THE URYSOHN SPHERE IS PSEUDOFINITE

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ABSTRACT. We show that the Urysohn sphere is pseudofinite. As a consequence, we derive an approximate 0-1 law for finite metric spaces of diameter at most 1.

1. INTRODUCTION

The Urysohn sphere $\mathfrak{U}$ is the unique, up to isometry, Polish metric space of diameter bounded by 1 that is both universal (every Polish metric space of diameter bounded by 1 embeds in $\mathfrak{U}$) and ultrahomogeneous (every isometry between finite subspaces of $\mathfrak{U}$ extends to a self-isometry of $\mathfrak{U}$). The (full) Urysohn space (introduced in [9]) is defined exactly as in $\mathfrak{U}$ without the bounded diameter requirement; the Urysohn sphere gets it name as it is isometric to any sphere of radius 1 in the Urysohn space.

The (continuous) model theory of the Urysohn sphere is relatively well-understood: it is ω-categorical, admits quantifier-elimination, is the model-completion of the empty theory of metric spaces, and is real rosy but not simple (see [4], [5, Lecture 4], [6], and [10]). Except for the neo-stability aspects, the model theory of $\mathfrak{U}$ resembles the classical model theory of an infinite set (the model-completion of the empty theory) or the random graph (the model-completion of a single, symmetric binary relation). In either case, the corresponding theory is *pseudofinite*. It has not yet been established that the Urysohn sphere is pseudofinite. It is the purpose of this note to remedy this.

In the continuous setting, a structure M is pseudofinite if a sentence true in M is approximately true in a finite structure, that is, given a sentence σ such that $\sigma^M = 0$ and $\epsilon > 0$, there is a finite structure A such that $\sigma^A < \epsilon$. As in classical logic, this is equivalent to saying that M is elementarily equivalent to an ultraproduct of finite structures. There are some subtleties to pseudofiniteness of continuous structures that do not appear in the classical setting as discussed in the article [2]; thankfully, these subtleties play no role here.

The plan of the proof is actually quite simple: mimic the proof of the pseudofiniteness of the random graph as given, for example, in [8, Section 2.4], where one establishes the fact that the extension axioms axiomatizing the random graph hold in almost every sufficiently large finite graph. Here, "almost every" is with respect to the counting measure on the finite set of finite graphs of a given size. The key here is to find the appropriate measure on the space of finite pseudometric

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spaces of a given size. This is done by using an appropriate measure on powers of the Urysohn sphere itself, an idea that was motivated when reading Vershik's influential article [11].

We obtain as a corollary to our result an approximate 0-1 law for finite metric spaces, which should be of independent combinatorial interest.

The results obtained here actually hold in greater generality: if M is a metric structure in a relational language whose theory is $\forall\exists$-axiomatizable and whose underlying metric space is perfect, then M is pseudofinite. We will present the proof in this more general setting in the next section.

In the last section, we obtain a version of our general theorem for not necessarily relational languages by showing that structures satisfying the same assumptions are almost pseudofinite in a certain precise sense.

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2. The main result

Throughout this section, we assume that L is a relational *metric language.*

Definition 2.1. A *kind* $\forall\exists$-sentence is one of the form

$$\sup_x \inf_y \min\left(\prod_{i\neq j} d(x_i,x_j),\varphi(x,y)\right),$$

where φ is a non-negative quantifier-free formula.[1]

The proof of the following lemma is left to the reader.

Lemma 2.2. *If T is $\forall\exists$-axiomatizable, then it has a kind $\forall\exists$-axiomatization, that is, an axiomatization consisting of kind $\forall\exists$-sentences.*

We call a Borel probability measure on a Hausdorff space X *reasonable* if it is atomless and strictly positive, that is, every nonempty open subset of X has positive measure. It is a fact that a Polish space admits a reasonable measure if and only if it is perfect. This fact must be known but since the only proof of this fact that we could find was a Math Stack Exchange post, we discuss the proof in an appendix at the end of this paper.

Suppose that μ is a reasonable measure on X. For any $m \geq 1$, we let μ^m denote the product measure on X^m. Since μ is atomless, we have that

$$\mu^2(\{(x,x) \ : \ x \in X\}) = 0,$$

whence $\mu^m(X^m_{\neq}) = 1$, where $X^m_{\neq} := \{(x_1,\ldots,x_m) \in X^m \ : \ x_i \neq x_j \text{ for } i \neq j\}$.

We call an L-structure perfect if its underlying metric space is perfect. Here is the main result of this paper:

[1] In this paper, when we say a sentence is $\forall\exists$, we always assume that its matrix is a non-negative quantifier-free formula.

Theorem 2.3. *Suppose that L is a relational language, M is a separable, perfect L-structure, and σ is a kind $\forall\exists$-sentence such that $\sigma^M = 0$. Then for any $\epsilon > 0$ and any reasonable measure μ on M, we have*

$$\lim_{m\to\infty} \mu^m\{(a_1, \ldots, a_m) \in M^m_{\neq} \ : \sigma^{\bar{a}} < \epsilon\} = 1$$

where $\bar{a}$ represents the substructure of M formed by $a_1, \ldots, a_m$.

Proof. Suppose that σ and $\varphi(x, y)$ are as in Definition 2.1 with $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_k)$. Fix $m \geq n + k$. Write $m = n + qk + r$ with $r < k$. For $i = 1, \ldots, q$, set

$$A_i := \{(a_1, \ldots, a_m) \in M^m_{\neq} \ : \ \varphi^M(a_1, \ldots, a_n, a_{n+(i-1)k+1}, \ldots, a_{n+ik}) < \epsilon\}.$$

Since A_i is a nonempty open subset of M^m (as $\sigma^M = 0$ and M is perfect) and μ is reasonable, we have that $\mu^m(A_i) > 0$. By S_m-invariance of μ^m, it follows that $\mu^m(A_i) = \mu^m(A_j)$ for all $i, j = 1, \ldots, q$; call this common value p. It is important to note that p is independent of m. Let $B := \bigcap_{i=1}^q (M^m_{\neq} \setminus A_i)$. Since the A_i's are independent, we have that

$$\mu^m(B) = (1 - p)^q = (1 - p)^{\lfloor \frac{m-n}{k} \rfloor}.$$

For $G = \{i_1, \ldots, i_n\} \in [m]^n$, let B_G be defined exactly as B except replacing $\{1, \ldots, n\}$ with $\{i_1, \ldots, i_n\}$. Again, by S_m-invariance of μ^m, we have that $\mu^m(B_G) = (1 - p)^{\lfloor \frac{m-n}{k} \rfloor}$. The set that we are really interested in is

$$C := \{(a_1, \ldots, a_m) \in M^m_{\neq} \ : \sigma^{\bar{a}} \geq \epsilon\}.$$

It follows that

$$\mu^m(C) \leq \mu^m \left(\bigcup_{G \in [m]^n} B_G \right) \leq \binom{m}{n} (1 - p)^{\lfloor \frac{m-n}{k} \rfloor}.$$

The right-hand side goes to 0 as $m \to \infty$, yielding the desired result. □

The following easy fact is [2, Lemma 2.23].

Fact 2.4. *Suppose that $\{\sigma = 0 \ : \ \sigma \in \Gamma\} \models \mathrm{Th}(M)$ for some collection Γ of L-sentences and that, for every $\sigma_1, \ldots, \sigma_n \in \Gamma$ and every $\epsilon > 0$, there is a finite L-structure A such that $A \models \max(\sigma_1, \ldots, \sigma_n) \leq \epsilon$. Then M is pseudofinite.*

Corollary 2.5. *Suppose that L is a relational language, M is a perfect L-structure, and $\mathrm{Th}(M)$ is $\forall\exists$-axiomatizable. Then M is pseudofinite.*

Proof. Let N be a separable elementary substructure of M. Note then that N is also perfect. By Lemma 2.2, Theorem 2.3, and Fact 2.4, it follows that N, and hence M, is pseudofinite. □

Corollary 2.6. $\mathfrak{U}$ *is pseudofinite.*

In [6], it is noted that there are certain expansions of $\mathfrak{U}$ by finitely many predicate symbols satisfying certain Lipshitz moduli of uniform continuity that have quantifier-elimination and are hence $\forall\exists$-axiomatizable. It follows that these expansions are also pseudofinite.

Let $L\text{-}\mathrm{Str}_m$ be the set of L-structures with universe $\{1,\ldots,m\}$. Let $L\text{-}\mathrm{Str}_m^M$ be the subset of $L\text{-}\mathrm{Str}_m$ consisting of those substructures that embed into M. In particular, if M is universal for finite L-structures, then $L\text{-}\mathrm{Str}_m = L\text{-}\mathrm{Str}_m^M$ for all m. There is a natural surjective map $\Phi_m : M^m_{\neq} \to L\text{-}\mathrm{Str}_m^M$. Given a reasonable measure μ on M, we let $\Phi_m^*\mu^m$ denote the pushforward of the measure μ^m via Φ_m.

A straightforward compactness argument yields the following:

Corollary 2.7 (Approximate 0-1 law)**.** *Suppose that L is a relational language, M is a perfect L-structure, and $\mathrm{Th}(M)$ is $\forall\exists$-axiomatizable. Fix an L-sentence σ and let $r := \sigma^M$. Then for any reasonable measure μ on M and any $\epsilon > 0$, we have*

$$\lim_{m\to\infty} \mu^m\{(a_1,\ldots,a_m)\in M^m_{\neq} \ : |\sigma - r|^{\bar{a}} < \epsilon\} = 1,$$

whence

$$\lim_{m\to\infty} \Phi_m^*\mu^m\{A \in L\text{-}\mathrm{Str}_m^M \ : |\sigma - r|^{A} < \epsilon\} = 1.$$

In particular, for any sentence σ in the language of pure metric spaces, setting $r := \sigma^{\mathfrak{U}}$, we have that, for sufficiently large m, with high probability, a metric space X of size m satisfies σ^X is approximately equal to r; here, the probability is calculated with respect to the pushforward measure obtained from any reasonable measure on $\mathfrak{U}$.

3. A version for arbitrary languages

In this section, L denotes an arbitrary metric language. To make matters simpler, we assume that L is countable. Let (r_k) be an enumeration of $\mathbb{Q}\cap[0,1]$ and let (f_k) be an enumeration of the L-terms. For $m\in\mathbb{N}$, we define a language $L(m)$ which only differs from L by changing the modulus of uniform continuity for $f_1,\ldots,f_m$ by declaring $\Delta_{f_j,L(m)}(r_k+\frac{1}{m}) := \Delta_{f_j,L}(r_k)$ for $j,k=1,\ldots,m$. The following lemma is immediate:

Lemma 3.1. *Suppose that $p:\mathbb{N}\to\mathbb{N}$ is such that $\lim_{m\to\infty} p(m)=\infty$. Further suppose that, for each m, A_m is an $L(p(m))$-structure. Then for any nonprincipal ultrafilter $\mathcal{U}$ on $\mathbb{N}$, there is a well-defined ultraproduct $\prod_{\mathcal{U}} A_m$ which is, moreover, an actual L-structure.*

Here is the general version of our theorem:

Theorem 3.2. *Suppose that M is a perfect L-structure such that $\mathrm{Th}(M)$ is $\forall\exists$-axiomatizable. Then there is $p:\mathbb{N}\to\mathbb{N}$ with $\lim_{m\to\infty} p(m)=\infty$ and finite $L(p(m))$-structures A_m such that, for any nonprincipal ultrafilter $\mathcal{U}$ on $\mathbb{N}$, we have $M\equiv\prod_{\mathcal{U}} A_m$.*

Proof. As before, we may assume that M is separable. We may assume that $\mathrm{Th}(M)$ is axiomatized by sentences σ of the form

$$\sup_x \inf_y g(P_1(f_1(x,y)),\ldots,P_l(f_l(x,y))) = 0,$$

where g is a continuous function, $P_1,\ldots,P_l$ are predicate symbols, and $f_1,\ldots,f_l$ are terms. Let (σ_k) enumerate such an axiomatization.

We now pass to a relational language L_r obtained from L by replacing every function symbol f with a predicate symbol Q_f. We view M as an L_r-structure M_r by interpreting $Q_f^M(a;b) := d(f^M(a),b)$.

For σ as above, we let σ_r be the L_r-sentence

$$\sup_x \inf_y \inf_z \max\left(\max_i Q_{f_i}(x,y;z_i), g(P_1(z_1),\ldots,P_l(z_l))\right).$$

Note that $\sigma_r^{M_r}=0$.

For any j,k, let α_j, $\beta_{j,k}$, and γ_j be the following L_r-sentences:

$$\sup_x \inf_y Q_{f_j}(x;y)$$

$$\sup_{x,x',y,y'} \min\left(\Delta_{f_j}(r_k) \dot{-} d(x,x'), d(y,y') \dot{-} (r_k + Q_{f_j}(x;y) + Q_{f_j}(x';y'))\right)$$

$$\sup_{x,y,y'} d(y,y') \dot{-} (Q_{f_j}(x;y) + Q_{f_j}(x;y')).$$

Note that $\alpha_j^{M_r}=\beta_{j,k}^{M_r}=\gamma_j^{M_r}=0$ for all j,k and that, in fact, these sentences axiomatize $\mathrm{Th}(M_r)$.

Now fix m and let $p(m)\geq m$ be such that all function symbols occurring in $\sigma_1,\ldots,\sigma_m$ are among $f_1,\ldots,f_{p(m)}$. Take $q\geq \max(p(m),3m)$ such that $\frac{3}{q} < \min_i \Delta_{\sigma_i}(\frac{1}{2m})$. By Corollary 2.7, there is a finite L_r-structure B_q such that

$$B_q \models \max\left(\max_{1\leq k\leq m}(\sigma_k)_r, \max_{1\leq j,k\leq q}\max(\alpha_j,\beta_{j,k},\gamma_j)\right) < \frac{1}{q}.$$

Turn B_q into a structure A_m by defining, for $1\leq j\leq p(m)$, $f_j^{A_m}(a) =$ some b such that $Q_{f_j}^{A_q}(a,b) < \frac{1}{q}$. One can interpret the other function symbols to be constant. Note that A_m is an $L(p(m))$-structure. Indeed, if $1\leq j\leq p(m)$ and $a,a'\in A_m$ are such that $d(a,a')<\Delta_{f_j}(r_k)$, then since $\beta_{j,k}^{B_q}<\frac{1}{q}$, we have

$$d(f_j^{A_m}(a), f_j^{A_m}(a')) \leq r_k + \frac{1}{q}+\frac{1}{q}+\frac{1}{q} \leq r_k + \frac{1}{m}.$$

We now show that $\sigma_k^{A_m} < \frac{1}{m}$ for $k=1,\ldots,m$, finishing the proof. Take $a\in A_m$. Take $b,c\in A_m$ such that $Q_{f_j}(a,b;c)<\frac{1}{q}$ for each $j=1,\ldots,p(m)$ and $g(P_1(c_1),\ldots,P_l(c_l))<\frac{1}{q}$. Suppose that $f_j^{A_m}(a,b)=c_j'$. Then

$$d(c_j,c_j') \leq Q_{f_j}(a,b;c) + Q_{f_j}(a,b;c') + \frac{1}{q} < \frac{3}{q},$$

whence, by choice of q, we have $g(P_1(c_1'),\ldots,P_l(c_l')) < \frac{1}{q}+\frac{1}{2m} < \frac{1}{m}$, as desired. □

Several examples of $\forall\exists$-axiomatizable perfect structures were shown to be pseudofinite in [2]: infinite-dimensional Hilbert spaces, atomless probability algebras, and randomizations of classical pseudofinite structures to name a few. However, there are some $\forall\exists$-axiomatizble perfect structures from analysis not yet known to be pseudofinite that are of particular interest, namely the Gurarij Banach space $\mathbb{G}$, whose model theory was been studied in [1]. One can view Theorem 3.2 as saying that $\mathbb{G}$ is *approximately pseudofinite* in a very precise sense. We leave it as an open question as to whether or not $\mathbb{G}$ is actually pseudocompact (that is, elementarily equivalent to an ultraproduct of compact structures in the language of Banach spaces), or even pseudo-finite-dimensional (that is, elementarily equivalent to an ultraproduct of finite-dimensional Banach spaces in the language of Banach spaces) and that an appropriate 0-1 law holds for it as well. In fact, such a result may be true more generally under an additional assumption of the form that finitely generated substructures are compact in some uniform manner.

APPENDIX A. REASONABLE MEASURES ON POLISH SPACES

Clearly an isolated point in a Polish space prevents the existence of a reasonable measure on that space. It turns out that it is the only obstruction:

Theorem A.1. *Suppose that X is a nonempty perfect Polish space. Then X admits a reasonable measure.*

As mentioned above, the only reference that we could find for the previous theorem is [3]. For the sake of the reader, we provide the proof in this appendix.

The following lemma is easy and left to the reader.

Lemma A.2. *Suppose that X and Y are Hausdorff spaces and $f : X \to Y$ is continuous. Let μ be a Borel probability measure on X and let $f_*\mu$ be the pushforward measure on Y.*

(1) *If μ is atomless and f is injective, then $f_*\mu$ is atomless.*
(2) *If μ is strictly positive and $f(X)$ is dense in Y, then $f_*\mu$ is strictly positive.*

In particular, if Y admits a dense subspace X that admits a reasonable measure, then Y also admits a reasonable measure.

Let $\mathcal{N}$ denote Baire space ω^ω.

Lemma A.3. *$\mathcal{N}$ admits a reasonable measure.*

Proof. Since $\mathcal{N}$ is homeomorphic to $\mathbb{R} \setminus \mathbb{Q}$ and any reasonable measure on $\mathbb{R}$ restricts to a reasonable measure on $\mathbb{R}\setminus\mathbb{Q}$, it suffices to show that $\mathbb{R}$ has a reasonable measure. But this follows from the fact that $\mathbb{R}$ and $(0, 1)$ are homemorphic. $\square$

Thus, to prove Theorem A.1, it suffices to prove the following well-known result whose proof we couldn't seem to find in the literature.

Theorem A.4. *Suppose that X is a nonempty perfect Polish space. Then there is a dense G_δ subspace Y of X such that Y is homeomorphic to $\mathcal{N}$.*

Proof. It is well-known that there is a dense G_δ subspace X_1 of X such that X_1 is 0-dimensional. (Indeed, let (U_n) be a countable base for X and let $X_1; = X \setminus \bigcup_n \partial U_n$. Then X_1 is G_δ and co-meager, whence dense by the Baire Category Theorem.) Note that X_1 is also perfect, whence there is a dense, co-dense G_δ subspace Y of X_1. (For example, let C be a countable dense subspace of X_1 and let $Y = X_1 \setminus C$. Y is co-meager as X_1 is perfect, whence dense by the Baire

Category Theorem.) We now recall the Alexandrov-Urysohn characterization of $\mathcal{N}$ as the unique nonempty, 0-dimensional Polish space such that every compact subset has empty interior. Y is clearly nonempty, 0-dimensional, and Polish; the latter property is left to the reader as an easy exercise. (This is also Exercise 7.13 in [7].) $\square$

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