BOUNDARY AMENABLEITY OF GROUPS VIA ULTRAPowers

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Abstract. We use C*-algebra ultrapowers to give a new construction of the Stone-Cech compactification of a separable, locally compact space. We use this construction to give a new proof of the fact that groups that act isometrically, properly, and transitively on trees are boundary amenable.

1. Introduction

Suppose that the discrete group \( \Gamma \) acts continuously on a compact space \( X \). We say that the action of \( \Gamma \) on \( X \) is amenable if there is a net of continuous functions \( x \mapsto \mu^x_n : X \to P(\Gamma) \) such that, for all \( \gamma \in \Gamma \), we have

\[
\sup_{x \in X} \| \gamma \cdot \mu^n_x - \mu^n_{\gamma x} \|_1 \to 0.
\]

We say that \( \Gamma \) is boundary amenable if \( \Gamma \) acts amenably on some compact space. Note that amenable groups are precisely the groups that act amenably on a one-point space, whence they are boundary amenable. A prototypical example of a boundary amenable group that is not amenable is any non-abelian finitely generated free group. Boundary amenable groups are sometimes referred to as exact groups for the reduced group C*-algebra \( C_r^*(\Gamma) \) is exact (meaning that the functor \( \otimes_{\text{min}} C_r^*(\Gamma) \) is exact) if and only if \( \Gamma \) is boundary amenable; see [7]. The study of exactness of group C*-algebras originated in [6].

In this note, we show how one can construct the Stone-Cech compactification of a separable, locally compact space using C*-algebra ultrapowers. When applied to the case of a tree, this construction gives a very natural proof of the fact that a group that acts isometrically, properly, and transitively on a tree is boundary amenable. It was our initial hope that this construction could be used to settle the boundary amenability of groups where the answer was unknown (most notably Thompson’s group) but we have thus far been unsuccessful (although remain optimistic). The naïve idea behind our optimism is that groups such as Thompson’s group “almost” act isometrically on a tree and it is often the case that ultrapower constructions can turn almost phenomena into exact ones.

In Section 2, we explain the needed background on groups acting on C*-algebras as well as ultrapowers of C*-algebras. In Section 3, we explain

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Goldbring’s work was partially supported by NSF CAREER grant DMS-1349399.
our main construction in the general setting of separable, locally compact spaces. Finally, in Section 4 we use our construction to prove the boundary amenability of groups acting isometrically, properly, and transitively on trees. This is a well-known result; it was established in [3] that free group C*-algebras have the completely bounded approximation property (CBAP). Later in [4] it was shown that the CBAP implies exactness of C*-algebras.

2. Preliminaries

2.1. Boundary amenability of groups acting on C*-algebras. We will verify that certain groups act amenably on a compact space by checking that the group acts amenably on a unital abelian C*-algebra as we now explain. Suppose that \( B \) is a unital C*-algebra and that \( \Gamma \) acts on \( B \). We consider the space \( C_c(\Gamma, B) \) of finitely supported functions \( \Gamma \to B \).

\( C_c(\Gamma, B) \) is naturally a \( \ast \)-algebra with respect to the convolution product

\[
(f \ast g)({\gamma}) = \sum_{\gamma_1 \cdot \gamma_2 = \gamma} f(\gamma_1) \cdot g(\gamma_2)
\]

and involution

\[
f^*({\gamma}) = \gamma \cdot f(\gamma^{-1})^*.
\]

We also view \( C_c(\Gamma, B) \) as a pre-Hilbert B-module with B-valued inner product \( \langle f, g \rangle_B = \sum_{\gamma \in \Gamma} f(\gamma)^* g(\gamma) \) and corresponding norm \( \|f\|_B := \|\langle f, f \rangle_B\|^{-1/2} \).

Recall also that an action of \( \Gamma \) on a compact space \( X \) induces an action of \( \Gamma \) on \( C(\mathbb{C}) \) by \( \gamma \cdot f(x) := f(\gamma^{-1}x) \).

Our approach to showing that groups are boundary amenable is via the following reformulation of amenable actions (see [5, Proposition 2.2]) and is originally due to [2].

**Fact 2.1.** The action \( G \acts X \) is amenable if and only if there exists a net \( T_i \in C_c(G, C(X)) \) such that, for each \( \gamma \in \Gamma \) and each \( i \), we have:

1. \( T_i(\gamma) \geq 0 \);
2. \( \langle T_i, T_i \rangle_{C(X)} = 1 \);
3. \( \|T_i - \delta_\gamma \ast T_i\|_{C(X)} \to 0 \).

In our proofs below, we will have an action of a group \( \Gamma \) on a unital, abelian C*-algebra \( B \) and we will prove that there exist \( T_i \in C_c(\Gamma, B) \) satisfying the clauses (1)-(3) in the aforementioned fact. By Gelfand theory, \( B \) is isomorphic to \( C(X) \) for some compact space \( X \). It remains to observe that Gelfand theory respects the group action, meaning that we obtain an induced action of \( \Gamma \) on \( X \) such that the corresponding action of \( \Gamma \) on \( C(X) \) “is” the corresponding action of \( \Gamma \) on \( B \). Thus, our criterion for boundary amenability of a group is the following:

**Fact 2.2.** A group \( \Gamma \) is boundary amenable if and only if there is a unital, abelian C*-algebra \( B \) and a net \( T_i \in C_c(G, B) \) such that, for each \( \gamma \in \Gamma \) and each \( i \), we have:

1. \( T_i(\gamma) \geq 0 \);
(2) \( \langle T_i, T_j \rangle_B = 1 \);
(3) \( \|T_i - \delta_n \ast T_i\|_B \to 0 \).

2.2. Ultrapowers of C*-algebras. Recall that a nonprincipal ultrafilter \( \mathcal{U} \) on \( \mathbb{N} \) is a \( \{0, 1\} \)-valued measure on all subsets of \( \mathbb{N} \) such that finite sets get measure 0. We usually identify a nonprincipal ultrafilter with its collection of measure 1 sets, whence we write \( A \in \mathcal{U} \) to indicate that \( A \) has measure 1. If \( P(n) \) is a property of natural numbers, we say that \( P(n) \) holds \( \mathcal{U} \)-almost everywhere or that \( \mathcal{U} \)-almost all \( n \) satisfy \( P \) if the set of \( n \) for which \( P(n) \) holds belongs to \( \mathcal{U} \). If \( (r_n) \) is a bounded sequence of real numbers, then the ultralimit of \( (r_n) \) with respect to \( \mathcal{U} \), denoted \( \lim_{n \in \mathcal{U}} r_n \) or even \( \lim_{\mathcal{U}} r_n \), is the unique real number \( r \) such that, for every \( \epsilon > 0 \), we have \( |r_n - r| < \epsilon \) \( \mathcal{U} \)-almost everywhere.

Suppose that \( A \) is a unital C*-algebra and \( \mathcal{U} \) is a nonprincipal ultrafilter on \( \mathbb{N} \). We can define a seminorm \( \| \cdot \|_{\mathcal{U}} \) on \( \ell^\infty(A) \) by setting \( \|(f_n)\|_{\mathcal{U}} := \lim_{\mathcal{U}} \|f_n\| \). We set \( A_{\mathcal{U}} \) to be the quotient of \( \ell^\infty(A) \) by those elements of \( \| \cdot \|_{\mathcal{U}} \)-norm 0; we refer to \( A_{\mathcal{U}} \) as the ultrapower of \( A \) with respect to \( \mathcal{U} \). It is well known that \( A_{\mathcal{U}} \) is once again a unital C*-algebra. For \( (f_n) \in \ell^\infty(A) \), we let \( (f_n)^* \) denote its image in \( A_{\mathcal{U}} \). The canonical diagonal embedding \( \Delta : A \to A_{\mathcal{U}} \) is given by \( \Delta(a) = (a)^* \).

3. The main construction

In this section, we consider a second countable, locally compact space \( X \) with fixed basepoint \( o \in X \). It is well-known that \( X \) admits a compatible proper metric \( d \) (see [8, Theorem 2]), and we fix such a metric in the rest of this section. For \( r \in \mathbb{R}_{>0} \), we set \( B(r) \) to be the closed ball of radius \( r \) around \( o \).

We set \( A = C_o(X) \), the space of complex-valued continuous functions on \( X \) that vanish at infinity. For \( (f_n) \in \ell^\infty(A) \), we say that \( (f_n) \) is \( \mathcal{U} \)-equicontinuous on bounded sets if, for every \( r, \epsilon > 0 \), there is \( \delta > 0 \) such that, for \( \mathcal{U} \)-almost all \( n \), we have for all \( s, t \in B(r) \) with \( d(s, t) < \delta \), that \( |f_n(s) - f_n(t)| \leq \epsilon \).

Given any \( (f_n) \in \ell^\infty(A) \), set \( f_{\mathcal{U}} : X \to \mathbb{C} \) by \( f_{\mathcal{U}}(t) := \lim_{\mathcal{U}} f_n(t) \). Note that \( f_{\mathcal{U}} \) is a bounded function. The following lemma is quite routine and left to the reader.

**Lemma 3.1.** If \( (f_n) \) is \( \mathcal{U} \)-equicontinuous on bounded sets, then \( f_{\mathcal{U}} \) is uniformly continuous on bounded sets.

**Lemma 3.2.** Suppose that \( (f_n)^* = (g_n)^* \) and \( (f_n) \) is \( \mathcal{U} \)-equicontinuous on bounded sets. Then so is \( (g_n) \).

**Proof.** Fix \( r, \epsilon > 0 \). Take \( \delta > 0 \) that witnesses \( \mathcal{U} \)-equicontinuity of \( (f_n) \) on \( B(r) \) for \( \epsilon/3 \). Then for \( \mathcal{U} \)-almost all \( n \), we have, for \( s, t \in B(r) \) with \( d(s, t) < \delta \), that
\[
|g_n(s) - g_n(t)| \leq 2\|g_n - f_n\| + |f_n(s) - f_n(t)| \leq \frac{2\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.
\]
The previous lemma allows us to consider the *continuous part of* \( A^d \)

\[ A^d := \{ (f_n)^\bullet \in A^d : (f_n) \text{ is } \mathcal{U}\text{-equicontinuous on bounded sets} \}. \]

**Lemma 3.3.**

(1) \( A^d \) is a C*-subalgebra of \( A^\mathcal{U} \).

(2) \( \Delta(A) \subseteq A^d \).

**Proof.** For (1), it is clear that \( A^d \) is a *-subalgebra of \( A^\mathcal{U} \). We must show that \( A^d \) is closed in \( A^\mathcal{U} \). Towards this end, suppose that \( (f_n)^\bullet \rightarrow (h_n)^\bullet \) as \( n \rightarrow \infty \). We need to show that \( (h_n)^\bullet \in A^d \). Fix \( r, \epsilon > 0 \). Let \( \delta > 0 \) be so that, if \( s, t \in B(r) \) and \( d(s, t) < \delta \), then for \( \mathcal{U}\)-almost all \( n \), we have \( |f_n(s) - f_n(t)| < \frac{\epsilon}{3} \). Choose \( m \in \mathbb{N} \) such that \( \|(f_n)^\bullet - (h_n)^\bullet\| < \epsilon/3 \). Suppose that \( s, t \in B(r) \) are such that \( d(s, t) < \delta \). Then we have that, for \( \mathcal{U}\)-almost all \( n \), that

\[ |h_n(s) - h_n(t)| \leq 2\|(f_n)^\bullet - (h_n)^\bullet\| + |f_n(s) - (f_n^m) - (f_n^m)(t)| \leq \frac{2\epsilon}{3} + \frac{\epsilon}{3} = \epsilon. \]

(2) follows from the fact that balls \( B(r) \) in \( X \) are compact, whence elements of \( A \) are uniformly continuous on such balls. \( \square \)

We now consider

\[ I := \{ (f_n)^\bullet \in A^d : (\exists r_n \in \mathbb{R})(\lim r_n = +\infty \text{ and } f_n|B(o, r_n) \equiv 0) \}. \]

It is clear from the definition that \( I \subseteq A^d \).

In the rest of this section, we fix continuous functions \( \chi_n : X \rightarrow \mathbb{R} \) such that:

(1) \( 0 \leq \chi_n \leq 1 \);

(2) \( \chi_n(t) = 0 \) for \( t \in B(n) \);

(3) \( \chi_n(t) = 1 \) when \( d(t, o) \geq n + 1 \).

**Proposition 3.4.**

(1) \( I \) is a closed ideal in \( A^\mathcal{U} \).

(2) \( A^d/I \) is unital.

(3) \( q \circ \Delta : A \rightarrow A^d/I \) is injective, where \( q : A^d \rightarrow A^d/I \) is the canonical quotient map.

(4) \( (q \circ \Delta)(A) \) is an essential ideal in \( A^d/I \).

**Proof.** For (1), suppose \( (f_n)^\bullet, (g_n)^\bullet \in I \), \( (h_n) \in A^d \), and \( \lambda \in \mathbb{C} \). Suppose that \( f_n|B(r_n), g_n|B(s_n) \equiv 0 \), where \( \lim_{\mathcal{U}} r_n = \lim_{\mathcal{U}} s_n = 0 \). Then

\[ \lambda f_n|B(r_n), (f_n + g_n)|B(min(r_n, s_n)), (f_n \cdot h_n)|B(r_n) \equiv 0; \]

since \( \lim_{\mathcal{U}} \min(r_n, s_n) = \infty \), we have \( \lambda f_n, f_n + g_n, f_n \cdot h_n \in I \) and \( I \) is an ideal.

We now prove that \( I \) is closed. Suppose that \( ((f_n^m)^\bullet : m \in \mathbb{N}) \) is a sequence from \( I \) such that \( \lim_{n\mathcal{U}} (f_n^m)^\bullet = (g_n)^\bullet \); we must show that \( (g_n)^\bullet \in I \).

Suppose that \( f_n^m|B(r_n^m) \equiv 0 \) with \( \lim_{n\mathcal{U}} r_n^m = \infty \) for each \( m \). Fix \( k \in \mathbb{N} \)
and take $m \in \mathbb{N}$ such that $\|(f_n^m)^\star - (g_n)^\star\| < \frac{1}{k}$. For $\mathcal{U}$-almost all $n$ we have $\|f_n^m - g_n\| < \frac{1}{k}$ and $r_n^m \geq k$. Thus, if we set

$$X_k := \{n \in \mathbb{N} : n \geq k \text{ and } |g_n(t)| < \frac{1}{k} \text{ for } t \in B(k)\},$$

we have that $X_k \in \mathcal{U}$. For $n \in \mathbb{N}$, set $l(n) := \max\{k \in \mathbb{N} : n \in X_k\}$. Note that $n \in X_k$ implies that $l(n) \geq k$, whence $\lim_{n,\mathcal{U}} l(n) = \infty$. Define $h_n := g_n \cdot \chi_{l(n)-1}$. Note that $(h_n)^\star \in I$; it remains to show that $(g_n)^\star = (h_n)^\star$.

For $n \in \mathbb{N}$, we have $\|g_n - h_n\| \leq \sup_{t \in B(l(n))} |g_n(t)| \leq \frac{1}{l(n)}$, whence

$$\|(g_n)^\star - (h_n)^\star\| = \lim_{\mathcal{U}} \|g_n - h_n\| \leq \lim_{\mathcal{U}} \frac{1}{l(n)} = 0.$$

For (2), consider any sequence $(g_n) \in \ell^\infty(\mathcal{A})$ such that $g_n \equiv 1$ on $B(n)$. (For example, take $g_n := 1 - \chi_{U_n}$.) We claim that $q(g_n)^\star$ is an identity for the larger algebra $A^{cl}/I$. Indeed, consider arbitrary $q(f_n)^\star \in A^{cl}/I$. Then $f_n g_n - f_n$ vanishes on $B(n)$, whence $(f_n g_n - f_n)^\star \in I$ and $q(f_n g_n)^\star = q(f_n)^\star$.

For (3), suppose that $(q \circ \Delta)(f) = 0$. Then there is $(g_n)^\star \in I$ such that $\Delta(f) = (g_n)^\star$. Suppose that $g_n|B(r_n) \equiv 0$ with $\lim_{\mathcal{U}} r_n = \infty$. Fix $t \in X$ and $\epsilon > 0$. Then for $\mathcal{U}$-almost all $n$, we have $\|f - g_n\| < \epsilon$ and $t \in B(r_n)$, whence $|f(t)| < \epsilon$. Since $t$ and $\epsilon$ were arbitrary, we have that $f \equiv 0$.

We now prove (4). We first show that $(q \circ \Delta)(A)$ is an ideal in $A^{cl}/I$. Towards this end, fix $f \in A$ and $q((g_n)^\star) \in A^{cl}/I$; we must show that $q((f g_n)^\star) \in q(\Delta(A))$. In fact, we will show that $q((f g_n)^\star) = q(\Delta(f g))$.

Recall that

$$\|[g((f g_n)^\star) - q(\Delta(f g))]| = \inf \{\lim_{\mathcal{U}} \|f g_n - f g_\mathcal{U} - h_n\| : (h_n)^\star \in I\}.$$

Set $M := \sup_n \|g_n\|$. Fix $\epsilon > 0$. Fix $m \in \mathbb{N}$ such that $|f(t)| < \frac{\epsilon}{\|f\|}$ when $t \in B(m)^c$. Let $\delta > 0$ witness the $\mathcal{U}$-equicontinuity of $(g_n)$ on $B(m)$ with respect to $\frac{\epsilon}{\|f\|}$ and fix a finite $\delta$-net $\{t_1, \ldots, t_k\}$ for $B(m)$. Fix $U \in \mathcal{U}$ such that $\{k \in \mathbb{N} : k \geq m\} \subseteq U$ and $|g_n(t_i) - g_\mathcal{U}(t_i)| < \frac{\epsilon}{\|f\|}$ for $i = 1, \ldots, k$ and $n \in U$. For $n \in U$, define $h_n \in A$ by $h_n := (f g_n - f g_\mathcal{U})\chi_n$. (Define $h_n \in A$ for $n \notin U$ in an arbitrary fashion). It suffices to show that $\lim_{\mathcal{U}} \|f g_n - f g_\mathcal{U} - h_n\| \leq \epsilon$. Suppose $n \in U$. First consider $t \in B(m)$. Then $|f g_n(t) - f g_\mathcal{U}(t) - h_n(t)| = |f g_n(t) - f g_\mathcal{U}(t)|$. Take $i$ such that $d(t, t_i) < \delta$. Then, for $\mathcal{U}$-almost all $n$, we have

$$|g_n(t) - g_\mathcal{U}(t)| \leq |g_n(t) - g_n(t_i)| + |g_n(t_i) - g_\mathcal{U}(t_i)| + |g_\mathcal{U}(t_i) - g_\mathcal{U}(t)| \leq \frac{\epsilon}{\|f\|},$$

whence $|f g_n(t) - f g_\mathcal{U}(t)| \leq \epsilon$. Now suppose that $t \in B(m)^c \cap B(n + 1)$. Then $|f g_n(t) - f g_\mathcal{U}(t) - h_n(t)| \leq |f g_n(t) - f g_\mathcal{U}(t)| < \epsilon$ by choice of $m$. If $t \in B(n + 1)^c$, then $f g_n(t) - f g_\mathcal{U}(t) - h_n(t) = 0$. It follows that $\lim_{\mathcal{U}} \|f g_n - f g_\mathcal{U} - h_n\| \leq \epsilon$, finishing the proof that $(q \circ \Delta)(A)$ is an ideal in $A^{cl}/I$.

We next show that $(q \circ \Delta)(A)$ is an essential ideal in $A^{cl}/I$. Suppose that $q(f_n)^\star \in A^{cl}/I$ is such that $q(f_n)^\star \cdot q(a)^\star = 0$ for all $a \in A$; we must show that $q(f_n)^\star = 0$. 
Fix \( t \in X \). Fix \( a \in A \) such that \( a(t) = 1 \). Then there is \((g_n) \in I \) such that \( \lim \|f_n a - g_n\| = 0 \). For \( U \)-most \( n \), we have \( t \in B(r_n) \), where \( g_n \) vanishes on \( B(r_n) \). It thus follows that
\[
\lim_{\mathcal{U}} |f_n(t)| \leq \lim_{\mathcal{U}} \|f_n a - g_n\| = 0.
\]

Set
\[
U_k := \{ n \in \mathbb{N} : n \geq k \text{ and } |f_n(t)| \leq \frac{1}{k} \text{ for } t \in B(k) \}.
\]

We claim that \( U_k \in \mathcal{U} \). Fix \( \delta > 0 \) that witnesses \( \mathcal{U} \)-equicontinuity of \((f_n)\) on \( B(k) \) with respect to \( \frac{1}{2\delta} \). Fix a finite \( \delta \)-net \( F \) for \( B(k) \). Then for \( \mathcal{U} \)-most \( n \), \( |f_n(t)| \leq \frac{1}{2\delta} \) for \( t \in F \). Thus, given any \( s \in B(k) \) and taking \( t \in F \) such that \( d(s,t) < \delta \), we have that \( |f_n(s)| \leq |f_n(s) - f_n(t)| + |f_n(t)| \leq \frac{1}{k} \) for \( \mathcal{U} \)-most \( n \).

For \( n \in \mathbb{N} \), set \( l(n) := \max\{k \in \mathbb{N} : n \in U_k \} \). For \( n \in U_k \), we have \( l(n) \geq k \), whence \( \lim_{\mathcal{U}} l(n) = \infty \). Define \( h_n \in A \) by \( h_n = f_n \cdot \chi_{l(n)-1} \). As above, we have that \((h_n) \in I \) and \( \|f_n - g_n\| \leq \frac{1}{l(n)} \) whence \( \lim_{\mathcal{U}} \|f_n - g_n\| \leq \lim_{\mathcal{U}} \frac{1}{l(n)} = 0 \). \( \square \)

Since \( q(\Delta(A)) \) is an essential ideal in the unital C*-algebra \( A^{\text{q}} / I \), we see that \( \Sigma(A^{\text{q}} / I) \) is a compactification of \( X \), where \( \Sigma(A^{\text{q}} / I) \) denotes the Gelfand spectrum of \( A^{\text{q}} / I \). It turns out that this compactification is indeed the Stone-Cech compactification of \( X \). Recall that \( C_0(X) \) denotes the unital C*-algebra of bounded, continuous, complex-valued functions on \( X \) and is naturally isomorphic to \( C(\beta X) \), where \( \beta X \) denotes the Stone-Cech compactification of \( X \).

**Proposition 3.5.** There is an isomorphism \( \Phi : A^{\text{q}} / I \to C_0(X) \) such that \( \Phi(q(\Delta(a))) = a \) for all \( a \in A \).

**Proof.** Define \( \Phi : A^{\text{q}} \to C_0(X) \) by \( \Phi((f_n)\) := \( f_\mathcal{U} \). It is clear that \( \Phi \) is a \(*\)-morphism. We next observe that \( \Phi \) is onto. Indeed, given \( f \in C_0(X) \) and \( n > 0 \), define \( f_n \in C_0(T) \) by \( f_n = (1 - \chi_n)f \). Since \( f \) is bounded, we have that \( (f_n) \in \ell^\infty(A) \). Since \( X \) is proper, \( f \) is uniformly continuous on bounded sets, whence \( (f_n) \) is \( \mathcal{U} \)-equicontinuous on bounded sets, that is, \((f_n) \in A^{\text{q}} \).

It is clear that \( \Phi((f_n)) = f \).

Now suppose that \((f_n) \in I \). Then by the definition of \( I \), we have that \( \Phi((f_n)) = 0 \), so \( \Phi \) induces a surjection \( \Phi : A^{\text{q}} / I \to C_0(X) \). Suppose now that \( \Phi((f_n)) = 0 \). For each \( n > 0 \), define a function \( g_n \in A \) by \( g_n = f_n \cdot \chi_{n-1} \). It is clear that \((g_n) \in I \). Since \( \lim_{\mathcal{U}} f_n(t) = 0 \) for all \( t \in X \) and \( \|f_n - g_n\| \leq \max_{t \in B(0,n)} |f_n(t)| \), it follows that \((f_n) = (g_n) \), whence \((f_n) \in I \), thus proving that \( \Phi : A^{\text{q}} / I \to C_0(X) \) is an isomorphism.

Finally, it is clear from the definition of \( \Phi \) that \( \Phi(q(\Delta(a))) = a \) for all \( a \in A \). \( \square \)

From now on, we set \( B := (A^{\text{q}} / I)/(q \circ \Delta)(A) \), a unital C*-algebra, and let \( r : A^{\text{q}} \to B \) denote the composition of \( q \) with the quotient map.
Lemma 3.6. Suppose that $A^d / I \to B$. Note that by the previous proposition, $B \cong C(X^*)$, where $X^*$ denotes the Stone-Cech remainder $\beta X \setminus X$ of $X$.

We now introduce a group action into the picture:

**Lemma 3.6.** Suppose that $\Gamma$ acts isometrically on $X$.

1. The induced action of $\Gamma$ on $A$ further induces an action of $\Gamma$ on $A^d$ by $\gamma \cdot (f_n)^* := (\gamma \cdot f_n)^*$.

2. Both $A^d$ and $I$ are invariant under the action of $\Gamma$ on $A^d$ from (1).

**Proof.** For (1), we need to verify that, for $(f_n), (g_n) \in \ell^\infty(A)$, if $\lim_{d} \|f_n - g_n\| = 0$, then $\lim_{d} \|\gamma \cdot f_n - \gamma \cdot g_n\| = 0$. However, this follows from the easy check that $\|\gamma \cdot f_n - \gamma \cdot g_n\| = \|f_n - g_n\|$ for each $n$.

We now prove (2). The fact that $A^d$ is invariant under the action of $\Gamma$ follows from the fact that $\Gamma$ acts by isometries and thus takes bounded sets to bounded sets. We now prove that $I$ is invariant under the action of $\Gamma$. Consider $\gamma \in \Gamma$ and $(f_n)^* \in I$; we must show that $(\gamma \cdot f_n)^* \in I$. Suppose that $f_n B(o, r_n) \equiv 0$ where $\lim_{d} r_n = \infty$. Set $k := d(\gamma^{-1} \cdot o, o)$. Then for $r_n > k$, we have that $(\gamma \cdot f_n) B(o, r_n - k) \equiv 0$: if $t \in B(o, r_n - k)$, then

$$d(\gamma^{-1} t, o) \leq d(\gamma^{-1} t, \gamma^{-1} o) + d(\gamma^{-1} o, o) = d(t, o) + k \leq r_n,$$

whence $f_n (\gamma^{-1} t) = 0$. Since $r_n > k$ for $d$-almost all $n$ and $\lim_{d} r_n - k = \infty$, it follows that $(\gamma \cdot f_n)^* \in I$.

By the previous lemma, we have an induced action of $\Gamma$ on $A^d / I$ by setting $\gamma \cdot q(f_n)^* := q(\gamma \cdot f_n)^*$, whence we also get an action of $\Gamma$ on $B$ by setting $\gamma \cdot r(f_n)^* := r(\gamma \cdot f_n)^*$.

4. **Groups acting properly and isometrically on a tree**

In this section, our locally compact space is simply a tree $T$ given the usual path metric, namely $d(x, y)$ is the length of the shortest path connecting $x$ and $y$. In this case, $A^d = A^d$. We further suppose that $\Gamma \curvearrowright T$ properly, isometrically and transitively. (Recall that the action is proper if the map $(g, t) \mapsto (gt, t) : G \times T \to T \times T$ is proper, meaning that inverse images of compact sets are compact.) In this case, $\text{Stab}(o)$ is finite, say of cardinality $m$. For a point $t \in T$, let $x_{[o,t]}$ denote the geodesic segment connecting $o$ and $t$.

**Theorem 4.1.** If $\Gamma$ acts properly, transitively, and isometrically on a simplicial tree $T$, then $\Gamma$ is exact.

**Proof.** For $t \in T$ and $i \in \mathbb{N}$, set

$$X(i, t) := \{ \gamma \in \Gamma : \gamma \cdot o \in B(i) \text{ and } \gamma \cdot o \in x_{[o,t]} \}$$

and $x(i, t) = |X(i, t)|^{-1/2}$. Note that $x(i, t) = (m \cdot \min(i, d(o, t)))^{-1/2}$. Define $T_i^{(n)} : \Gamma \to A$ by

$$T_i^{(n)}(\gamma)(t) = \begin{cases} x(i, t) & \text{if } t \in B(2n) \text{ and } \gamma \in X(i, t); \\ 0 & \text{otherwise.} \end{cases}$$
A ∑ ∑ functions satisfy the criteria of Fact 2.2, whence the action of $T_i$ on $X^*$ is amenable.

Certainly each $(T_i^{(n)}(\gamma))^*$ is a positive element of $A^{\mathcal{U}}$; since $r$ is a $C^*$-algebra homomorphism, we have that each $T_i(\gamma) \geq 0$ in $B$.

We now verify that $(T_i, T_i)_B = 1_B$; in other words, we must show that $\sum_{\gamma \in \Gamma} T_i(\gamma)^2 = 1_B$. First observe that there is a finite $\Gamma_i \subseteq \Gamma$ such that $\sum_{\gamma \in \Gamma} T_i^{(n)}(\gamma) = \sum_{\gamma \in \Gamma_i} T_i^{(n)}(\gamma) = \chi_{B(2n)}$. Since $(\chi_{B(2n)})^* + I$ is the unit of $A^{\mathcal{U}}/I$, it follows that $r((\chi_{B(2n)})^*)$ is the identity of $B$. Now compute:

$$
\sum_{\gamma \in \Gamma} T_i(\gamma)^2 = \sum_{\gamma \in \Gamma_i} T_i(\gamma)^2 \\
= \sum_{\gamma \in \Gamma_i} (r((T_i^{(n)}(\gamma))^*))^2 \\
= \sum_{\gamma \in \Gamma_i} r((T_i^{(n)}(\gamma))^* \cdot (T_i^{(n)}(\gamma))^*) \\
= r(\sum_{\gamma \in \Gamma_i} ((T_i^{(n)}(\gamma))^2)^*) \\
= r(\sum_{\gamma \in \Gamma_i} T_i^{(n)}(\gamma)^2)^* \\
= r((\chi_{B(2n)})) \\
= 1_B.
$$

It remains to prove that, for each $\gamma_1 \in \Gamma$, we have $\lim_{i \to \infty} \|T_i - \delta_{\gamma_1} \cdot T_i\|_B = 0$. It is straightforward to compute that $\delta_{\gamma_1} \cdot T_i = \gamma_1 \cdot T_i(\gamma_1^{-1} \gamma)$. It follows that $\|T_i - \delta_{\gamma_1} \cdot T_i\|_B^2$ is equal to

$$
\| \sum_{\gamma \in \Gamma} (T_i(\gamma)^2 + (\gamma_1 \cdot T_i(\gamma_1^{-1} \gamma))^2 - 2T_i(\gamma) \gamma_1 \cdot T_i(\gamma_1^{-1} \gamma)) \|.
$$

Now $\gamma_1 T_i^{(n)}(\gamma_1^{-1} \gamma)(t) = T_i^{(n)}(\gamma_1^{-1} \gamma)(\gamma_1^{-1} \gamma(t))$, which is only nonzero if:

1. $\gamma_1^{-1} t \in B(2n)$;
2. $\gamma_1^{-1} \gamma \cdot o \in B(i)$;
3. $\gamma_1^{-1} \gamma \cdot o \in x_{[o, \gamma_1^{-1} t]}$.

Also notice that $\sum_{\gamma \in \Gamma} (\gamma_1 \cdot T_i^{(n)}(\gamma_1^{-1} \gamma))^2 = \chi_{\gamma_1 \cdot B(2n)}$, so (i) equals

$$
\| r((\chi_{B(2n)})^* + r((\chi_{B(2n)}))^* - 2 \sum_{\gamma \in \Gamma} (T_i^{(n)}(\gamma) \gamma_1 \cdot T_i^{(n)}(\gamma_1^{-1} \gamma))^*) \|_B.
$$
which in turn equals
\[ \inf \left\{ \lim_{n \to \infty} \| \chi_B(2n) + \chi_{\gamma_1 \cdot B(2n)} - 2 \sum_{\gamma \in \Gamma} T_i^{(n)}(\gamma) \gamma_1 \cdot T_i^{(n)}(\gamma^{-1} \gamma) \right\}, \]  
where \((g_n)^*\) ranges over \(I\) and \(a\) ranges over \(A\).

Set
\[ a(t) = \left( \chi_B(2n) + \chi_{\gamma_1 \cdot B(2n)} - 2 \sum_{\gamma \in \Gamma} T_i^{(n)}(\gamma) \gamma_1 \cdot T_i^{(n)}(\gamma^{-1} \gamma) \right)^* \cdot \chi_B(i \cup \gamma_1 \cdot B(i)). \]

\(a(t)\) is certainly an element of \(A\) since \(B(i) \cup \gamma_1 \cdot B(i)\) is compact. Set \(g_n(t) = \chi_B(2n) \Delta \gamma_1 \cdot B(2n)\). Finally set
\[ O(i, n) = (B(2n) \cap \gamma_1 \cdot B(2n)) \setminus (B(i) \cup \gamma_1 \cdot B(i)). \]

Then \(a \in A\), \((g_n)^* \in I\) (as \(g_n\left| B(n) \equiv 0\right.)\) and the value in (††) is bounded by
\[ \lim_{n \to \infty} \sup_{t \in O(i, n)} |2 - 2 \sum_{\gamma \in \Gamma} T_i^{(n)}(\gamma) \gamma_1 \cdot T_i^{(n)}(\gamma^{-1} \gamma)(t)|. \]  

Let \(Z(i, t)\) be the set
\[ \{ \gamma \in \Gamma : \gamma \cdot o \in B(i), \gamma \cdot o \in x_{[\gamma, i]}, \gamma^{-1} \gamma \cdot o \in B(i), \gamma^{-1} \gamma \cdot o \in x_{[\gamma, i]} \}. \]

Set \(k := d(\gamma_1 \cdot o, o)\). For \(n\) sufficiently large and for \(t \in O(i, n)\), we have
\[ |Z(i, t)| \geq m(i - k + 1) \]  
and
\[ 2 \sum_{\gamma \in \Gamma} (T_i^{(n)}(\gamma) \gamma_1 \cdot T_i^{(n)})(t) \geq \frac{2m(i - k + 1)}{m_i}, \]
whence the quantity appearing in († † †) is bounded above by
\[ 2 - 2 \cdot \frac{2m(i - k + 1)}{m_i}, \]
goese to 0 as \(i \to \infty\) as desired.

\[ \square \]

REFERENCES

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