# BOUNDARY AMENABILITY OF GROUPS VIA ULTRAPOWERS

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Abstract. We use C*-algebra ultrapowers to give a new construction of the Stone-Cech compactification of a separable, locally compact space. We use this construction to give a new proof of the fact that groups that act isometrically, properly, and transitively on trees are boundary amenable.

## 1. Introduction

Suppose that the discrete group $\Gamma$ acts continuously on a compact space $X$. We say that the action of $\Gamma$ on $X$ is *amenable* if there is a net of continuous functions $x \mapsto \mu_n^x : X \to P(\Gamma)$ such that, for all $\gamma \in \Gamma$, we have

$$\sup_{x\in X} \|\gamma \cdot \mu_n^x - \mu_n^{\gamma\cdot x}\|_1 \to 0.$$

We say that $\Gamma$ is *boundary amenable* if $\Gamma$ acts amenably on some compact space. Note that amenable groups are precisely the groups that act amenably on a one-point space, whence they are boundary amenable. A prototypical example of a boundary amenable group that is not amenable is any non-abelian finitely generated free group. Boundary amenable groups are sometimes referred to as exact groups for the reduced group C*-algebra $\mathrm{C}^*_r(\Gamma)$ is exact (meaning that the functor $\otimes_{\min}\mathrm{C}^*_r(\Gamma)$ is exact) if and only if $\Gamma$ is boundary amenable; see [7]. The study of exactness of group C*-algebras originated in [6].

In this note, we show how one can construct the Stone-Cech compactification of a separable, locally compact space using C*-algebra ultrapowers. When applied to the case of a tree, this construction gives a very natural proof of the fact that a group that acts isometrically, properly, and transitively on a tree is boundary amenable. It was our initial hope that this construction could be used to settle the boundary amenability of groups where the answer was unknown (most notably *Thompson's group*) but we have thus far been unsuccessful (although remain optimistic). The naïve idea behind our optimism is that groups such as Thompson's group "almost" act isometrically on a tree and it is often the case that ultrapower constructions can turn almost phenomena into exact ones.

In Section 2, we explain the needed background on groups acting on C*-algebras as well as ultrapowers of C*-algebras. In Section 3, we explain

Goldbring's work was partially supported by NSF CAREER grant DMS-1349399.

our main construction in the general setting of separable, locally compact spaces. Finally, in Section 4 we use our construction to prove the boundary amenability of groups acting isometrically, properly, and transitively on trees. This is a well-known result; it was established in [3] that free group C*-algebras have the completely bounded approximation property (CBAP). Later in [4] it was shown that the CBAP implies exactness of C*-algebras.

## 2. Preliminaries

### 2.1. Boundary amenability of groups acting on C*-algebras.

We will verify that certain groups act amenably on a compact space by checking that the group acts amenably on a unital abelian C*-algebra as we now explain. Suppose that $B$ is a unital C*-algebra and that $\Gamma$ acts on $B$. We consider the space $C_c(\Gamma, B)$ of finitely supported functions $\Gamma \to B$. $C_c(\Gamma, B)$ is naturally a $*$-algebra with respect to the convolution product

$$(f * g)(\gamma) = \sum_{\gamma_1 \cdot \gamma_2 = \gamma} f(\gamma_1)(\gamma_1 \cdot g(\gamma_2))$$

and involution

$$f^*(\gamma) = \gamma \cdot f(\gamma^{-1})^*.$$

We also view $C_c(\Gamma, B)$ as a pre-Hilbert $B$-module with $B$-valued inner product $\langle f, g\rangle_B = \sum_{\gamma \in \Gamma} f(\gamma)^* g(\gamma)$ and corresponding norm $\|f\|_B := \|\langle f, f\rangle_B\|^{-1/2}$.

Recall also that an action of $\Gamma$ on a compact space $X$ induces an action of $\Gamma$ on $C(X)$ by $(\gamma \cdot f)(x) := f(\gamma^{-1}x)$.

Our approach to showing that groups are boundary amenable is via the following reformulation of amenable actions (see [5, Proposition 2.2]) and is originally due to [2].

**Fact 2.1.** *The action $G \curvearrowright X$ is amenable if and only if there exists a net $T_i \in C_c(G, C(X))$ such that, for each $\gamma \in \Gamma$ and each $i$, we have:*

1. $T_i(\gamma) \geq 0$;
2. $\langle T_i, T_i\rangle_{C(X)} = 1$;
3. $\|T_i - \delta_\gamma * T_i\|_{C(X)} \to 0$.

In our proofs below, we will have an action of a group $\Gamma$ on a unital, abelian C*-algebra $B$ and we will prove that there exist $T_i \in C_c(\Gamma, B)$ satisfying the clauses (1)-(3) in the aforementioned fact. By Gelfand theory, $B$ is isomorphic to $C(X)$ for some compact space $X$. It remains to observe that Gelfand theory respects the group action, meaning that we obtain an induced action of $\Gamma$ on $X$ such that the corresponding action of $\Gamma$ on $C(X)$ "is" the corresponding action of $\Gamma$ on $B$. Thus, our criterion for boundary amenability of a group is the following:

**Fact 2.2.** *A group $\Gamma$ is boundary amenable if and only if there is a unital, abelian C*-algebra $B$ and a net $T_i \in C_c(G, B)$ such that, for each $\gamma \in \Gamma$ and each $i$, we have:*

1. $T_i(\gamma) \geq 0$;

(2) $\langle T_i, T_i\rangle_B = 1$;
(3) $\|T_i - \delta_\gamma * T_i\|_B \to 0$.

2.2. **Ultrapowers of $C^*$ algebras.** Recall that a nonprincipal ultrafilter $\mathcal{U}$ on $\mathbb{N}$ is a $\{0,1\}$-valued measure on *all* subsets of $\mathbb{N}$ such that finite sets get measure 0. We usually identify a nonprincipal ultrafilter with its collection of measure 1 sets, whence we write $A \in \mathcal{U}$ to indicate that $A$ has measure 1. If $P(n)$ is a property of natural numbers, we say that $P(n)$ holds *$\mathcal{U}$-almost everywhere* or that *$\mathcal{U}$-almost all $n$ satisfy $P$* if the set of $n$ for which $P(n)$ holds belongs to $\mathcal{U}$. If $(r_n)$ is a bounded sequence of real numbers, then the *ultralimit of $(r_n)$ with respect to $\mathcal{U}$*, denoted $\lim_{n,\mathcal{U}} r_n$ or even $\lim_\mathcal{U} r_n$, is the unique real number $r$ such that, for every $\epsilon > 0$, we have $|r_n - r| < \epsilon$ $\mathcal{U}$-almost everywhere.

Suppose that $A$ is a unital C*-algebra and $\mathcal{U}$ is a nonprincial ultrafilter on $\mathbb{N}$. We can define a seminorm $\|\cdot\|_\mathcal{U}$ on $\ell^\infty(A)$ by setting $\|(f_n)\|_\mathcal{U} := \lim_\mathcal{U} \|f_n\|$. We set $A^\mathcal{U}$ to be the quotient of $\ell^\infty(A)$ by those elements of $\|\cdot\|_\mathcal{U}$-norm 0; we refer to $A^\mathcal{U}$ as the *ultrapower of $A$ with respect to $\mathcal{U}$*. It is well known that $A^\mathcal{U}$ is once again a unital C*-algebra. For $(f_n) \in \ell^\infty(A)$, we let $(f_n)^\bullet$ denote its image in $A^\mathcal{U}$. The canonical *diagonal embedding* $\Delta : A \to A^\mathcal{U}$ is given by $\Delta(a) = (a)^\bullet$.

## 3. The main construction

In this section, we consider a second countable, locally compact space $X$ with fixed basepoint $o \in X$. It is well-known that $X$ admits a compatible proper metric $d$ (see [8, Theorem 2]), and we fix such a metric in the rest of this section. For $r \in \mathbb{R}^{>0}$, we set $B(r)$ to be the closed ball of radius $r$ around $o$.

We set $A = C_o(X)$, the space of complex-valued continuous functions on $X$ that vanish at infinity. For $(f_n) \in \ell^\infty(A)$, we say that $(f_n)$ is *$\mathcal{U}$-equicontinuous on bounded sets* if, for every $r, \epsilon > 0$, there is $\delta > 0$ such that, for $\mathcal{U}$-almost all $n$, we have for all $s, t \in B(r)$ with $d(s,t) < \delta$, that $|f_n(s) - f_n(t)| \leq \epsilon$.

Given any $(f_n) \in \ell^\infty(A)$, set $f_\mathcal{U} : X \to \mathbb{C}$ by $f_\mathcal{U}(t) := \lim_\mathcal{U} f_n(t)$. Note that $f_\mathcal{U}$ is a bounded function. The following lemma is quite routine and left to the reader.

**Lemma 3.1.** *If $(f_n)$ is $\mathcal{U}$-equicontinuous on bounded sets, then $f_\mathcal{U}$ is uniformly continuous on bounded sets.*

**Lemma 3.2.** *Suppose that $(f_n)^\bullet = (g_n)^\bullet$ and $(f_n)$ is $\mathcal{U}$-equicontinuous on bounded sets. Then so is $(g_n)$.*

*Proof.* Fix $r, \epsilon > 0$. Take $\delta > 0$ that witnesses $\mathcal{U}$-equicontinuity of $(f_n)$ on $B(r)$ for $\epsilon/3$. Then for $\mathcal{U}$-almost all $n$, we have, for $s, t \in B(r)$ with $d(s,t) < \delta$, that

$$|g_n(s) - g_n(t)| \leq 2\|g_n - f_n\| + |f_n(s) - f_n(t)| \leq \frac{2\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.$$

□

The previous lemma allows us to consider the *continuous part of* $A^{\mathcal{U}}$

$$A^{c\mathcal{U}} := \{(f_n)^\bullet \in A^{\mathcal{U}} \ : \ (f_n) \text{ is } \mathcal{U}\text{-equicontinuous on bounded sets}\}.$$

**Lemma 3.3.**

(1) $A^{c\mathcal{U}}$ *is a* $\mathrm{C}^*$*-subalgebra of* $A^{\mathcal{U}}$.
(2) $\Delta(A) \subseteq A^{c\mathcal{U}}$.

*Proof.* For (1), it is clear that $A^{c\mathcal{U}}$ is a $*$-subalgebra of $A^{\mathcal{U}}$. We must show that $A^{c\mathcal{U}}$ is closed in $A^{\mathcal{U}}$. Towards this end, suppose that $(f_n^m)^\bullet$ is a sequence in $A^{c\mathcal{U}}$ such that $(f_n^m)^\bullet \to (h_n)^\bullet$ as $n \to \infty$. We need to show that $(h_n)^\bullet \in A^{c\mathcal{U}}$. Fix $r, \epsilon > 0$. Let $\delta > 0$ be so that, if $s, t \in B(r)$ and $d(s,t) < \delta$, then for $\mathcal{U}$-almost all $n$, we have $|f_n(s) - f_n(t)| < \frac{\epsilon}{3}$. Choose $m \in \mathbb{N}$ such that $\|(f_n^m)^\bullet - (h_n)^\bullet\| < \epsilon/3$. Suppose that $s, t \in B(r)$ are such that $d(s,t) < \delta$. Then we have that, for $\mathcal{U}$-almost all $n$, that

$$|h_n(s) - h_n(t)| \leq 2\|f_n^m - h_n\| + |f_n^m(s) - f_n^m(t)| \leq \frac{2\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.$$

(2) follows from the fact that balls $B(r)$ in $X$ are compact, whence elements of $A$ are uniformly continuous on such balls. □

We now consider

$$I := \{(f_n)^\bullet \in A^{\mathcal{U}} \ : \ (\exists r_n \in \mathbb{R})(\lim_{\mathcal{U}} r_n = +\infty \text{ and } f_n|B(o, r_n) \equiv 0)\}.$$

It is clear from the definition that $I \subseteq A^{c\mathcal{U}}$.

In the rest of this section, we fix continuous functions $\chi_n : X \to \mathbb{R}$ such that:

(1) $0 \leq \chi_n \leq 1$;
(2) $\chi_n(t) = 0$ for $t \in B(n)$;
(3) $\chi_n(t) = 1$ when $d(t, o) \geq n + 1$.

**Proposition 3.4.**

(1) *$I$ is a closed ideal in* $A^{\mathcal{U}}$.
(2) $A^{c\mathcal{U}}/I$ *is unital.*
(3) $q \circ \Delta : A \to A^{c\mathcal{U}}/I$ *is injective, where* $q : A^{c\mathcal{U}} \to A^{c\mathcal{U}}/I$ *is the canonical quotient map.*
(4) $(q \circ \Delta)(A)$ *is an essential ideal in* $A^{c\mathcal{U}}/I$.

*Proof.* For (1), suppose $(f_n)^\bullet, (g_n)^\bullet \in I$, $(h_n) \in A^{\mathcal{U}}$, and $\lambda \in \mathbb{C}$. Suppose that $f_n|B(r_n), g_n|B(s_n) \equiv 0$, where $\lim_{\mathcal{U}} r_n = \lim_{\mathcal{U}} s_n = 0$. Then

$$\lambda f_n|B(r_n), (f_n + g_n)|B(\min(r_n, s_n)), (f_n \cdot h_n)|B(r_n) \equiv 0;$$

since $\lim_{\mathcal{U}} \min(r_n, s_n) = \infty$, we have $\lambda f_n, f_n + g_n, f_n \cdot h_n \in I$ and $I$ is an ideal.

We now prove that $I$ is closed. Suppose that $((f_n^m)^\bullet \ : \ m \in \mathbb{N})$ is a sequence from $I$ such that $\lim_m (f_n^m)^\bullet = (g_n)^\bullet$; we must show that $(g_n)^\bullet \in I$. Suppose that $f_n^m|B(r_n^m) \equiv 0$ with $\lim_{n,\mathcal{U}} r_n^m = \infty$ for each $m$. Fix $k \in \mathbb{N}$

and take $m \in \mathbb{N}$ such that $\|(f_n^m)^\bullet - (g_n)^\bullet\| < \frac{1}{k}$. For $\mathcal{U}$-almost all $n$ we have $\|f_n^m - g_n\| < \frac{1}{k}$ and $r_n^m \geq k$. Thus, if we set

$$X_k := \{n \in \mathbb{N} \ : n \geq k \text{ and } |g_n(t)| < \frac{1}{k} \text{ for } t \in B(k)\},$$

we have that $X_k \in \mathcal{U}$. For $n \in \mathbb{N}$, set $l(n) := \max\{k \in \mathbb{N} \ : \ n \in X_k\}$. Note that $n \in X_k$ implies that $l(n) \geq k$, whence $\lim_{n,\mathcal{U}} l(n) = \infty$. Define $h_n := g_n \cdot \chi_{l(n)-1}$. Note that $(h_n)^\bullet \in I$; it remains to show that $(g_n)^\bullet = (h_n)^\bullet$. For $n \in \mathbb{N}$, we have $\|g_n - h_n\| \leq \sup_{t \in B(l(n))} |g_n(t)| \leq \frac{1}{l(n)}$, whence

$$\|(g_n)^\bullet - (h_n)^\bullet\| = \lim_{\mathcal{U}} \|g_n - h_n\| \leq \lim_{\mathcal{U}} \frac{1}{l(n)} = 0.$$

For (2), consider any sequence $(g_n) \in \ell^\infty(A)$ such that $g_n \equiv 1$ on $B(n)$. (For example, take $g_n := 1 - \chi_n$.) We claim that $q(g_n)^\bullet$ is an identity for the larger algebra $A^{\mathcal{U}}/I$. Indeed, consider arbitrary $q(f_n)^\bullet \in A^{\mathcal{U}}/I$. Then $f_n g_n - f_n$ vanishes on $B(n)$, whence $(f_n g_n - f_n)^\bullet \in I$ and $q(f_n g_n)^\bullet = q(f_n)^\bullet$.

For (3), suppose that $(q \circ \Delta)(f) = 0$. Then there is $(g_n)^\bullet \in I$ such that $\Delta(f) = (g_n)^\bullet$. Suppose that $g_n | B(r_n) \equiv 0$ with $\lim_{\mathcal{U}} r_n = \infty$. Fix $t \in X$ and $\epsilon > 0$. Then for $\mathcal{U}$-almost all $n$, we have $\|f - g_n\| < \epsilon$ and $t \in B(r_n)$, whence $|f(t)| < \epsilon$. Since $t$ and $\epsilon$ were arbitrary, we have that $f \equiv 0$.

We now prove (4). We first show that $(q \circ \Delta)(A)$ is an ideal in $A^{c\mathcal{U}}/I$. Towards this end, fix $f \in A$ and $q((g_n)^\bullet) \in A^{c\mathcal{U}}/I$; we must show that $q((fg_n)^\bullet) \in q(\Delta(A))$. In fact, we will show that $q((fg_n)^\bullet) = q(\Delta(fg_{\mathcal{U}}))$. Recall that

$$\|q((fg_n)^\bullet) - q(\Delta(fg_{\mathcal{U}}))\| = \inf\{\lim_{\mathcal{U}} \|fg_n - fg_{\mathcal{U}} - h_n\| \ : \ (h_n)^\bullet \in I\}.$$

Set $M := \sup_n \|g_n\|$. Fix $\epsilon > 0$. Fix $m \in \mathbb{N}$ such that $|f(t)| < \frac{\epsilon}{2M}$ when $t \in B(m)^c$. Let $\delta > 0$ witness the $\mathcal{U}$-equicontinuity of $(g_n)$ on $B(m)$ with respect to $\frac{\epsilon}{3\|f\|}$ and fix a finite $\delta$-net $\{t_1, \ldots, t_k\}$ for $B(m)$. Fix $U \in \mathcal{U}$ such that $\{k \in \mathbb{N} \ : \ k \geq m\} \subseteq U$ and $|g_n(t_i) - g_{\mathcal{U}}(t_i)| < \frac{\epsilon}{3\|f\|}$ for $i = 1, \ldots, k$ and $n \in U$. For $n \in U$, define $h_n \in A$ by $h_n := (fg_n - fg_{\mathcal{U}})\chi_n$. (Define $h_n \in A$ for $n \notin U$ in an arbitrary fashion). It suffices to show that $\lim_{\mathcal{U}} \|fg_n - fg_{\mathcal{U}} - h_n\| \leq \epsilon$. Suppose $n \in U$. First consider $t \in B(m)$. Then $|fg_n(t) - fg_{\mathcal{U}}(t) - h_n(t)| = |fg_n(t) - fg_{\mathcal{U}}(t)|$. Take $i$ such that $d(t, t_i) < \delta$. Then, for $\mathcal{U}$-almost all $n$, we have

$$|g_n(t) - g_{\mathcal{U}}(t)| \leq |g_n(t) - g_n(t_i)| + |g_n(t_i) - g_{\mathcal{U}}(t_i)| + |g_{\mathcal{U}}(t_i) - g_{\mathcal{U}}(t)| \leq \frac{\epsilon}{\|f\|},$$

whence $|fg_n(t) - fg_{\mathcal{U}}(t)| \leq \epsilon$. Now suppose that $t \in B(m)^c \cap B(n+1)$. Then $|fg_n(t) - fg_{\mathcal{U}}(t) - h_n(t)| \leq |fg_n(t) - fg_{\mathcal{U}}(t)| < \epsilon$ by choice of $m$. If $t \in B(n+1)^c$, then $fg_n(t) - fg_{\mathcal{U}}(t) - h_n(t) = 0$. It follows that $\lim_{\mathcal{U}} \|fg_n - fg_{\mathcal{U}} - h_n\| \leq \epsilon$, finishing the proof that $(q \circ \Delta)(A)$ is an ideal in $A^{c\mathcal{U}}$.

We next show that $(q \circ \Delta)(A)$ is an essential ideal in $A^{c\mathcal{U}}/I$. Suppose that $q(f_n)^\bullet \in A^{c\mathcal{U}}/I$ is such that $q(f_n)^\bullet \cdot q(a)^\bullet = 0$ for all $a \in A$; we must show that $q(f_n)^\bullet = 0$.

Fix $t \in X$. Fix $a \in A$ such that $a(t) = 1$. Then there is $(g_n)^\bullet \in I$ such that $\lim_{\mathcal{U}} \|f_n a - g_n\| = 0$. For $\mathcal{U}$-most $n$, we have $t \in B(r_n)$, where $g_n$ vanishes on $B(r_n)$. It thus follows that

$$\lim_{\mathcal{U}} |f_n(t)| \leq \lim_{\mathcal{U}} \|f_n a - g_n\| = 0.$$

Set

$$U_k := \{n \in \mathbb{N} \ : \ n \geq k \text{ and } |f_n(t)| \leq \frac{1}{k} \text{ for } t \in B(k)\}.$$

We claim that $U_k \in \mathcal{U}$. Fix $\delta > 0$ that witnesses $\mathcal{U}$-equicontinuity of $(f_n)$ on $B(k)$ with respect to $\frac{1}{2k}$. Fix a finite $\delta$-net $F$ for $B(k)$. Then for $\mathcal{U}$-most $n$, $|f_n(t)| \leq \frac{1}{2k}$ for $t \in F$. Thus, given any $s \in B(k)$ and taking $t \in F$ such that $d(s,t) < \delta$, we have that $|f_n(s)| \leq |f_n(s) - f_n(t)| + |f_n(t)| \leq \frac{1}{k}$ for $\mathcal{U}$-most $n$.

For $n \in \mathbb{N}$, set $l(n) := \max\{k \in \mathbb{N} \ : \ n \in U_k\}$. For $n \in U_k$, we have $l(n) \geq k$, whence $\lim_{\mathcal{U}} l(n) = \infty$. Define $h_n \in A$ by $h_n = f_n \cdot \chi_{l(n)-1}$. As above, we have that $(h_n)^\bullet \in I$ and $\|f_n - g_n\| \leq \frac{1}{l(n)}$ whence $\lim_{\mathcal{U}} \|f_n - g_n\| \leq \lim_{\mathcal{U}} \frac{1}{l(n)} = 0$. □

Since $q(\Delta(A))$ is an essential ideal in the unital C*-algebra $A^{c\mathcal{U}}/I$, we see that $\Sigma(A^{c\mathcal{U}}/I)$ is a compactification of $X$, where $\Sigma(A^{c\mathcal{U}}/I)$ denotes the Gelfand spectrum of $A^{c\mathcal{U}}/I$. It turns out that this compactification is indeed the Stone-Cech compactification of $X$. Recall that $C_b(X)$ denotes the unital C*-algebra of bounded, continuous, complex-valued functions on $X$ and is natrually isomorphic to $C(\beta X)$, where $\beta X$ denotes the Stone-Cech compactification of $X$.

**Proposition 3.5.** *There is an isomorphism* $\Phi : A^{c\mathcal{U}}/I \to C_b(X)$ *such that* $\Phi(q(\Delta(a)) = a$ *for all* $a \in A$.

*Proof.* Define $\Phi : A^{c\mathcal{U}} \to C_b(X)$ by $\Phi((f_n)^\bullet) := f_{\mathcal{U}}$. It is clear that $\Phi$ is a *-morphism. We next observe that $\Phi$ is onto. Indeed, given $f \in C_b(X)$ and $n > 0$, define $f_n \in C_0(T)$ by $f_n = (1 - \chi_n)f$. Since $f$ is bounded, we have that $(f_n) \in \ell^\infty(A)$. Since $X$ is proper, $f$ is uniformly continuous on bounded sets, whence $(f_n)$ is $\mathcal{U}$-equicontinuous on bounded sets, that is, $(f_n)^\bullet \in A^{c\mathcal{U}}$. It is clear that $\Phi((f_n)^\bullet) = f$.

Now suppose that $(f_n)^\bullet \in I$. Then by the definition of $I$, we have that $\Phi((f_n)^\bullet) = 0$, so $\Phi$ induces a surjection $\Phi : A^{c\mathcal{U}}/I \to C_b(X)$. Suppose now that $\Phi((f_n)^\bullet) = 0$. For each $n > 0$, define a function $g_n \in A$ by $g_n = f_n\chi_{n-1}$. It is clear that $(g_n)^\bullet \in I$. Since $\lim_{\mathcal{U}} f_n(t) = 0$ for all $t \in X$ and $\|f_n - g_n\| \leq \max_{t \in B(o,n)} |f_n(t)|$, it follows that $(f_n)^\bullet = (g_n)^\bullet$, whence $(f_n)^\bullet \in I$, thus proving that $\Phi : A^{\mathcal{U}}/I \to C_b(X)$ is an isomorphism.

Finally, it is clear from the definition of $\Phi$ that $\Phi(q(\Delta(a)) = a$ for all $a \in A$. □

From now on, we set $B := (A^{c\mathcal{U}}/I)/((q \circ \Delta)(A))$, a unital C*-algebra, and let $r : A^{\mathcal{U}} \to B$ denote the composition of $q$ with the quotient map

$A^{c\mathcal{U}}/I \to B$. Note that by the previous proposition, $B \cong C(X^*)$, where $X^*$ denotes the *Stone-Cech remainder* $\beta X \setminus X$ of $X$.

We now introduce a group action into the picture:

**Lemma 3.6.** *Suppose that $\Gamma$ acts isometrically on $X$.*

(1) *The induced action of $\Gamma$ on $A$ further induces an action of $\Gamma$ on $A^{\mathcal{U}}$ by $\gamma \cdot (f_n)^\bullet := (\gamma \cdot f_n)^\bullet$.*
(2) *Both $A^{c\mathcal{U}}$ and $I$ are invariant under the action of $\Gamma$ on $A^{\mathcal{U}}$ from (1).*

*Proof.* For (1), we need to verify that, for $(f_n), (g_n) \in \ell^\infty(A)$, if $\lim_{\mathcal{U}} \|f_n - g_n\| = 0$, then $\lim_{\mathcal{U}} \|\gamma \cdot f_n - \gamma \cdot g_n\| = 0$. However, this follows from the easy check that $\|\gamma \cdot f_n - \gamma \cdot g_n\| = \|f_n - g_n\|$ for each $n$.

We now prove (2). The fact that $A^{c\mathcal{U}}$ is invariant under the action of $\Gamma$ follows from the fact that $\Gamma$ acts by isometries and thus takes bounded sets to bounded sets. We now prove that $I$ is invariant under the action of $\Gamma$. Consider $\gamma \in \Gamma$ and $(f_n)^\bullet \in I$; we must show that $(\gamma \cdot f_n)^\bullet \in I$. Suppose that $f_n|B(o, r_n) \equiv 0$ where $\lim_{\mathcal{U}} r_n = \infty$. Set $k := d(\gamma^{-1} \cdot o, o)$. Then for $r_n > k$, we have that $(\gamma \cdot f_n)|B(o, r_n - k) \equiv 0$: if $t \in B(o, r_n - k)$, then

$$d(\gamma^{-1}t, o) \leq d(\gamma^{-1}t, \gamma^{-1}o) + d(\gamma^{-1}o, o) = d(t, o) + k \leq r_n,$$

whence $f_n(\gamma^{-1}t) = 0$. Since $r_n > k$ for $\mathcal{U}$-almost all $n$ and $\lim_u r_n - k = \infty$, it follows that $(\gamma \cdot f_n)^\bullet \in I$. □

By the previous lemma, we have an induced action of $\Gamma$ on $A^{c\mathcal{U}}/I$ by setting $\gamma \cdot q(f_n)^\bullet := q(\gamma \cdot f_n)^\bullet$, whence we also get an action of $\Gamma$ on $B$ by setting $\gamma \cdot r(f_n)^\bullet := r(\gamma \cdot f_n)^\bullet$.

## 4. Groups acting properly and isometrically on a tree

In this section, our locally compact space is simply a tree $T$ given the usual path metric, namely $d(x, y) =$ the length of the shortest path connecting $x$ and $y$. In this case, $A^{c\mathcal{U}} = A^{\mathcal{U}}$. We further suppose that $\Gamma \curvearrowright T$ properly, isometrically and transitively. (Recall that the action is proper if the map $(g, t) \mapsto (gt, t) : G \times T \to T \times T$ is proper, meaning that inverse images of compact sets are compact.) In this case, $\mathrm{Stab}(o)$ is finite, say of cardinality $m$. For a point $t \in T$, we let $x_{[o,t]}$ denote the geodesic segment connecting $o$ and $t$.

**Theorem 4.1.** *If $\Gamma$ acts properly, transitively, and isometrically on a simplicial tree $T$, then $\Gamma$ is exact.*

*Proof.* For $t \in T$ and $i \in \mathbb{N}$, set

$$X(i, t) := \{\gamma \in \Gamma \ : \ \gamma \cdot o \in B(i) \text{ and } \gamma \cdot o \in x_{[o,t]}\}$$

and $x(i, t) = |X(i, t)|^{-1/2}$. Note that $x(i, t) = (m \cdot \min(i, d(o, t)))^{-1/2}$. Define $T_i^{(n)} : \Gamma \to A$ by

$$T_i^{(n)}(\gamma)(t) = \begin{cases} x(i, t) & \text{if } t \in B(2n) \text{ and } \gamma \in X(i, t); \\ 0 & \text{otherwise.} \end{cases}$$

Now define $T_i : \Gamma \to B$ by $T_i(\gamma) := r((T_i^{(n)}(\gamma)^\bullet)$. We claim that these functions satisfy the criteria of Fact 2.2, whence the action of $\gamma$ on $X^*$ is amenable.

Certainly each $(T_i^{(n)}(\gamma))^\bullet$ is a positive element of $A^{\mathcal{U}}$; since $r$ is a C*-algebra homomorphism, we have that each $T_i(\gamma) \geq 0$ in $B$.

We now verify that $\langle T_i, T_i\rangle_B = 1_B$; in other words, we must show that $\sum_{\gamma\in\Gamma} T_i(\gamma)^2 = 1_B$. First observe that there is a finite $\Gamma_i \subseteq \Gamma$ such that $\sum_{\gamma\in\Gamma} T_i^{(n)}(\gamma)^2 = \sum_{\gamma\in\Gamma_i} T_i^{(n)}(\gamma)^2 = \chi_{B(2n)}$. Since $(\chi_{B(2n)})^\bullet + I$ is the unit of $A^{\mathcal{U}}/I$, it follows that $r((\chi_{B(2n)})^\bullet)$ is the identity of $B$. Now compute:

$$\begin{aligned}
\sum_{\gamma\in\Gamma} T_i(\gamma)^2 &= \sum_{\gamma\in\Gamma_i} T_i(\gamma)^2 \\
&= \sum_{\gamma\in\Gamma_i} (r((T_i^{(n)}(\gamma))^\bullet)^2 \\
&= \sum_{\gamma\in\Gamma_i} r(((T_i^{(n)}(\gamma))^\bullet)^2) \\
&= r(\sum_{\gamma\in\Gamma_i} ((T_i^{(n)}(\gamma))^\bullet)^2)) \\
&= r(\sum_{\gamma\in\Gamma_i} ((T_i^{(n)}(\gamma))^2)^\bullet)) \\
&= r((\sum_{\gamma\in\Gamma_i} T_i^{(n)}(\gamma)^2)^\bullet) \\
&= r((\chi_{B(o,2n)})^\bullet) \\
&= 1_B.
\end{aligned}$$

It remains to prove that, for each $\gamma_1 \in \Gamma$, we have $\lim_{i\to\infty} \|T_i - \delta_{\gamma_1} * T_i\|_B = 0$. It is straightforward to compute that $\delta_{\gamma_1} * T_i = \gamma_1 \cdot T_i(\gamma_1^{-1}\gamma)$. It follows that $\|T_i - \delta_{\gamma_1} * T_i\|_B^2$ is equal to

$$\|\sum_{\gamma\in\Gamma}(T_i(\gamma)^2 + (\gamma_1 \cdot T_i(\gamma_1^{-1}\gamma))^2 - 2T_i(\gamma)\gamma_1 \cdot T_i(\gamma_1^{-1}\gamma))\|. \quad (\dagger)$$

Now $\gamma_1 T_i^{(n)}(\gamma_1^{-1}\gamma)(t) = T_i^{(n)}(\gamma_1^{-1}\gamma)(\gamma_1^{-1}(t))$, which is only nonzero if:

(1) $\gamma_1^{-1}t \in B(2n)$;
(2) $\gamma_1^{-1}\gamma \cdot o \in B(i)$;
(3) $\gamma_1^{-1}\gamma \cdot o \in x_{[o,\gamma_1^{-1}t]}$.

Also notice that $\sum_{\gamma\in\Gamma}(\gamma_1 \cdot T_i^{(n)}(\gamma_1^{-1}\gamma))^2 = \chi_{\gamma_1 \cdot B(2n)}$, so ($\dagger$) equals

$$\|r(\chi_{B(2n)})^\bullet + r(\chi_{\gamma_1 B(2n)})^\bullet - 2\sum_{\gamma\in\Gamma}(T_i^{(n)}(\gamma)\gamma_1 \cdot T_i^{(n)}(\gamma_1^{-1}\gamma))^\bullet)\|_B,$$

which in turn equals

$$\inf\{\lim_{n,\mathcal{U}} \|\chi_{B(2n)} + \chi_{\gamma_1\cdot B(2n)} - 2\sum_{\gamma\in\Gamma} T_i^{(n)}(\gamma)\gamma_1 \cdot T_i^{(n)}(\gamma_1^{-1}\gamma)) - g_n - a\|\}, \quad (\dagger\dagger)$$

where $(g_n)^\bullet$ ranges over $I$ and $a$ ranges over $A$.

Set

$$a(t) = \left( (\chi_{B(2n)} + \chi_{\gamma_1\cdot B(2n)} - 2\sum_{\gamma\in\Gamma} T_i^{(n)}(\gamma)\gamma_1 \cdot T_i^{(n)(\gamma_1^{-1}\gamma)}) \cdot \chi_{B(i)\cup\gamma_1\cdot B(i)} \right)^\bullet .$$

$a(t)$ is certainly an element of $A$ since $B(i) \cup \gamma_1 \cdot B(i)$ is compact. Set $g_n(t) = \chi_{B(2n)\triangle\gamma_1\cdot B(2n)}$. Finally set

$$O(i,n) = (B(2n) \cap \gamma_1 \cdot B(2n)) \setminus (B(i) \cup \gamma_1 \cdot B(i)).$$

Then $a \in A$, $(g_n)^\bullet \in I$ (as $g_n|B(n) \equiv 0$) and the value in $(\dagger\dagger)$ is bounded by

$$\lim_{n,\mathcal{U}} \sup_{t\in O(i,n)} |2 - 2\sum_{\gamma\in\Gamma} T_i^{(n)}(\gamma)\gamma_1 \cdot T_i^{(n)}(\gamma_1^{-1}\gamma)(t)|. \quad (\dagger\dagger\dagger)$$

Let $Z(i,t)$ be the set

$$\{\gamma \in \Gamma \ : \ \gamma\cdot o \in B(i), \gamma \cdot o \in x_{[o,t]}, \gamma_1^{-1}\gamma \cdot o \in B(i), \gamma_1^{-1}\gamma\cdot o \in x_{[o,\gamma_1^{-1}t]}\}.$$

Set $k := d(\gamma_1 \cdot o, o)$. For $n$ sufficiently large and for $t \in O(i,n)$, we have $|Z(i,t)| \geq m(i-k+1)$ and $2\sum_{\gamma\in\Gamma}(T_i^{(n)}(\gamma)\gamma_1 \cdot T_i^{(n)})(t) \geq 2\frac{m(i-k+1)}{mi}$, whence the quantity appearing in $(\dagger\dagger\dagger)$ is bounded above by $2 - 2\frac{m(i-k+1)}{mi}$, which goes to 0 as $i \to \infty$ as desired. □

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