Definable Functions in Urysohn’s Metric Space

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1 Continuous Logic

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3 Definable functions
A (bounded) metric structure is a (bounded) complete metric space \((M, d)\), together with distinguished elements, functions (mapping \(M^n\) into \(M\) for various \(n\)), and predicates (mapping \(M^n\) into a bounded interval in \(\mathbb{R}\) for various \(n\)). Each function and predicate is required to be uniformly continuous. For the sake of simplicity, we suppose that the metric is bounded by 1 and the predicates all take values in \([0, 1]\).
Examples of Metric Structures

1. If $\mathcal{M}$ is a structure from classical model theory, then we can consider $\mathcal{M}$ as a metric structure by equipping it with the discrete metric. If $P \subseteq M^n$ is a distinguished predicate, then we consider it as a mapping $P : M^n \rightarrow \{0, 1\} \subseteq [0, 1]$ by

$$P(a) = 0 \text{ if and only if } \mathcal{M} \models P(a).$$

2. Suppose $X$ is a Banach space with unit ball $B$. Then $(B, 0_X, \| \cdot \|, (f_{\alpha, \beta})_{\alpha, \beta})$ is a metric structure, where $f_{\alpha, \beta} : B^2 \rightarrow B$ is given by $f(x, y) = \alpha \cdot x + \beta \cdot y$ for all scalars $\alpha$ and $\beta$ with $|\alpha| + |\beta| \leq 1$.

3. If $H$ is a Hilbert space with unit ball $B$, then $(B, 0_H, \| \cdot \|, \langle \cdot, \cdot \rangle, (f_{\alpha, \beta})_{\alpha, \beta})$ is a metric structure.
As in classical logic, a signature $L$ for continuous logic consists of constant symbols, function symbols, and predicate symbols, the latter two coming also with arities.

**New to continuous logic:** For every function symbol $F$, the signature must specify a *modulus of uniform continuity* $\Delta_F$, which is a function $\Delta_F : (0, 1] \rightarrow (0, 1]$. Likewise, a modulus of uniform continuity is specified for each predicate symbol.

The metric $d$ is included as a (logical) predicate in analogy with $=$ in classical logic.
An *L-structure* is a metric structure $\mathcal{M}$ whose distinguished constants, functions, and predicates are interpretations of the corresponding symbols in $L$. Moreover, the uniform continuity of the functions and predicates is witnessed by the moduli of uniform continuity specified by $L$.

e.g. If $P$ is a unary predicate symbol, then for all $\epsilon > 0$ and all $x, y \in M$, we have:

$$d(x, y) < \Delta_P(\epsilon) \Rightarrow |P^\mathcal{M}(x) - P^\mathcal{M}(y)| \leq \epsilon.$$
Formulae

- Terms are defined as in classical logic.
- Atomic formulae are of the form $d(t_1, t_2)$ and $P(t_1, \ldots, t_n)$ where $P$ is an $n$-ary predicate symbol and $t_1, \ldots, t_n$ are terms.
- Connectives: If $\varphi_1, \ldots, \varphi_n$ are formulae and $u : [0, 1]^n \to [0, 1]$ is any continuous function, then $u(\varphi_1, \ldots, \varphi_n)$ is a formula.
- Quantifiers: If $\varphi$ is a formula, then so is $\sup_x \varphi$ and $\inf_x \varphi$. ($\sup = $"$\forall$" and $\inf = $"$\exists$"

- If $\varphi(x_1, \ldots, x_n)$ is an $L$-formula, $\mathcal{M}$ an $L$-structure, and $a_1, \ldots, a_n$ elements of $M$, then $\mathcal{M}$ gives a value $\varphi^\mathcal{M}(a_1, \ldots, a_n)$, which is a number in $[0, 1]$ measuring “how true” $\varphi$ is when $a_1, \ldots, a_n$ are plugged in for the free variables.
- $t^\mathcal{M} : M^n \to M$ and $\varphi^\mathcal{M} : M^n \to [0, 1]$ are uniformly continuous for any term $t$ and any formula $\varphi$ (with $\Delta_t$ and $\Delta_\varphi$ calculable from the moduli in the signature.)
A condition is an expression of the form “$\varphi = 0$”, where $\varphi$ is a formula. If $\varphi$ is a sentence, then the condition “$\varphi = 0$” is called a closed condition.

Example

In the signature for Hilbert spaces, the condition $\langle x, y \rangle = 0$ expresses that $x$ and $y$ are orthogonal. The closed condition

$$\inf_{x_1} \ldots \inf_{x_n} \max_{i,j} |\langle x_i, x_j \rangle - \delta_{ij}| = 0$$

expresses that, for any $\epsilon > 0$, there are $x_1, \ldots, x_n$ such that $\langle x_i, x_j \rangle < \epsilon$ and $\|x_i\| - 1 < \epsilon$. In an $\omega_1$-saturated structure, where inf’s are realized, it will express that there are $n$ mutually orthogonal unit vectors.
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We can express weak inequalities as conditions: \( \varphi \leq \psi \) can be expressed as \( \varphi - \psi = 0 \), where \( a - b = \max(0, a - b) \).

An \( L \)-theory is a set of closed \( L \)-conditions.

If \( \mathcal{M} \) is an \( L \)-structure, then the theory of \( \mathcal{M} \) is the theory

\[
\text{Th}(\mathcal{M}) := \{ \varphi = 0 \mid \varphi \text{ a sentence, } \varphi^\mathcal{M} = 0 \}.
\]

If \( \varphi^\mathcal{M} = r \), then \( |\varphi^\mathcal{M} - r| = 0 \), so “\( |\varphi - r| = 0 \)” will be in the theory of \( \mathcal{M} \).

An \( L \)-theory is complete if it is of the form \( \text{Th}(\mathcal{M}) \) for some \( L \)-structure \( \mathcal{M} \).
Examples of Complete Continuous Theories

1. Infinite-dimensional Hilbert spaces (over $\mathbb{R}$)
2. Probability algebras based on atomless probability spaces
3. $L^p$-Banach lattices
4. Richly branching $\mathbb{R}$-trees
Continuous Logic

Definable and algebraic closure

Definition

Suppose that $\mathcal{M}$ is a structure and $A \subseteq M$. If $b \in M$, we say:

- $b \in dcl(A)$ if $\{b\}$ is an $A$-definable set.
- $b \in acl(A)$ if $b$ lives in a *compact* $A$-definable set.
Saturated Structures

**Definition**
If $\mathcal{M}$ is an $L$-structure and $A \subseteq M$ is a parameterset, then a collection $p(x)$ of $L(A)$-conditions is a \textit{(complete) type over $A$} if there is $\mathcal{M} \preceq \mathcal{N}$ and $b \in N^{|x|}$ such that $p(x) = \{ \varphi(x) = 0 : \varphi^\mathcal{N}(b) = 0, \varphi(x) \in L(A) \}$.

**Definition**
If $\kappa$ is an infinite cardinal, a structure $\mathcal{M}$ is said to be \textit{$\kappa$-saturated} if every type over a parameterset of cardinality $< \kappa$ is realized in $M$.

**Fact**
Given any infinite cardinal $\kappa$ and any structure $\mathcal{M}$, there is an elementary extension $\mathcal{M} \preceq \mathcal{N}$ such that $\mathcal{N}$ is $\kappa$-saturated.
Definable and algebraic closure—restated

**Definition**

Suppose that $\mathcal{M}$ is a $\omega_1$-saturated structure and $A \subseteq M$. If $b \in M$, we say:

- $b \in \text{dcl}(A)$ if $\sigma(b) = b$ for all $\sigma \in \text{Aut}(\mathcal{M}/A)$.
- $b \in \text{acl}(A)$ if the orbit of $b$ under the action of $\text{Aut}(\mathcal{M}/A)$ is compact.

It is clear from the above description that $\bar{A} \subseteq \text{dcl}(A) \subseteq \text{acl}(A)$ for all $A \subseteq M$, even if $\mathcal{M}$ is not saturated.

**Remark**

For my next talk, it will be relevant to note that dcl and acl have *countable character*: $b \in \text{dcl}(A)$ if and only if $b \in \text{dcl}(A_0)$ for some countable $A_0 \subseteq A$. 

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Definable functions in Urysohn space

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Continuous Logic

Definable and algebraic closure-restated

Definition

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The Urysohn Sphere

Definition

A *Polish metric space* is a separable, complete metric space.

Definition

The *Urysohn sphere* $\mathcal{U}$ is the unique (up to isometry) Polish metric space of diameter 1 which is:

1. **universal** - all Polish metric spaces of diameter $\leq 1$ admit an isometric embedding into $\mathcal{U}$;

2. **ultrahomogeneous** - if $\phi : X_1 \to X_2$ is an isometry between finite subspaces of $\mathcal{U}$, then there is an isometry $\tilde{\phi} : \mathcal{U} \to \mathcal{U}$ extending $\phi$.

Existence: Urysohn, Katětov; alternatively, it is the Fraisse limit of finite metric spaces of diameter $\leq 1$ (in the sense of continuous logic)
In this slide, a formula $\theta(x_1, \ldots, x_n)$ denotes a formula of the form $\max_{i,j} |d(x_i, x_j) - r_{ij}|$, where $(r_{ij})$ is a distance matrix for a finite metric space of diameter $\leq 1$.

Then for any such formula $\theta(x_1, \ldots, x_n, x_{n+1})$ and any $\epsilon > 0$, there is a $\delta > 0$ such that, for $a_1, \ldots, a_n \in \mathcal{U}$ satisfying $(\theta \upharpoonright n)(a_1, \ldots, a_n) < \delta$, there exists $a_{n+1} \in \mathcal{U}$ such that $\theta(a_1, \ldots, a_n, a_{n+1}) \leq \epsilon$.

We let $T_{\mathcal{U}}$ denote the set of axioms of the form:

$$\forall \vec{x} \exists y ((\theta \upharpoonright n)(\vec{x}) < \delta \rightarrow \theta(\vec{x}, y) \leq \epsilon).$$

More precisely,

$$\sup_{\vec{x}} \inf_{y} \left( \min \left( \frac{\epsilon}{1 - \delta (1 - (\theta \upharpoonright n)(\vec{x}))}, \theta(\vec{x}, y) \right) \div \epsilon \right) = 0.$$
Theorem (Folklore/Henson/Usvyatsov)

1. $T\mathcal{U}$ is $\aleph_0$-categorical, whence equal to $\text{Th}(\mathcal{U})$;
2. $T\mathcal{U}$ admits QE;
3. $T\mathcal{U}$ is the model completion of the empty $L$-theory (so is the theory of existentially closed metric spaces of diameter $\leq 1$);
4. for all $A \subseteq \mathcal{U}$, we have $\text{acl}(A) = \overline{A}$, so $\text{dcl}$ and $\text{acl}$ are trivial.

So $T\mathcal{U}$ is like a continuous analogue of the theory of the infinite set in classical logic. (And in other ways, it’s drastically different!-See next talk.)
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Proof of Fact 4

Lemma

\[ \text{acl}(A) = \bar{A}. \]

Proof.

Work in an \( \omega_1 \)-saturated elementary extension \( \mathcal{U} \) of \( \mathcal{U} \). Suppose that \( b \notin \bar{A} \). Consider the following collection of formulae:

\[
\{ d(x_i, a) = d(b, a) : i < \omega, \ a \in A \} \cup \{ d(x_i, x_j) = 2 \odot d(b, \bar{A}) : i < j < \omega \}. 
\]

Any finite subset defines a metric space, so can be realized in \( \mathcal{U} \). By \( \omega_1 \)-saturation, we can find \( (b_i : i < \omega) \) in \( \mathcal{U} \) realizing this partial type. By quantifier-elimination, \( \text{tp}(b_i/A) = \text{tp}(b/A) \) for all \( i < \omega \). But \( (b_i) \) has no convergent subsequence, so the orbit of \( b \) under \( \text{Aut}(\mathcal{U}/A) \) is not compact, whence \( b \notin \text{acl}(A) \). \qed
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Definable predicates

For purposes of definability in continuous logic, formulae aren’t expressive enough. It turns out that we need to consider *uniform limits of formulae*, which we call definable predicates:

**Definition**

Suppose that $\mathcal{M}$ is a structure and $A \subseteq M$. Then $P : M^n \rightarrow [0, 1]$ is said to be a *definable predicate in $\mathcal{M}$ over $A$* if there are formulae $\varphi_n(x)$ with parameters from $A$ such that the sequence $(\varphi_n^M)$ converges uniformly to $P$.

**Remark**

Although each $\varphi_n$ can only mention finitely many parameters from $A$, the sequence $(\varphi_n)$ can mention *countably* many parameters from $A$. Thus, definable things (sets, functions, . . .) are always definable over countably many parameters, but not necessarily finitely many parameters.
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For $A \subseteq M$, a function $f : M^n \to M$ is $A$-definable if the predicate $(x, y) \mapsto d(f(x), y) : M^{n+1} \to [0, 1]$ is an $A$-definable predicate.

Given an elementary extension $M \preceq N$, such a function admits a canonical extension $\tilde{f} : N^n \to N$, which is also $A$-definable: We have $(\varphi_n^M)$ converging uniformly to $d(f(x), y)$. Then $(\varphi_n^N)$ will converge uniformly to some $Q(x, y)$. One then checks that the zero set of $Q$ defines a function, which will be $\tilde{f}$. 
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Definable functions

1. For $A \subseteq M$, a function $f : M^n \rightarrow M$ is $A$-definable if the predicate $(x, y) \mapsto d(f(x), y) : M^{n+1} \rightarrow [0, 1]$ is an $A$-definable predicate.

2. Given an elementary extension $M \preceq N$, such a function admits a canonical extension $\tilde{f} : N^n \rightarrow N$, which is also $A$-definable.

3. Definable functions are uniformly continuous.

4. If $f : M^n \rightarrow M$ is $A$-definable, then for every $x = (x_1, \ldots, x_n) \in M^n$, we have $f(x) \in dcl(A \cup \{x_1, \ldots, x_n\})$. 
Definable functions

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Again, $\mathbb{U}$ is an $\omega_1$-saturated elementary extension of $\mathcal{U}$.

**Theorem (G.)**

If $f : \mathbb{U}^n \to \mathbb{U}$ is $A$-definable, then either $\tilde{f}$ is a projection function $(x_1, \ldots, x_n) \mapsto x_i$ or else $\tilde{f}$ has compact image contained in $\tilde{A} \subseteq \mathbb{U}$. Consequently, either $f$ is a projection function or else has relatively compact image.
Definable functions

Corollaries

**Corollary**

1. If $f : \mathcal{U} \to \mathcal{U}$ is a definable surjective/open/proper map, then $f = \text{id}_\mathcal{U}$.

2. If $f : \mathcal{U} \to \mathcal{U}$ is a definable isometric embedding, then $f = \text{id}_\mathcal{U}$.

3. If $n \geq 2$, then there are no definable isometric embeddings $\mathcal{U}^n \to \mathcal{U}$.

Reason: Compact sets in $\mathcal{U}$ have no interior.
Isometric Embeddings $\mathcal{U} \to \mathcal{U}$

There are many natural isometric embeddings $\mathcal{U} \to \mathcal{U}$, none of which (other than $\text{id}_\mathcal{U}$) are definable in $\mathcal{U}$.

## Examples

1. Suppose that $X_1$ and $X_2$ are compact subspaces of $\mathcal{U}$. Then any isometry $\phi : X_1 \to X_2$ can be extended to an isometry $\tilde{\phi} : \mathcal{U} \to \mathcal{U}$.

2. Suppose that $x_1, \ldots, x_n \in \mathcal{U}$. Define

   $$\text{Med}(x_1, \ldots, x_n) := \{ z \in \mathcal{U} \mid d(z, x_i) = d(z, x_j) \text{ for all } i, j \}.$$  

   Then $\text{Med}(x_1, \ldots, x_n)$ is isometric to $\mathcal{U}$.

3. Suppose that $M$ is a Polish subspace of $\mathcal{U}$ which is a Heine-Borel subspace. Then for any $R \in (0, 1]$, $\{ x \in \mathcal{U} \mid d(x, M) \geq R \}$ is isometric to $\mathcal{U}$. 
Corollary

*There are no definable group operations on $U$.*

Cameron and Vershik introduced a group operation on $U$ for which there is a dense cyclic subgroup. This group operation allows one to introduce a notion of translation in $U$. By the above corollary, this group operation is not definable.
Suppose that $f : \mathcal{U} \to \mathcal{U}$ is an $A$-definable function, where $A \subseteq \mathcal{U}$ is countable. Let $\tilde{f} : \mathcal{U} \to \mathcal{U}$ denote its canonical extension.

1. By triviality of $\text{dcl}$, for any $x \in \mathcal{U}$, we have $\tilde{f}(x) \in \text{dcl}(Ax) = \bar{A} \cup \{x\}$.

2. Let $X = \{x \in \mathcal{U} \mid f(x) = x\}$. Show that $\tilde{f}^{-1}(\bar{A}) \setminus X \subseteq \text{int}(\tilde{f}^{-1}(\bar{A}))$.

3. Prove a general lemma showing that if $F \subseteq \mathcal{U}$ is a closed subset and $G \subseteq F$ is a closed, separable subset of $F$ for which $F \setminus G \subseteq \text{int}(F)$, then either $F = G$ or $F = \mathcal{U}$. This involves a bit of “Urysohn-esque” arguing.

4. Finally, a saturation argument shows that if $\tilde{f}(\mathcal{U}) \subseteq \mathcal{U}$, then $\tilde{f}(\mathcal{U})$ is compact.
Proof of Step 2

Lemma

\[ X = \{ x \in \mathcal{U} \mid f(x) = x \}. \text{ Then } \tilde{f}^{-1}(\bar{A}) \setminus X \subseteq \text{int}(\tilde{f}^{-1}(\bar{A})). \]

Proof.

Suppose \( \tilde{f}(x) \in \bar{A} \) and \( \tilde{f}(x) \neq x \). Let \( r := d(\tilde{f}(x), x) > 0 \). Let \( \delta = \min(\frac{r}{2}, \Delta_f(\frac{r}{2})) \). Suppose \( d(x, y) < \delta \). Then \( d(\tilde{f}(x), \tilde{f}(y)) \leq \frac{r}{2} \). If \( \tilde{f}(y) = y \), then

\[
d(x, \tilde{f}(x)) \leq d(x, y) + d(\tilde{f}(x), y) < r,
\]

a contradiction. Thus \( y \in \tilde{f}^{-1}(\bar{A}) \). \( \square \)
Definable functions

Urysohn-esque arguing

Lemma

Let \((x_i \mid i < \omega)\) be a sequence from \(U\) and \((r_i \mid i < \omega)\) a sequence from \((0, 1)\). Set \(B := \bigcup_{i<\omega} B(x_i; r_i)\). Then \(U \setminus B\) is finitely injective.

Proof

Fix \(a_1, \ldots, a_n \in U \setminus B\) and let \(\{a_1, \ldots, a_n, y\}\) be a one-point metric extension. By saturation, it is enough to find, for each \(m < \omega\), a \(z \in U\) such that \(d(y, a_i) = d(z, a_i)\) for \(i = 1, \ldots, n\) and such that \(d(z, x_i) \geq r_i\) for each \(i = 1, \ldots, m\).
Proof (cont’d)

Consider the one-point metric extension

\[
\{a_1, \ldots, a_n, x_1, \ldots, x_m, z\}
\]

of \{a_1, \ldots, a_n, x_1, \ldots, x_m\} given by:

- \(d(z, a_i) = d(y, a_i)\) for each \(i \in \{1, \ldots, n\}\), and
- \(d(z, x_j) = \min_{1 \leq k \leq n}(d(y, a_k) + d(a_k, x_j))\) for each \(j \in \{1, \ldots, m\}\).

Such a \(z\) can be found in \(U\) and this \(z\) is as desired.

Corollary

\(U \setminus B\) is path-connected.
Lemma

Suppose that $F \subseteq U$ is closed and $G \subseteq F$ is a closed, separable subset of $F$ for which $F \setminus G \subseteq \text{int}(F)$. Then either $F = G$ or $F = U$.

Proof.

Suppose $F \neq G$. Let $0 < r < d(y, G)$. Cover $G$ with countably many balls of radius $r$ and call the union of these balls $B$. Set $Y = U \setminus B$, which is path-connected by the previous lemma. Now $F \cap Y = \text{int}(F) \cap Y$ is a nonempty, clopen subset of $Y$, implying that $F \cap Y = Y$. It follows that $Y \subseteq F$. Since $r$ can be taken to be arbitrarily small, this shows that $U \setminus G \subseteq F$, whence $F = U$. 

$\square$
Suppose that $\tilde{f}(U) \subseteq \mathcal{U}$. Then $\tilde{f}(U)$ is compact.

Proof.

It is a fact that $\tilde{f}(U)$ is closed, so we only need to show that it is totally bounded. Fix $\delta > 0$. Let $\varphi(x, y)$ be a formula that approximates $d(f(x), y)$ with error $\frac{\delta}{4}$. Let $(a_i : i < \omega)$ be a dense subset of $\mathcal{U}$. Then the collection $\{\varphi(x, a_i) \geq \frac{\delta}{2} : i < \omega\}$ of conditions is inconsistent. By $\omega_1$-saturation, there are $a_1, \ldots, a_n$ such that $\{\varphi(x, a_i) \geq \frac{\delta}{2} : 1 \leq i \leq n\}$ is inconsistent. It follows that $\tilde{f}(U) \subseteq \bigcup_{i=1}^{n} B(a_i; \delta)$. 

$\square$
Question

Can we improve the theorem on definable functions to read: If $f : \mathcal{U}^n \to \mathcal{U}$ is definable, then either $f$ is a projection or a constant function?

I can show that a positive solution to the above question follows from a positive solution to the $n = 1$ case.
The Case of Relatively Compact Image

In the hopes of answering this question, we can say some things about \( \tilde{f}(\mathbb{U}^n) \) in the case that it is relatively compact:

- \( \tilde{f}(\mathbb{U}^n) \) is a continuum (connected, compact space).
- Consequently, if \( \bar{A} \) is totally disconnected, then \( \tilde{f} \) is a constant function.
- \( \tilde{f}(\mathbb{U}^n) \) is a perfect space unless it is a singleton.
- If \( \tilde{f}(\mathbb{U}^n) \) is not a singleton, then \( \tilde{f}(\mathbb{U}^n) \) is either a Peano space (continuous image of \([0, 1]\)) or else a reducible continuum (every two points are contained in a proper subcontinuum.)
- Consequently, \( \tilde{f}(\mathbb{U}^n) \) is a decomposable continuum. Since the generic continuum is (hereditarily) indecomposable, we see that \( \tilde{f}(\mathbb{U}^n) \) is a special kind of continuum.
- \( \tilde{f}(\mathbb{U}^n) \) contains \textit{arbitrarily small path-connected subcontinua.}
Question 4

Are there any definable injections $f : \mathcal{U} \to \mathcal{U}$ other than the identity?

There can exist injective functions $\mathcal{U} \to \mathcal{U}$ which have relatively compact image, so our theorem doesn’t immediately help us: Consider $(x_n) \mapsto \left( \frac{x_n}{2^n} \right) : (0, 1)^\infty \to \ell^2$.

and use the fact that $\mathcal{U} \cong \ell^2 \cong (0, 1)^\infty$.

Observe that a positive answer to Question 3 yields a negative answer to this question.
Lemma

If $f : \mathbb{U} \to \mathbb{U}$ is injective and definable, then $f = \text{id}_{\mathbb{U}}$.

Proof.

One can show that the complement of an open ball in $\mathbb{U}$ is definable. Since $f$ maps definable sets to definable sets (which is a fact we are unsure of in $\mathbb{U}$), it follows that $f$ is a closed map, whence a topological embedding. By our main theorem, we see that $f$ is the identity.

Remark

This doesn’t immediately help us, for an injective definable map $\mathbb{U} \to \mathbb{U}$ need not induce an injective definable map $\mathbb{U} \to \mathbb{U}$. (Continuous logic is a positive logic!)
Upwards Transfer

Lemma (BBHU, Ealy-G.)

Suppose that $M$ is $\omega$-saturated and $P, Q : M^n \to [0, 1]$ are definable predicates such that $P$ is defined over a finite parameterset. Then the statement “for all $a \in M^n (P(a) = 0 \Rightarrow Q(a) = 0)” is expressible in continuous logic.

- It follows that the natural extension of an isometric embedding is also an isometric embedding:

$$|d(x, y) - r| = 0 \Rightarrow |d(f(x), f(y)) - r| = 0.$$

- It also follows that if $f : M^n \to M$ is an $A$-definable injection, where $A$ is finite, then $\tilde{f}$ is also an injection:

$$d(f(x), f(y)) = 0 \Rightarrow d(x, y) = 0.$$
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Suppose that $M$ is $\omega$-saturated and $P, Q : M^n \rightarrow [0, 1]$ are definable predicates such that $P$ is defined over a finite parameterset. Then the statement “for all $a \in M^n (P(a) = 0 \Rightarrow Q(a) = 0)” is expressible in continuous logic.

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- It also follows that if $f : M^n \rightarrow M$ is an $A$-definable injection, where $A$ is finite, then $\tilde{f}$ is also an injection:

  $$d(f(x), f(y)) = 0 \Rightarrow d(x, y) = 0.$$
Musings on Definable Sets

A closed set $X \subseteq \mathcal{U}^m$ is $A$-definable if the predicate $x \mapsto d(x, X) : \mathcal{U}^m \to [0, 1]$ is $A$-definable.

By the strong $\omega$-categoricity of $T_\mathcal{U}$, we have that, for finite $A \subseteq \mathcal{U}$, $X \subseteq \mathcal{U}^m$ is $A$-definable if and only if $X$ is invariant under $\text{Aut}(\mathcal{U}/A)$.

Consequently, for $A$-definable $X$, $Y \subseteq \mathcal{U}$, we have:

- $\partial X$, $\text{int}(X)$, $\mathcal{U} \setminus X$, $X \cap Y$, and $\text{Ker}(X)$ are $A$-definable.
- If $X$ is connected, then $X$ is a “generalized annulus”.
- The connected components of $X$ are $A$-definable and any 1-element connected subset of $X$ must be an element of $A$.
  Moreover, if there are infinitely many connected components of $X$, then they cannot be a uniform distance apart.
- If $X$ is compact, then $X$ is a (finite) subset of $A$. 
Question 5

What can we say about arbitrary definable subsets of $\mathcal{U}^m$?

There probably is no nice “geometric” description of the definable sets. Indeed, any compact set is definable in any metric structure, so any compact metric space is a definable subset of $\mathcal{U}$. However, maybe we can obtain results along the lines of the preceding slide saying that certain topological and geometric constructions preserve definability...
References

