$ (irreducible) iff $\phi \in \mathcal{P}(A)$.

Proof of GNS

Proof.

- $(x, y)_\phi := \phi(y^* x)$ positive semidefinite on A.
- complete to Hilbert space K_ϕ, $A \ni a \mapsto \hat{a} \in K_\phi$ dense.
- $\pi_\phi(a)\hat{b} := \widehat{ab}$, well-defined by C-S.
- $\|\pi_\phi(a)\hat{b}\|_\phi = \phi(b^* a^* ab)^{1/2} \leq \|a^* a\|^{1/2} \phi(b^* 1 b) = \|a\| \cdot \|b\|_\phi$.
- $\xi_\phi = \hat{1}$.

□

Universal Representation

Definition

$\pi_u : A \to \mathcal{B}(H_u)$ is the universal representation of A where

$$H_u := \bigoplus_{\phi \in \mathcal{S}(A)} H_\phi, \;\; \pi_u := \bigoplus_{\phi \in \mathcal{S}(A)} \pi_\phi$$

Ultraproducts of Representations

- $\{A_i : i \in I\}$, I a directed set.
- $\{\pi_i : i \in I\}$, $\pi_i : A_i \to \mathcal{B}(H_i)$.
- $\mathcal{U}$ ultrafilter on I.

$$\prod_{\mathcal{U}} A_i := \prod_I A_i / \{(a_i) : \lim_{\mathcal{U}} \|a_i\| = 0\}$$

Fact

The direct product representation $\prod_I$ descends to a representation

$$\pi_{\mathcal{U}} : \prod_{\mathcal{U}} A_i \to \prod_{\mathcal{U}} \mathcal{B}(H_i) \subset \mathcal{B}(H_{\mathcal{U}})$$

which is faithful iff π_i is $\mathcal{U}$-almost always faithful.

Building C*-Algebras from Relations

- $G = \{x_i : i \in I\}$ set of variables. "generators"
- $R = \{p_j : j \in J\}$ set of noncommuting polynomials in x_i's and x_i^*'s "relations"

Definition

A a C*-algebra <u>models</u> $(G|R)$, $A \models (G|R)$ if there is a map $x_i \to T_i \in A$ s.t. $\{T_i, T_i^*\}$ generates A and $p_j(T) = 0$ for all $j \in J$.

Definition

We say $(G|R)$ <u>consistent</u> if it admits a model.

Building C*-Algebras from Relations

Theorem (Compactness)

$(G|R)$ is consistent iff every subcollection of finitely many generators and relations is consistent.

Proof.

F finite subcollection of generators and relations. $\mathcal{F}$ directed set of all finite subcollections. $\mathcal{U}$ ultrafilter on $\mathcal{F}$.

If $A_F \models F$, then

$$\prod_{\mathcal{U}} A_F \models (G|R).$$

□

Universal Models

Theorem

If $(G|R)$ is consistent, then there is a unique C^*-algebra $C^*(G|R)$ s.t.

- $C^*(G|R) \models (G|R)$.
- *if $B \models (G|R)$ then there is a $*$-epimorphism $\pi : C^*(G|R) \to B$.*

Proof.

K isomorphism classes of models of $(G|R)$, then

$$x_i \mapsto (T_i^{(k)})_{k\in K} \in \mathcal{B}(\bigoplus_{k\in K} H_k)$$

generates $C^*(G|R)$. □

Group C*-Algebras

Theorem

Generators and relations defining a group are consistent.

Proof.

$g \in G$, define $u_g \in \mathcal{U}(\ell^2 G)$

$$u_g \xi(h) := \xi(gh), \ \xi \in \ell^2 G.$$

□

We define the reduced group C*-algebra

$$C_r^*(G) := C^*(u_g, g \in G) \subset \mathcal{B}(\ell^2 G)$$

Group C*-Algebras

$C^*(G)$ the universal C*-algebra given by group G generators and relations.

Fact

Any unitary rep'n $\pi : G \to \mathcal{U}(H)$ extends to a rep'n $\pi : C^*(G) \to \mathcal{B}(H)$.

Fact

$C_r^*(\mathbb{Z}) \cong C(\mathbb{T})$.

Proof.

Fourier transform □

Fact

$C^*(\mathbb{Z}) \cong C^*(1, u | 1 = 1^* = 1^2, u^*u = 1 = uu^*) \cong C_r^*(\mathbb{Z})$

- $C^*(G) \cong C_r^*(G)$ iff G is amenable. (Fell)

Fact

$\mathbb{F}_\infty$ free group on countably many generators. $C^*(\mathbb{F}_\infty)$ is the universal separable C^*-algebra.

- $C_r^*(\mathbb{F}_\infty)$ is simple. (Powers)
- $C^*(\mathbb{F}_\infty \times \mathbb{F}_\infty) \cong C^*(\mathbb{F}_\infty) \otimes C^*(\mathbb{F}_\infty) \subset \mathcal{B}(H_u \otimes H_u)$?? (Kirchberg, Connes)

More Universal Algebras

Example

$C^*(x|x = x^*) \cong C[0,1)$

Theorem (Coburn)

$C^*(1, v|1 = 1^* = 1^2, v^*v = 1)$ $*$-isomorphic to the Toeplitz algebra $C^*(S)$, $S : \ell^2\mathbb{N} \to \ell^2\mathbb{N}$ shift.

Example

$G = \{1, v_1, \ldots, v_n\}$, $R = \{1 = 1^* = 1^2, v_i^* v_i = 1, \sum_i v_i v_i^* = 1\}$ The Cuntz algebra $\mathcal{O}_n$ is $C^*(G|R)$.

Cuntz Algebras

Theorem (Cuntz)

Let $S_1, \ldots, S_n \in \mathcal{B}(\ell^2 N)$ *isometries such that* $S_i^* S_j = 0$ $i \neq j$ *and* $\sum_i S_i S_i^* = I$*. Then* $C^*(S_1, \ldots, S_n) \cong \mathcal{O}_n$*.*

Theorem (Cuntz)

$\mathcal{O}_n$*,* $n \geq 2$*, is separable, simple,* *purely infinite*. *("simple + purely infinite"* $\Leftrightarrow \forall a \in A_+,\ a \neq 0,\ \exists x(x^* a x = 1)$*.)*

- Cuntz algebras can be generalized to Cuntz–Kreiger algebras, etc.

Stable Relations

Definition

$(G|R) = (x_i|p_j)$ is stable if for any $\delta > 0$, there exists $\epsilon > 0$ such that for any C*-algebra A and any map $x_i \mapsto a_i$ such that $\|p_j(a)\| < \epsilon$ for all j, there exist (b_i) in A s.t.

$$\sup_i \|a_i - b_i\| \leq \delta$$

such that

$$p_j(b) = 0 \text{ for all } j.$$

Fact

This is equivalent to:
$\pi : C^*(G|R) \to \prod_{\mathcal{U}} A_i \Rightarrow \exists \pi_i : C^*(G|R) \to A_i$, *$\mathcal{U}$-almost always.*

Stable Relations

Theorem

$(p|p = p^* = p^2)$ *is stable.*

Proof.

Wlog $p^* = p$, so $C^*(p)$ is abelian. $p \sim p^2 \Rightarrow \sigma(p) \subset [0, \epsilon)(\epsilon, 1]$. $p \leftrightarrow \mathrm{id}_{\sigma(p)}$. Define $f = 0$ on $[0, \epsilon) \cap \sigma(p)$, $f = 1$ on $(\epsilon, 1] \cap \sigma(p)$. f corresponds to an element $q \in C^*(p)$ s.t. $q = q^* = q^2$ and $\|p - q\| = \|\mathrm{id} - f\|_\infty$.

□

Stable Relations

Theorem

$(1, u | 1 = 1^* = 1^2, u^* u = 1 = uu^*)$ *is stable.*

Proof.

Wlog 1 is $1 \in A$. $1 \sim u^* u$ implies $u^* u$ invertible. Let $v = u(u^* u)^{-1/2}$.

$$v^* v = (u^* u)^{-1/2} u^* u (u^* u)^{-1/2} = u^* u (u^u *)^{-1} = 1.$$

This implies that $(vv^*)^2 = vv^* =: p$. But $p \leq 1$, $p \sim 1$ implies that $p = 1$.

□