The theory of tracial von Neumann algebras does not have a model companion

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Hilbert spaces

Definition

A *Hilbert space* H is a complex inner product space such that the induced metric is complete.

Examples

- $\mathbb{C}^n$, where $\langle \vec{x}, \vec{y} \rangle := \sum_{i=1}^n x_i \overline{y_i}$.
- $\ell^2 = \{(x_n) \in \mathbb{C}^{\mathbb{N}} \ : \ \sum_n |x_n|^2 < \infty\}$ where $\langle (x_n), (y_n) \rangle := \sum_{n=1}^\infty x_n \overline{y_n}$.
- $L^2(X, \mu) := \{f : X \to \mathbb{C} \ : \ f$ is measurable and $\int_X |f|^2 d\mu < \infty\}$, where $\langle f, g \rangle := \int_X f \bar{g} d\mu$ (for (X, μ) a finite measure space).

Bounded operators

Definition

If X, Y are normed spaces (over $\mathbb{C}$), then a linear transformation $T : X \to Y$ is *bounded* if the image of the unit ball of X under T is bounded.

- If T is bounded, then we set $\|T\| := \sup\{\|Tx\| \ : \ \|x\| = 1\}$, called the *operator norm of* T, and observe that $\|T\|$ is the least upper bound for the image of the unit ball of X under T.
- The set of bounded linear operators $\mathcal{B}(X, Y)$ from X to Y forms a normed space with the above notion of $\|T\|$.
- If $X = Y$, we write $\mathcal{B}(X)$ instead of $\mathcal{B}(X, X)$.
- T is bounded if and only if T is (uniformly) continuous.

Examples of bounded operators

Examples

- Every linear transformation between finite-dimensional normed spaces is bounded.
- Fix $(d_n) \in \mathbb{C}^n$ and consider $T : \mathbb{C}^n \to \mathbb{C}^n$ given by $T((x_n)) := (d_n x_n)$. Then $T \in \mathcal{B}(\ell^2)$ if and only if (d_n) is bounded.
- If $f \in L^\infty(X, \mu)$, then $m_f : L^2(X, \mu) \to L^2(X, \mu)$ defined by $m_f(g) := fg$ is a bounded linear transformation.

Topologies on Operator spaces

Suppose that H is a Hilbert space. We consider the following topologies on $\mathcal{B}(H)$:

- The *operator norm topology*.
- The *strong topology*: a subbasis of open sets is given by

$$\{T \in B(H) \ : \ \|T(v) - T_0(v)\| < \epsilon\},$$

  where $T_0 \in B(H)$, $v \in H$ and $\epsilon > 0$.
- The *weak topology*: a subbasis of open sets is given by

$$\{T \in B(H) \ : \ |\langle T(v) - T_0(v), w\rangle| < \epsilon\},$$

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Notice norm convergence $\Rightarrow$ strong convergence $\Rightarrow$ weak convergence.

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Why other topologies?

Lemma

- *The map $T \to T^*$ is weakly continuous but not strongly continuous.*
- *The map $(S, T) \to ST$ is separately strongly continuous but not jointly strongly continuous.*
- *If $A \subseteq \mathcal{B}(H)$ is a $*$-subalgebra, then so is the weak closure of A.*

von Neumann's bicommutant theorem

Given a subset S of $\mathcal{B}(H)$, we let
$S' := \{T \in \mathcal{B}(H) \ : \ TU = UT$ for all $U \in S\}$. Notice that S' is always a subalgebra of $\mathcal{B}(H)$ and $S \subseteq S''$ is always true.

Theorem (von Neumann)

Suppose that $A \subseteq \mathcal{B}(H)$ is a unital $*$-subalgebra. The following are equivalent:

- *$A = S'$ for some $S \subseteq \mathcal{B}(H)$;*
- *$A = A''$;*
- *A is closed with respect to the weak topology;*
- *A is closed with respect to the strong topology.*

A unital $*$-subalgebra of $\mathcal{B}(H)$ satisfying any of the equivalent conditions of the above theorem is called a *von Neumann algebra*.

Examples of vNas

Example

$\mathcal{B}(H)$ is a von Neumann algebra.

Example

Suppose that (X, μ) is a finite measure space. Then $L^\infty(X, \mu)$ acts on the Hilbert space $L^2(X, \mu)$ by left multiplication, yielding an embedding

$$L^\infty(X, \mu) \hookrightarrow \mathcal{B}(L^2(X, \mu)),$$

the image of which is a von Neumann algebra. (Actually, all abelian von Neumann algebras are isomorphic to some $L^\infty(X, \mu)$, whence von Neumann algebra theory is sometimes dubbed "noncommutative measure theory.")

Group von Neumann algebras

Example

Suppose that G is a locally compact group and $\alpha : G \to \mathcal{B}(H)$ is a unitary group representation. Then the *group von Neumann algebra of* α is $\alpha(G)''$. (Understanding $\alpha(G)''$ is tantamount to understanding the invariant subspaces of α.)

In the important special case that $\alpha : G \to \mathcal{B}(L^2(G))$ (where G is equipped with its haar measure) is given by left translations

$$\alpha(g)(f)(x) := f(g^{-1}x),$$

we call $\alpha(G)''$ the *group von Neumann algebra of* G and denote it by $L(G)$.

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$\mathcal{R}$

Example

Let M_2 denote the set of 2×2 matrices with entries from $\mathbb{C}$. We consider the canonical embeddings

$$M_2 \hookrightarrow M_2 \otimes M_2 \hookrightarrow M_2 \otimes M_2 \otimes M_2 \hookrightarrow \cdots$$

and set $M := \bigcup_{n=1}^{\infty} \bigotimes_n M_2$.

- The normalized traces on $\bigotimes_n M_2$ form a cohesive family of traces, yielding a trace $\mathrm{tr} : M \to \mathbb{C}$.
- We can define an inner product on M by $\langle A, B \rangle := \mathrm{tr}(B^* A)$. Set H to be the completion of M with respect to this inner product.
- M acts on H by left multiplication, whence we can view M as a $*$-subalgebra of $\mathcal{B}(H)$. We set $\mathcal{R}$ to be the von Neumann algebra generated by M. $\mathcal{R}$ is called *the hyperfinite II_1 factor*.

Tracial von Neumann algebras

Suppose that A is a von Neumann algebra. A *tracial state* (or just *trace*) on A is a linear functional $\tau : A \to \mathbb{C}$ satisfying:

- $\tau(1) = 1$;
- $\tau(x^*x) \geq 0$ for all $x \in A$;
- $\tau(xy) = \tau(yx)$ for all $x, y \in A$.

A *tracial von Neumann algebra* is a pair (A, τ), where A is a von Neumann algebra and τ is a trace on A.

In the case that τ is also *faithful*, meaning that $\tau(x^*x) = 0 \Rightarrow x = 0$, the function $\langle x, y \rangle_\tau := \tau(y^*x)$ is an inner product on A, yielding the so-called *2-norm* $\| \cdot \|_2$ on A. The associated metric is complete on any bounded subset of A.

(A, τ) is called *separable* if the metric associated to the 2-norm is separable.

II_1 Factors

A von Neumann algebra A is said to be a *factor* if $A \cap A' = \mathbb{C} \cdot 1$.

Fact

If A is a von Neumann algebra, then $A \cong \int_X^{\oplus} A_x$ (a *direct integral*) where each A_x is a factor.

A factor is said to be of type II_1 if it is infinite-dimensional and admits a trace.

Fact

A II_1 factor admits a unique weakly continuous trace, which is automatically faithful.

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Examples-revisited

- $\mathcal{B}(H)$ is a factor. If $\dim(H) < \infty$, then $\mathcal{B}(H)$ admits a trace, but is not a II_1 factor. If $\dim(H) = \infty$, then $\mathcal{B}(H)$ admits no trace. Thus, $\mathcal{B}(H)$ is never a II_1 factor.
- $L^\infty(X, \mu)$ admits a trace $f \mapsto \int_X f \, d\mu$ but is not a factor.
- If G is a countable group that is *ICC*, namely all conjugacy classes (other than $\{1\}$) are infinite, then $L(G)$ is a II_1 factor; the trace is given by $T \mapsto \langle T\delta_e, \delta_e \rangle$. In particular, if $n \geq 2$, then $L(\mathbb{F}_n)$ is a II_1 factor.
- $\mathcal{R}$ is a II_1 factor; the trace $\mathrm{tr} : \bigcup_n \bigotimes_n M_2 \to \mathbb{C}$ extends uniquely to the completion. Moreover, $\mathcal{R}$ embeds into any II_1 factor.

Why model theorists care about II_1 factors

- It is straightforward to check that the class of tracial von Neumann algebras (in the correct signature for continuous logic) is a universally axiomatizable class. We let T_{vNa} denote the theory of tracial von Neumann algebras.
- Moreover, it is a fact that the class of II_1 factors is $\forall\exists$-axiomatizable.
- Note that any tracial von Neumann algebra embeds into a II_1 factor: $A \subseteq A * L(\mathbb{Z})$ (free product).
- It follows that an existentially closed tracial von Neumann algebra is a II_1 factor.

Ultrapowers of von Neumann algebras

Suppose that (A,τ) is a tracial von Neumann algebra and $\mathcal{U}$ is a nonprincipal ultrafilter on $\mathbb{N}$. We set

$$\ell^\infty(A) := \{(a_n) \in A^{\mathbb{N}} \ : \ \|a_n\| \text{ is bounded}\}.$$

Unfortunately, if we quotient this out by the ideal

$$\{(a_n) \in A^{\mathbb{N}} \ : \ \lim_{\mathcal{U}} \|a_n\| = 0\},$$

the resulting quotient is usually never a von Neumann algebra. Rather, we have to quotient out by the smaller ideal

$$\{(a_n) \in A^{\mathbb{N}} \ : \ \lim_{\mathcal{U}} \|a_n\|_2 = 0\},$$

yielding the *tracial ultrapower* $A^{\mathcal{U}}$ of A.
Continuous logic provided a logical framework for the study of these ultrapowers.

$\mathcal{R}^\omega$-embeddability

Definition

We say that a separable II_1 factor A is *$\mathcal{R}^\omega$-embeddable* if there is a nonprincipal ultrafilter $\mathcal{U}$ on $\mathbb{N}$ such that A embeds into $\mathcal{R}^{\mathcal{U}}$.

Remarks

1. If A is $\mathcal{R}^\omega$-embeddable, then A embeds into $\mathcal{R}^{\mathcal{U}}$ for *any* nonprincipal ultrafilter on $\mathbb{N}$.
2. A is $\mathcal{R}^\omega$-embeddable if and only if $A \models \mathrm{Th}_\forall(\mathcal{R})$, the universal theory of $\mathcal{R}$.

Connes' Embedding Problem

- In 1976, Connes proved that $L(\mathbb{F}_2)$ is $\mathcal{R}^\omega$-embeddable.
- He then remarked "Apparently such an embedding ought to exist for all II_1 factors..."
- This remark is now known as the *Connes Embedding Problem* (CEP) and is *the* central question in operator algebras. It has zillions of equivalent reformulations.
- For example, it is known that $L(G)$ is $\mathcal{R}^\omega$-embeddable if and only if G is hyperlinear. So settling the CEP for group von Neumann algebras would settle the question of whether or not all groups are hyperlinear (a serious question in group theory).
- Call a separable II_1 factor A *locally universal* if every separable II_1 factor is A^ω-embeddable. (So CEP asks whether or not $\mathcal{R}$ is locally universal.) Hart, Farah, and Sherman proved the existence of one (and therefore many) locally universal II_1 factors ("Poor man's CEP").

Model companions

- Recall that a theory T is *model complete* if any embedding between models of T is elementary.
- If T' is a theory, then a model complete theory T is a *model companion* for T' if any model of T' embeds in a model of T and vic-versa (that is, if $T'_\forall = T_\forall$). A theory can have at most one model companion.
- If T' is universal, then T' has a model companion T if and only if the class of its existentially closed structures is elementary; in this case T is their theory.

Theorem (G., Hart, Sinclair)

T_{vNa} does not have a model companion.

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Crossed products of tracial vNas

Suppose that M is a von Neumann algebra, G is a countable group and $\alpha : G \to \mathrm{Aut}(M)$ is a group homomorphism. Then there is another von Neumann algebra $M \rtimes_\alpha G$ satisfying the following

Proposition

1. There is an embedding $I : M \to M \rtimes_\alpha G$;
2. $L(G)$ is naturally a subalgebra of $M \rtimes_\alpha G$;
3. The action of G on M, inside of $M \rtimes_\alpha G$, is given by unitary conjugation:

$$I(\alpha_g(x)) = \lambda(h) \circ I(x) \circ \lambda(g^{-1}), \quad x \in M, g \in G.$$

4. If M is tracial, then so is $M \rtimes_\alpha G$.
5. If M is $\mathcal{R}^\omega$-embeddable and G is amenable, then $M \rtimes_\alpha G$ is also $\mathcal{R}^\omega$-embeddable.

$\mathcal{R}$ does not have QE

Theorem

$\mathcal{R}$ does not have QE.

Proof.

- It is enough to find $\mathcal{R}^\omega$-embeddable von Neumann algebras M and N with $M \subset N$ and an embedding $\pi : M \hookrightarrow \mathcal{R}^\mathcal{U}$ that does not extend to an embedding $N \hookrightarrow \mathcal{R}^\mathcal{U}$.
- Towards this end, it is enough to find a countable group G such that $L(G)$ is $\mathcal{R}^\omega$-embeddable, an embedding $\pi : L(G) \hookrightarrow \mathcal{R}^\mathcal{U}$, and $\alpha \in \mathrm{Aut}(L(G))$ such that there exists no unitary $u \in \mathcal{R}^\mathcal{U}$ satisfying $(\pi \circ \alpha)(x) = u\pi(x)u^*$ for all $x \in L(G)$. (We'll explain this on the next slide.)
- By nontrivial work of Nate Brown, we can take $G = \mathrm{SL}(3, \mathbb{Z}) * \mathbb{Z}$ and $\alpha = \mathrm{id} * \theta$ for any nontrivial $\theta \in \mathrm{Aut}(L(\mathbb{Z}))$.

$\mathcal{R}$ does not have QE (cont'd)

Proof.

- Suppose that G, π, and α are as above. Set $M := L(G)$ and $N := M \rtimes_\alpha \mathbb{Z}$. Then N is $\mathcal{R}^\omega$-embeddable.
- Suppose, towards a contradiction, that π extends to $\tilde{\pi} : N \hookrightarrow \mathcal{R}^\mathcal{U}$.
- Let $u \in N$ be the generator of $\mathbb{Z}$ and set $\tilde{u} := \tilde{\pi}(u)$. We then have, for $x \in M$:
$$\tilde{u}\pi(x)\tilde{u}^* = \pi(uxu^*) = \pi(\alpha(x)),$$
contradicting our choice of π and α.

$\square$

Other non-QE results

Definition

If A is a separable II_1 factor, we say that A is *McDuff* if $A \otimes \mathcal{R} \cong A$.

- For example, $\mathcal{R}$ is McDuff.
- Any II_1 factor A embeds into a McDuff factor: $A \subseteq A \otimes \mathcal{R}$.
- It is a fact that McDuffness is $\forall\exists$-axiomatizable, whence a separable existentially closed tracial von Neumann algebra is a McDuff II_1 factor.

We noticed that Brown's work would apply if instead of $\mathcal{R}$ we had a locally universal, McDuff II_1 factor. We thus have:

Theorem

If $\mathcal{S}$ is a locally universal, McDuff II_1 factor, then $\mathcal{S}$ does not have QE.

Proof of the Main Theorem

- Suppose, towards a contradiction, that T_{vNa} has a model companion T. Since T_{vNA} is $\forall$-axiomatizable and has the amalgamation property, we have that T has QE.
- Fix a separable model $\mathcal{S}$ of T. As discussed earlier, models of T are then existentially closed tracial von Neumann algebras, whence $\mathcal{S}$ is a McDuff II_1 factor.
- Moreover, $\mathcal{S}$ is a locally universal II_1 factor: if A is an arbitrary separable tracial vNA, then A embeds in some separable $\mathcal{S}_1 \models T$. Since $\mathcal{S}^{\mathcal{U}}$ is ω_1-saturated, $\mathcal{S}_1$ embeds in $\mathcal{S}^{\mathcal{U}}$, whence A embeds in $\mathcal{S}^{\mathcal{U}}$.
- By our previous theorem, $\mathcal{S}$ does not have QE, a contradiction. $\square$

Are there model complete theories of tracial vNas?

Just because there is no model companion of T_{vNA} does not prevent there from being a model-complete theory of tracial von Neumann algebras, so we raise the question: Is there a model-complete theory of tracial von Neumann algebras (whose models would automatically be II_1 factors)?

Theorem (G., Hart, Sinclair)

If the CEP has a positive solution, then there is no model-complete theory of tracial von Neumann algebras.

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A preliminary result

Fact (Jung)

Any embedding $\mathcal{R} \to \mathcal{R}^{\mathcal{U}}$ is unitarily equivalent to the diagonal embedding (whence elementary).

Remark

Jung's result shows that $\mathcal{R}$ is the prime model of its theory.

$\mathcal{R}$ is the only possibility

Proposition

Suppose that A is an $\mathcal{R}^\omega$-embeddable II_1 factor such that $\mathrm{Th}(A)$ is model-complete. Then $A \equiv \mathcal{R}$.

Proof.

Draw crude diagram on the board. □

We now see how CEP implies that there is no model-complete theory of II_1 factors. Indeed, if T were a model-complete theory of II_1 factors, then by CEP, models of T would be $\mathcal{R}^\omega$-embeddable, whence the above proposition shows $T = \mathrm{Th}(\mathcal{R})$. Another use of CEP shows that $T_\forall = T_{vNa}$, whence T is a model companion for T_{vNa}, which we know has no model companion.

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Proposition

Suppose that A is an $\mathcal{R}^\omega$-embeddable II_1 factor such that Th(A) is model-complete. Then $A \equiv \mathcal{R}$.

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Free Group Factors

- Murray and von Neumann showed that $L(\mathbb{F}_n) \not\cong \mathcal{R}$ by showing that $\mathcal{R}$ has a certain property, called (Γ), that $L(\mathbb{F}_n)$ does not have.
- It is not too difficult to show that (Γ) is axiomatizable by a set of sentences in continuous logic, whence $L(\mathbb{F}_n) \not\equiv \mathcal{R}$.
- Since $L(\mathbb{F}_n)$ is $\mathcal{R}^\omega$-embeddable, we see that $\mathrm{Th}(\mathbb{F}_n)$ is not model-complete.
- **Big Open Question:** For distinct $m, n \geq 2$, is $L(\mathbb{F}_m) \cong L(\mathbb{F}_n)$?
- **Weaker, but still difficult, Open Question:** For distinct $m, n \geq 2$, is $L(\mathbb{F}_m) \equiv L(\mathbb{F}_n)$?
- If the above question has an affirmative answer, we see that this common theory is not model-complete. But are the natural embeddings $L(\mathbb{F}_m) \hookrightarrow L(\mathbb{F}_n)$ (for $m < n$) elementary (like in the case of $\mathrm{Th}(\mathbb{F}_n)$)?

The order property

Definition

Suppose that M is a metric structure, $\varphi(x; y)$ is a formula with $|x| = |y| = n$, and $\epsilon > 0$.

- For $a, b \in M^n$, we write $a \prec_{\varphi,\epsilon} b$ if $\varphi(a, b) \leq \epsilon$ and $\varphi(b, a) \geq 1 - \epsilon$.
- A *φ-ϵ chain of length k in M* is a sequence $a_1, \ldots, a_k$ from M^n such that $a_i \prec_{\varphi,\epsilon} a_j$ for $1 \leq i < j \leq k$.
- M has the *order property* or is *unstable* if there exists φ such that, for every $\epsilon > 0$, M has arbitrarily long finite φ-ϵ chains.
- A sequence $(M_i : i \in \mathbb{N})$ of structures has the order property if there exists φ such that, for every $\epsilon > 0$ and every $k \in \mathbb{N}$, all but finitely many of the M_i have a φ-ϵ chain of length k.

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- A φ-ϵ *chain of length k in M* is a sequence $a_1, \ldots, a_k$ from M^n such that $a_i \prec_{\varphi,\epsilon} a_j$ for $1 \leq i < j \leq k$.
- M has the *order property* or is *unstable* if there exists φ such that, for every $\epsilon > 0$, M has arbitrarily long finite φ-ϵ chains.
- A sequence $(M_i \ : \ i \in \mathbb{N})$ of structures has the order property if there exists φ such that, for every $\epsilon > 0$ and every $k \in \mathbb{N}$, all but finitely many of the M_i have a φ-ϵ chain of length k.

The order property in matrix algebras

Theorem (Hart. Farah, Sherman)

The sequence $(M_{2^n} \ : \ n \in \mathbb{N})$ *has the order property.*

Proof.

Let $x = \begin{pmatrix} 0 & \sqrt{2} \\ 0 & 0 \end{pmatrix}$ and for $1 \leq i \leq n-1$, let

$$a_i = \bigotimes_{j=0}^{i} x \otimes \bigotimes_{j=i+1}^{n-1} 1 \text{ and } b_i = \bigotimes_{j=0}^{i} 1 \otimes x^* \otimes \bigotimes_{j=i+2}^{n-1} 1.$$

Set $\varphi(x_1, x_2; y_1 y_2) := \|[x_1, y_2]\|_2$ and observe that, for $i < j$, we have $\varphi(a_i, b_i; a_j, b_j) = 0$ and $\varphi(a_j, b_j; a_i, b_i) = 2$. $\square$

II_1 factors are unstable

Corollary (Hart, Farah, Sherman)

Every II_1 factor has the order property.

Proof.

Every II_1 factor contains a copy of M_{2^n}. □

Corollary (Hart, Farah, Sherman)

Assuming $(\neg \mathrm{CH})$, any separable II_1 factor has two nonisomorphic ultrapowers.

Folkloric Theorem (Hart)

Any II_1 factor is not (model-theoretically) simple.

Rosiness?

We had hoped that, although II_1 factors are not even simple, perhaps there could be a well-behaved notion of independence. Here is a natural candidate:

- Suppose that M is a II_1 factor. Let L^2M be the completion of the inner product space $(M, \langle \cdot, \cdot \rangle_\tau)$.
- For a subalgebra N of M, we let $E_N : L^2M \to L^2N$ be the orthogonal projection map ("conditional expectation").
- For $D \subseteq M$, let $\langle D \rangle$ denote the von Neumann subalgebra of M generated by D.
- Define $A \mathop{\smile\!\!\!\!\!\!\mid}_C B$ to hold if and only if, for all $a \in A$, we have $E_{\langle BC \rangle}(a) = E_{\langle C \rangle}(a)$.
- We had an idea how to prove that $\smile\!\!\!\!\!\!\mid$ is an independence relation for $\mathrm{Th}(\mathcal{R})$ assuming QE. Without QE, this seems very difficult.

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